Real solutions to equations from geometry

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ABSTRACT. See Preface.

Dedicated to the memory of my first teacher, Samuel Sottile, who died as I began these notes in 2005.

Contents

Preface

Understanding, finding, or even deciding on the existence of real solutions to a system of equations is a very difficult problem with many applications outside of mathematics. While it is hopeless to expect much in general, we know a surprising amount about these questions for systems which possess additional structure coming from geometry. Such equations from geometry for which we have information about their real solutions are the subject of this book.

This book focuses on equations from toric varieties and Grassmannians. Not only is much known in these cases, but they encompass some of the most common applications. The results may be grouped into three themes:

(I) Upper bounds on the number of real solutions.
(II) Geometric problems that can have all solutions be real.
(III) Lower bounds on the number of real solutions.

Upper bounds (I) bound the complexity of the set of real solutions—they are one of the sources for the theory of o-minimal structures which are an important topic in real algebraic geometry. The existence (II) of geometric problems that can have all solutions be real was initially surprising, but this phenomenon now appears to be ubiquitous. Lower bounds (III) give existence proofs of real solutions. Their most spectacular manifestation is the nontriviality of the Welschinger invariant, which was computed via tropical geometry. One of the most surprising manifestations of this phenomenon is when the upper bound equals the lower bound, which is the subject of the Shapiro Conjecture and the focus of the last five chapters.

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CHAPTER 1

Overview

In mathematics and its applications, we are often faced with a system of multivariate polynomial equations whose solutions we need to study or to find. Systems that arise naturally typically possess some geometric or combinatorial structure that may be exploited to better understand their solutions. Such structured systems are studied in enumerative algebraic geometry, which has given us the deep and powerful tools of intersection theory [**54**] to count and analyze their *complex* solutions. A companion to this theoretical work are algorithms, both symbolic (based on Gröbner bases [**154**] or resultants) and numerical (many based on numerical homotopy continuation [**137**]) for solving and analyzing systems of polynomial equations. An elegant and elementary introduction into algebraic geometry, algorithms, and its applications is given in the two-volume series [**31, 30**].

Despite these successes, this line of research largely sidesteps the often primary goal of formulating problems as solutions to systems of equations—namely to determine or study their real solutions. This deficiency is particularly acute in applications, from control [**25**], to kinematics [**22**], statistics [**114**], and computational biology [**110**], for it is typically the real solutions that are needed in applications. One reason that traditional algebraic geometry ignores the real solutions is that there are few elegant theorems or general results available to study real solutions. Nevertheless, the demonstrated importance of understanding the real solutions to systems of equations demands our attention.

In the 19th century and earlier, many elegant and powerful methods were developed to study the real roots of univariate polynomials (Sturm sequences, Budan-Fourier Theorem, Routh-Hurwitz criterion), which are now standard tools in some applications of mathematics. These and other results lead to a rich algorithmic theory of real algebraic geometry, which is developed in [**4**]. In contrast, it has only been in the past few decades that serious attention has been paid toward understanding the real solutions to systems of multivariate polynomial equations.

This recent work has concentrated on systems possessing some, often geometric, structure. The reason for this is two-fold: Not only do systems from nature typically possess some special structure that should be exploited in their study, but it is highly unlikely that any results of substance hold for general or unstructured systems. From this work, a story has emerged of bounds (both upper and lower) on the number of real solutions to certain classes of systems, as well as the discovery and study of systems that have only real solutions. This overview chapter will sketch this emerging landscape and the subsequent chapters will treat these ongoing developments in more detail.

We will use the notations $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, to denote the natural numbers, integers, rational numbers, real numbers, and complex numbers. We write $\mathbb{R}_>$ for the positive real numbers, and $\mathbb{R}^*$ (or $\mathbb{T}_\mathbb{R}$) and $\mathbb{C}^*$ (or $\mathbb{T}$) for the nonzero real and complex numbers, respectively. For a positive integer n, write $[n]$ for the set $\{1,\ldots,n\}$, and let $\mathbb{Z}^n$ be the free abelian group of rank n (a lattice), and $\mathbb{Q}^n$, $\mathbb{R}^n$, and $\mathbb{C}^n$, vector spaces of dimension n over the indicated fields. Likewise $\mathbb{P}^n$ and $\mathbb{RP}^n$ are complex and projective spaces of dimension n, and $\mathbb{R}^n_>$, $(\mathbb{R}^*)^n$, and $(\mathbb{C}^*)^n = \mathbb{T}^n$ for the groups of n-tuples of positive, nonzero real, and nonzero complex numbers, respectively. These groups, vector spaces, and projective spaces, all have distinguished ordered bases. We will use $\mathbb{Z}^{\mathcal{A}}, \mathbb{R}^{\mathcal{A}}, \mathbb{T}^{\mathcal{A}}, \mathbb{P}^{\mathcal{A}}, \ldots$ to denote the groups and spaces with distinguished bases indexed by the elements of a set $\mathcal{A}$.

1.1. Introduction

Our goal is to say something meaningful about the real solutions to a system of multivariate polynomial equations. For example, consider a system

$$f_1(x_1,\ldots,x_n) \;=\; f_2(x_1,\ldots,x_n) \;=\; \cdots \;=\; f_N(x_1,\ldots,x_n) \;=\; 0\,, \tag{1.1}$$

of N real polynomials in n variables. Let r be its number of real solutions and let d be its number of complex solutions. We always assume that our systems are *generic* in the sense that all of their solutions are *nondegenerate*. Specifically, the differentials df_i of the polynomials at each solution span $\mathbb{C}^n$, so that each solution has algebraic multiplicity 1. Our systems will come in families whose generic member is nondegenerate and has d complex solutions. Since every real number is complex, and since nonreal solutions come in complex conjugate pairs, we have the following trivial inequality,

$$d \;\geq\; r \;\geq\; d \mod 2 \;\in\; \{0,1\}\,. \tag{1.2}$$

We can say nothing more unless the equations have some structure, and a particularly fruitful class of structures are those which come from geometry. The main point of this book is that we can identify structures in equations that will allow us to do better than this trivial inequality (1.2).

Our discussion will have three themes:

(I) Sometimes, there is a smaller bound on r than d.
(II) For many problems from enumerative geometry, the upper bound is sharp.
(III) The lower bound on r may be significantly larger than $d \mod 2$.

A major theme will be the Shapiro Conjecture (Mukhin, Tarasov, and Varchenko Theorem [**104**]) and its generalizations, which is a situation where the upper bound of d is also the lower bound—all solutions to our system are real. This also occurs in Example 9.7.

We will not describe how to actually find the solutions to a system (1.1) and there will be little discussion of algorithms and no complexity analysis. The book of Basu, Pollack, and Roy [**4**] is an excellent place to learn about algorithms for computing real algebraic varieties and finding real solutions. We remark that some techniques employed to study real solutions underlie numerical algorithms to compute the solutions [**137**]. Also, ideas from toric geometry [**52, 61**], Gröbner bases [**154**], combinatorial commutative algebra [**100**], and Schubert Calculus [**53**] permeate this book. Other background material may be found in [**31, 30**].

1.2. Polyhedral bounds

When $N = n$, the most fundamental bound on the number of complex solutions is due to Bézout: d is at most the product of the degrees of the polynomials f_i. When the polynomials have a sparse, or polyhedral structure, the smaller BKK bound applies.

Integer vectors $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ are exponents for (Laurent) monomials

$$\mathbb{Z}^n \ni a \;\leftrightarrow\; x^a \;:=\; x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \;\in\; \mathbb{C}[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]\,.$$

We will often identify a monomial with its exponent vector and thus will just call elements of $\mathbb{Z}^n$ *monomials*. Let $\mathcal{A} \subset \mathbb{Z}^n$ be a finite set of monomials. A linear combination

$$\sum_{a \in \mathcal{A}} c_a x^a \qquad c_a \in \mathbb{R}$$

of monomials from $\mathcal{A}$ is a *sparse polynomial* with *support $\mathcal{A}$*. Sparse polynomials naturally define functions on the complex torus $\mathbb{T}^n := (\mathbb{C}^*)^n$. A system (1.1) of $N = n$ polynomials in n variables, where each polynomial has support $\mathcal{A}$, will be called a *system* (of polynomials) with *support $\mathcal{A}$*. These are often called *unmixed systems* in contrast to *mixed systems* where each polynomial may have different support. While sparse systems occur naturally—multilinear or multihomogeneous polynomials are an example—they also occur in problem formulations for the simple reason that we humans seek simple formulations of problems, and this often means polynomials with few terms.

A fundamental result about unmixed systems is the Kushnirenko bound on their number of complex solutions. The *Newton polytope* of a polynomial f with support $\mathcal{A}$ is the convex hull $\Delta_{\mathcal{A}}$ of the set $\mathcal{A}$ of monomials of f. Write $\mathrm{volume}(\Delta)$ for the Euclidean volume of a polytope Δ.

THEOREM 1.1 (Kushnirenko [**10**]). *A system of n polynomials in n variables with support $\mathcal{A}$ has at most $n!\,\mathrm{volume}(\Delta_{\mathcal{A}})$ isolated solutions in $(\mathbb{C}^*)^n$, and exactly this number when the polynomials are generic polynomials with support $\mathcal{A}$.*

Bernstein generalized this to mixed systems. The Minkowski sum $P + Q$ of two polytopes in $\mathbb{R}^n$ is their pointwise sum as sets of vectors in $\mathbb{R}^n$. Let $P_1, \ldots, P_n \subset \mathbb{R}^n$ be polytopes. The volume

$$\mathrm{volume}(t_1 P_1 \;+\; t_2 P_2 \;+\; \cdots \;+\; t_n P_n)$$

is a homogeneous polynomial of degree n in the variables $t_1, \ldots, t_n$ [**63**, Exercise 15.2.6]. The *mixed volume* $\mathrm{MV}(P_1, \ldots, P_n)$ of $P_1, \ldots, P_n$ is the coefficient of the monomial $t_1 \cdots t_n$ in this polynomial.

THEOREM 1.2 (Bernstein [**11**]). *A system of n polynomials in n variables where the polynomials have supports $\mathcal{A}_1, \ldots, \mathcal{A}_n$ has at most $\mathrm{MV}(\Delta_{\mathcal{A}_1}, \ldots, \Delta_{\mathcal{A}_n})$ isolated solutions in $(\mathbb{C}^*)^n$, and exactly this number when the polynomials are generic for their given support.*

Since $\mathrm{MV}(P_1, \ldots, P_n) = n!\,\mathrm{volume}(P)$ when $P_1 = \cdots = P_n = P$, this generalizes Kushnirenko's Theorem. We will prove these theorems in Chapters 3 and 4.

The bound of Theorem 1.1 and its generalization Theorem 1.2 is often called the *BKK bound* for Bernstein, Khovanskii, and Kushnirenko [**10**].

1.3. Upper bounds

While the number of complex roots of a univariate polynomial is typically equal to its degree, the number of real roots depends upon the length of the expression for the polynomial. Indeed, by Descartes's rule of signs [**34**] (see Section 2.1), a univariate polynomial with $m{+}1$ terms has at most m positive roots, and thus at most $2m$ nonzero real roots. For example, the polynomial $x^d - a$ with $a \neq 0$ has 0, 1, or 2 real roots, but always has d complex roots. Khovanskii generalized this type of bound to multivariate polynomials with his fundamental *fewnomial bound*.

Theorem 1.3 (Khovanskii [**83**]). *A system of n polynomials in n variables having a total of $1{+}l{+}n$ distinct monomials has at most*

$$2^{\binom{l+n}{2}}(n+1)^{l+n}$$

nondegenerate positive solutions.

There are two reasons for this restriction to positive solutions. Most fundamentally is that Khovanskii's proof requires this restriction. This restriction also excludes the following type of trivial zeroes: Under the substitution $x_i \mapsto x_i^2$, each positive solution becomes 2^n real solutions, one in each of the 2^n orthants. More subtle substitutions lead to similar trivial zeroes which differ from the positive solutions only by some sign patterns.

This is the first of many results verifying the principle of Bernstein and Kushnirenko that the topological complexity of a set defined by real polynomials should depend on the number of terms in the polynomials and not on the degrees of the polynomials. Khovanskii's work was also a motivation for the notion of o-minimal structures [**160, 113**]. The main point of Khovanskii's theorem is the existence of such a bound and not the actual bound itself.

Nevertheless, it raises interesting questions about such bounds. For each $l, n \geq 1$, we define the *Khovanskii number $X(l,n)$* to be the maximum number of nondegenerate positive solutions to a system of n polynomials in n variables with $1{+}l{+}n$ monomials. Khovanskii's Theorem gives a bound on $X(l,n)$, but that bound is enormous. For example, when $l = n = 2$, the bound is 5184. Because of this, Khovanskii's bound was expected to be far from sharp. Despite this expectation, the first nontrivial improvement was only given in 2003.

Theorem 1.4 (Li, Rojas, and Wang [**94**]). *Two trinomials in two variables have at most five nondegenerate positive solutions.*

This bound is sharp. Haas [**64**] had shown that the system of two trinomials in x and y

$$10x^{106} + 11y^{53} - 11y \;=\; 10y^{106} + 11x^{53} - 11x \;=\; 0\,, \tag{1.3}$$

has five positive solutions.

Since we may multiply one of the trinomials in (1.3) by an arbitrary monomial without changing the solutions, we can assume that the two trinomials (1.3) share a common monomial, and so there are at most $3+3-1 = 5 = 2+2+1$ monomials between the two trinomials, and so two trinomials give a fewnomial system with $l = n = 2$. While five is less than 5184, Theorem 1.4 does not quite show that $X(2,2) = 5$ as two trinomials do not constitute a general fewnomial system with $l = n = 2$. Nevertheless, Theorem 1.4 gave strong evidence that Khovanskii's fewnomial bound may be improved. Such an improved bound was given in [**17**].

THEOREM 1.5. $X(l,n) < \frac{e^2+3}{4}2^{\binom{l}{2}}n^l$.

For small values of l, it is not hard to improve this. For example, when $l=0$, the support $\mathcal{A}$ of the system is a simplex, and there will be at most one positive real solution, so $X(0,n)=1$. Theorem 1.5 was inspired by the sharp bound of Theorem 1.7 when $l=1$ [**15**]. A set $\mathcal{A}$ of exponents is *primitive* if $\mathcal{A}$ affinely spans the full integer lattice $\mathbb{Z}^n$. That is, the differences of vectors in $\mathcal{A}$ generate $\mathbb{Z}^n$.

THEOREM 1.6. *If $l=1$ and the set $\mathcal{A}$ of exponents is primitive, then there can be at most $2n+1$ nondegenerate nonzero real solutions, and this is sharp in that for any n there exist systems with $n+2$ monomials and $2n+1$ nondegenerate real solutions whose exponent vectors affinely span $\mathbb{Z}^n$.*

Observe that this bound is for all nonzero real solutions, not just positive solutions. We will discuss this in Section 5.3. Further analysis by Bihan gives the sharp bound on $X(1,n)$.

THEOREM 1.7 (Bihan [**15**]). $X(1,n)=n+1$.

These fewnomial bounds are discussed and proven in Chapters 5 and 6.

In contrast to these results establishing absolute upper bounds on the number of real solutions which improve the trivial bound of the number d of complex roots, there are a surprising number of problems that come from geometry for which all solutions can be real. For example, Sturmfels [**153**] proved the following. (Regular triangulations are defined in Section 4.2, and we give his proof in Section 4.4.)

THEOREM 1.8. *Suppose that a lattice polytope $\Delta\subset\mathbb{Z}^n$ admits a regular triangulation with each simplex having minimal volume $\frac{1}{n!}$. Then there is a system of sparse polynomials with support $\Delta\cap\mathbb{Z}^n$ having all solutions real.*

For many problems from enumerative geometry, it is similarly possible that all solutions can be real. This will be discussed in Chapter 9.

1.4. The Wronski map and the Shapiro Conjecture

The *Wronskian* of univariate polynomials $f_1(t),\ldots,f_m(t)$ is the determinant

$$\mathrm{Wr}(f_1,f_2,\ldots,f_m) \;:=\; \det\Big(\big(\tfrac{d}{dt}\big)^{i-1}f_j(t)\Big)_{i,j=1,\ldots,m}\,.$$

When the polynomials f_i have degree $m+p-1$ and are linearly independent, the Wronskian has degree at most mp. For example, if $m=2$, then $\mathrm{Wr}(f,g)=f'g-fg'$, which has degree $2p$ as the coefficients of t^{2p+1} in this expression cancel. Up to a scalar, the Wronskian depends only upon the linear span of the polynomials $f_1,\ldots,f_m$. Removing these ambiguities gives the *Wronski map*,

$$\mathrm{Wr}\,:\;\; \mathrm{Gr}(m,\mathbb{C}_{m+p-1}[t]) \;\longrightarrow\; \mathbb{P}(\mathbb{C}_{mp}[t])\simeq\mathbb{P}^{mp}\,, \tag{1.4}$$

where $\mathrm{Gr}(m,\mathbb{C}_{m+p-1}[t])$ is the *Grassmannian* of m-dimensional subspaces of the linear space $\mathbb{C}_{m+p-1}[t]$ of complex polynomials of degree $m+p-1$ in the variable t, and $\mathbb{P}(\mathbb{C}_{mp}[t])$ is the projective space of complex polynomials of degree at most mp, which has dimension mp, equal to the dimension of the Grassmannian.

Work of Schubert in 1886 [**130**], combined with a result of Eisenbud and Harris in 1983 [**40**] shows that the Wronski map is surjective and the general polynomial $\Phi\in\mathbb{P}^{mp}$ has

$$\#_{m,p} \;:=\; \frac{1!2!\cdots(m-1)!\cdot(mp)!}{m!(m+1)!\cdots(m+p-1)!} \tag{1.5}$$

preimages under the Wronski map. These results concern the complex Grassmannian and complex projective space.

Boris Shapiro and Michael Shapiro made a conjecture in 1993/4 about the Wronski map from the real Grassmannian to real projective space.

THEOREM 1.9. *If the polynomial $\Phi \in \mathbb{P}^{mp}$ has only real zeroes, then every point in* $\mathrm{Wr}^{-1}(\Phi)$ *is real. Moreover, if Φ has mp simple real zeroes then there are $\#_{m,p}$ real points in* $\mathrm{Wr}^{-1}(\Phi)$.

This was proven when $\min(m,p) = 2$ by Eremenko and Gabrielov [**46**], who subsequently found a second, elementary proof [**42**], which we present in Chapter 11. It was finally settled by Mukhin, Tarasov, and Varchenko [**104**], who showed that every point in the fiber is real. We sketch their proof in Chapter 12. The second statement, about there being the expected number of real roots, follows from this by an argument of Eremenko and Gabrielov that we reproduce in Chapter 13 (Theorem 13.2). It also follows from a second proof of Mukhin, Tarasov, and Varchenko, in which they directly show transversality [**106**], which is equivalent to the second statement. This *Shapiro Conjecture* has appealing geometric interpretations, enjoys links to several areas of mathematics, and has many theoretically satisfying generalizations which we will discuss in Chapters 10, 11, 13, and 14. We now mention two of its interpretations.

EXAMPLE 1.10 (The problem of four lines). A geometric interpretation of the Wronski map and the Shapiro Conjecture when $m = p = 2$ is a variant of the classical problem of the lines in space which meet four given lines. Points in $\mathrm{Gr}(2, \mathbb{C}_3[t])$ correspond to lines in $\mathbb{C}^3$ as follows. The *moment curve γ* in $\mathbb{C}^3$ is the curve with parameterization

$$\gamma(t) \ := \ (t, t^2, t^3)\,.$$

A cubic polynomial f is the composition of γ and an affine-linear map $\mathbb{C}^3 \to \mathbb{C}$, and so a two-dimensional space of cubic polynomials is a two-dimensional space of affine-linear maps whose common kernel is the corresponding line in $\mathbb{C}^3$. (This description is not exact, as some points in $\mathrm{Gr}(2, \mathbb{C}_3[t])$ correspond to lines at infinity.)

Given a polynomial $\Phi(t)$ of degree four with distinct real roots, points in the fiber $\mathrm{Wr}^{-1}(\Phi)$ correspond to the lines in space which meet the four lines tangent to the moment curve γ at its points coming from the roots of Φ. There will be two such lines, and the Shapiro Conjecture asserts that both will be real.

It is not hard to see this directly. Any fractional linear change of parameterization of the moment curve is realized by a projective linear transformation of three-dimensional space which stabilizes the image of the moment curve. Thus we may assume that the polynomial $\Phi(t)$ is equal to $(t^3 - t)(t - s)$, which has roots $-1, 0, 1$, and s, where $s \in (0, 1)$. Applying an affine transformation to three-dimensional space, the moment curve becomes the curve with parameterization

$$\gamma \ : \ t \ \longmapsto \ (6t^2 - 1,\ \tfrac{7}{2}t^3 + \tfrac{3}{2}t,\ \tfrac{3}{2}t - \tfrac{1}{2}t^3)\,. \qquad (1.6)$$

Then the lines tangent to γ at the roots $-1, 0, 1$ of Φ have parameterizations

$$(5 - s,\ -5 + s,\ -1)\,,\ (-1,\ s,\ s)\,,\ (5 + s\ , 5 + s\ , 1) \qquad s \in \mathbb{R}\,.$$

These lie on a hyperboloid of one sheet, which is defined by

$$1 - x_1^2 + x_2^2 - x_3^2 \ = \ 0\,. \qquad (1.7)$$

We display this geometric configuration in Figure 1.1. There, $\ell(i)$ is the line tangent to γ at the point $\gamma(i)$. The hyperboloid has two rulings. One ruling contains our

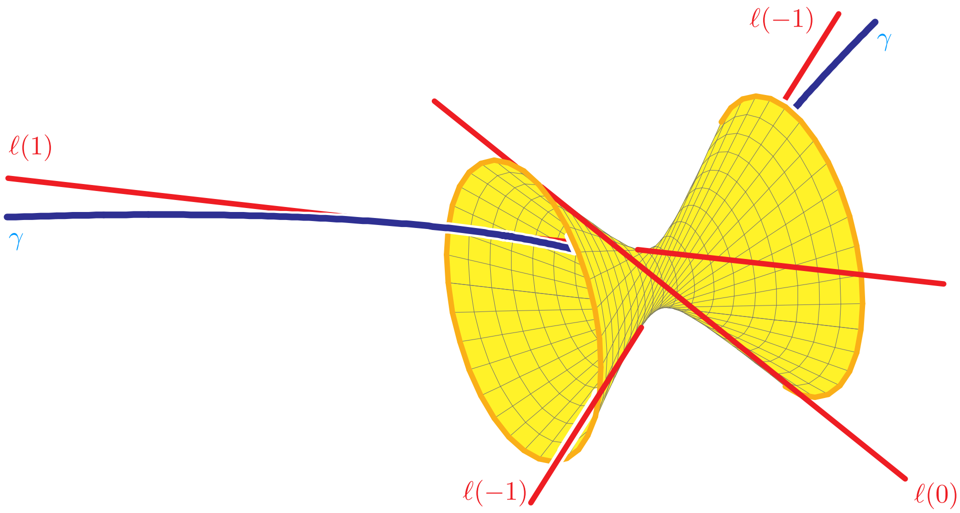


FIGURE 1.1. hyperboloid containing three lines tangent to γ.

three tangent lines and the other ruling (which is drawn on hyperboloid) consists of the lines which meet our three tangent lines.

Now consider the fourth line $\ell(s)$ which is tangent to γ at the point $\gamma(s)$. This has the parameterization

$$\ell(s) \;=\; \left(6s^2-1\,,\ \tfrac{7}{2}s^3+\tfrac{3}{2}s\,,\ \tfrac{3}{2}s-\tfrac{1}{2}s^3\right) \;+\; t\left(12s\,,\ \tfrac{21}{2}s^2+\tfrac{3}{2}\,,\ \tfrac{3}{2}-\tfrac{3}{2}s^2\right).$$

We compute the intersection of the fourth line with the hyperboloid. Substituting its parameterization into (1.7) and dividing by -12 gives the equation

$$(s^3-s)(s^3-s+t(6s^2-2)+9st^2) \;=\; 0\,.$$

The first (nonconstant) factor s^3-s vanishes when $\ell(s)$ is equal to one of $\ell(-1)$, $\ell(-0)$, or $\ell(1)$–for these values of s every point of $\ell(s)$ lies on the hyperboloid. The second factor has solutions

$$t \;=\; -\,\frac{3s^2-1\pm\sqrt{3s^2+1}}{9s}\,.$$

Since $3s^2+1>0$ for all s, both solutions will be real. In fact, for $s\neq\sqrt{-1/3}$, this will have exactly two solutions.

We may also see this geometrically. Consider the fourth line $\ell(s)$ for $0<s<1$. In Figure 1.2, we look down the throat of the hyperboloid at the interesting part of this configuration. This picture demonstrates that $\ell(s)$ must meet the hyperboloid in two real points. Through each point, there is a real line in the second ruling which meets all four tangent lines, and this proves the Shapiro Conjecture for $m=p=2$.

EXAMPLE 1.11 (Rational functions with real critical points). When $m=2$, the Shapiro Conjecture may be interpreted in terms of rational functions. A rational function $\rho(t)=f(t)/g(t)$ is a quotient of two univariate polynomials, f and g. This

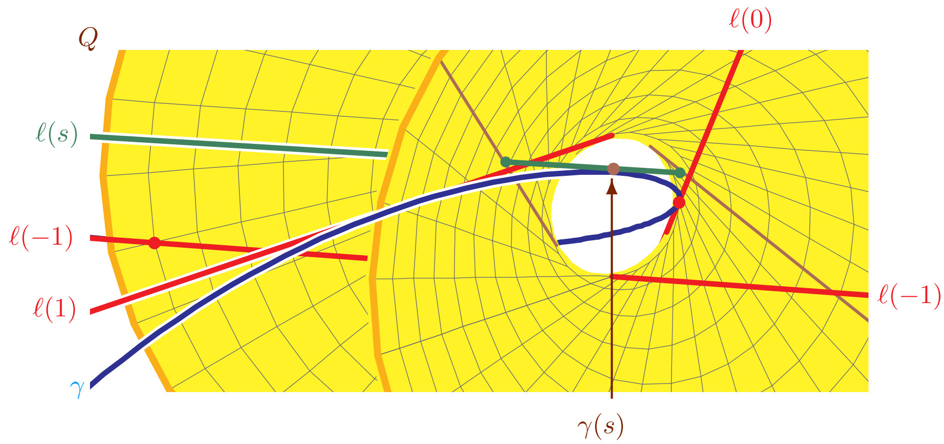


FIGURE 1.2. The fourth tangent line meets hyperboloid in two real points.

defines a map $\rho\colon \mathbb{P}^1 \to \mathbb{P}^1$ whose critical points are those t for which $\rho'(t) = 0$. Since $\rho'(t) = (f'g - g'f)/g^2$, we see that the critical points are the roots of the Wronskian of f and g. Composing the rational function $\rho\colon \mathbb{P}^1 \to \mathbb{P}^1$ with an automorphism of the target $\mathbb{P}^1$ gives an equivalent rational function, and the equivalence class of ρ is determined by the linear span of its numerator and denominator. Thus the Shapiro Conjecture asserts that a rational function having only real critical points is equivalent to a real rational function.

Eremenko and Gabrielov [**46**] proved exactly this statement in 2002, thereby establishing the Shapiro Conjecture in the case $m = 2$.

THEOREM 1.12. *A rational function with only real critical points is equivalent to a real rational function.*

In Chapter 11 we will present an elementary proof of this result that Eremenko and Gabrielov gave in 2005 [**42**].

1.5. Lower bounds

We begin with perhaps the most exciting recent development in real algebraic geometry. This starts with the fundamental observation of Euclid that two points determine a line. Slightly less elementary is that five points in the plane with no three collinear determine a conic. In general, if you have n general points in the plane and you want to pass a rational curve of degree d through all of them, there may be no solution to this interpolation problem (if n is too big), or an infinite number of solutions (if n is too small), or a finite number of solutions (if n is just right). It turns out that "n just right" means $n = 3d-1$ ($n = 2$ for lines where $d = 1$, and $n = 5$ for conics where $d = 2$).

A harder question is, if $n = 3d-1$, *how many* rational curves of degree d interpolate the points? Call this number N_d, so that $N_1 = 1$ and $N_2 = 1$ because the line and conic of the previous paragraph are unique. It has long been known that $N_3 = 12$ (see Example 9.3 for a proof), and in 1873 Zeuthen [**164**] showed that $N_4 = 620$. That was where matters stood until 1989, when Ran [**118**] gave a recursion for these numbers. In the 1990's, Kontsevich and Manin [**88**] used associativity in quantum cohomology of $\mathbb{P}^2$ to give the elegant recursion

$$N_d \;=\; \sum_{a+b=d} N_a N_b \left(a^2b^2\binom{3d-4}{3a-2} - a^3b\binom{3d-4}{3a-1} \right) , \tag{1.8}$$

which begins with the Euclidean declaration that two points determine a line ($N_1 = 1$). These numbers grow quite fast, for example $N_5 = 87304$.

The number of real rational curves which interpolate a given $3d-1$ points in the real plane $\mathbb{RP}^2$ will depend rather subtly on the configuration of the points. To say anything about the real rational curves would seem impossible. However this is exactly what Welschinger [**162**] did, by finding an invariant which does not depend upon the choice of points.

A rational curve in the plane is necessarily singular—typically it has $\binom{d-1}{2}$ ordinary double points. Real curves have three types of ordinary double points. Only two types are visible in $\mathbb{RP}^2$, and we are familiar with them from rational cubics, which typically have an ordinary double point. The curve on the left below has a *node* with two real branches, and the curve on the right has a *solitary point* '$\bullet$', where two complex conjugate branches meet.

The third type of ordinary double point is a pair of complex conjugate ordinary double points, which are not visible in $\mathbb{RP}^2$.

THEOREM 1.13 (Welschinger [**162**]). *The sum,*

$$\sum (-1)^{\#\{solitary\ points\ in\ C\}} , \tag{1.9}$$

over all real rational curves C of degree d interpolating $3d-1$ general points in $\mathbb{RP}^2$ does not depend upon the choice of points.

Set W_d to be the sum (1.9). The absolute value of this *Welschinger invariant* is then a lower bound on the number of real rational curves of degree d interpolating $3d-1$ points in $\mathbb{RP}^2$. Since $N_1 = N_2 = 1$, we have $W_1 = W_2 = 1$. Prior to Welschinger's discovery, Kharlamov [**33**, Proposition 4.7.3] (see also Example 9.3) showed that $W_3 = 8$. The question remained whether any other Welschinger invariants were nontrivial. This was settled in the affirmative by Itenberg, Kharlamov, and Shustin [**77, 78**], who used Mikhalkin's Tropical Correspondence Theorem [**99**] to show

(1) If $d > 0$, then $W_d \geq \frac{d!}{3}$. (Hence W_d is positive.)

(2) $\displaystyle\lim_{d\to\infty} \frac{\log N_d}{\log W_d} = 1$. (In fact for d large, $\log N_d \sim 3d\log d \sim \log W_d$.)

In particular, there are always quite a few real rational curves of degree d interpolating $3d-1$ points in $\mathbb{RP}^2$. Since then, Itenberg, Kharlamov, and Shustin [**79**] gave a recursive formula for the Welschinger invariant which is based upon Gathmann and Markwig's [**60**] tropicalization of the Caporaso-Harris [**26**] formula. This shows that $W_4 = 240$ and $W_5 = 18264$. Solomon [**136**] has also found an intersection-theoretic interpretation for these invariants.

These ideas have also found an application. Gahleitner, Jüttler, and Schicho [**58**] proposed a method to compute an approximate parameterization of a plane curve using rational cubics. Later, Fiedler-Le Touzé [**49**] used the result of Kharlamov (that $W_3 = 8$), and an analysis of pencils of plane cubics to prove that this method works.

While the story of this interpolation problem is fairly well-known, it was not the first instance of lower bounds in enumerative real algebraic geometry. In their investigation of the Shapiro Conjecture, Eremenko and Gabrielov found a similar invariant $\sigma_{m,p}$ which gives a lower bound on the number of real points in the inverse image $\mathrm{Wr}^{-1}(\Phi)$ under the Wronski map of a real polynomial $\Phi \in \mathbb{RP}^{mp)}$. Assume that $p \leq m$. If $m+p$ is odd, set $\sigma_{m,p}$ to be

$$\frac{1!2!\cdots(m{-}1)!(p{-}1)!(p{-}2)!\cdots(p{-}m{+}1)!(\frac{mp}{2})!}{(p{-}m{+}2)!(p{-}m{+}4)!\cdots(p{+}m{-}2)!\left(\frac{p-m+1}{2}\right)!\left(\frac{p-m+3}{2}\right)!\cdots\left(\frac{p+m-1}{2}\right)!}\,. \tag{1.10}$$

If $m+p$ is even, then set $\sigma_{m,p} = 0$. If $p > m$, then set $\sigma_{m,p} := \sigma_{p,m}$.

THEOREM 1.14 (Eremenko-Gabrielov [**45**]). *If a polynomial $\Phi(t) \in \mathbb{RP}^{mp}$ of degree mp is a regular value of the Wronski map, then there are at least $\sigma_{m,p}$ real m-dimensional subspaces of polynomials of degree $m{+}p{-}1$ with Wronskian Φ.*

REMARK 1.15. The number of complex points in $\mathrm{Wr}^{-1}(\Phi)$ is $\#_{m,p}$ (1.5). It is instructive to compare these numbers. We show them for $m{+}p = 11$ and $m = 2, \ldots, 5$.

m	2	3	4	5
$\sigma_{m,p}$	14	110	286	286
$\#_{m,p}$	4862	23371634	13672405890	396499770810

We also have $\sigma_{7,6} \approx 3.4 \cdot 10^4$ and $\#_{7,6} \approx 9.5 \cdot 10^{18}$. Despite this disparity in their magnitudes, the asymptotic ratio of $\log(\sigma_{m,p})/\log(\#_{m,p})$ appears to be close to $1/2$. We display this ratio in the table below, for different values of m and p.

$\frac{\log(\sigma_{m,p})}{\log(\#_{m,p})}$		m					
		2	$\frac{m+p-1}{10}$	$2\frac{m+p-1}{10}$	$3\frac{m+p-1}{10}$	$4\frac{m+p-1}{10}$	$5\frac{m+p-1}{10}$
$m+p-1$	100	0.47388	0.45419	0.43414	0.41585	0.39920	0.38840
	1000	0.49627	0.47677	0.46358	0.45185	0.44144	0.43510
	10000	0.49951	0.48468	0.47510	0.46660	0.45909	0.45459
	100000	0.49994	0.48860	0.48111	0.47445	0.46860	0.46511
	1000000	0.49999	0.49092	0.48479	0.47932	0.47453	0.47168
	10000000	0.50000	0.49246	0.48726	0.48263	0.47857	0.47616

Thus, the lower bound on the number of real points in a fiber of the Wronski map appears asymptotic to the square root of the number of complex solutions.

It is interesting to compare this to the the result of Shub and Smale [**132**] that the expected number of real solutions to a system of n Gaussian random polynomials in n variables of degrees $d_1, \ldots, d_n$ is $\sqrt{d_1 \cdots d_n}$, which is the square root of the number of complex solutions to such a system of polynomials. Thus $\frac{1}{2}$ is the ratio of the logarithm of the expected number of complex solutions to the logarithm of the expected number of real solutions.

The idea behind the proof of Theorem 1.14 is to compute the topological degree of the real Wronski map, which is the restriction of the Wronski map to real subspaces of polynomials,

$$\mathrm{Wr}_{\mathbb{R}} \;:=\; \mathrm{Wr}|_{\mathrm{Gr}(m,\mathbb{R}_{m+p-1}[t])} \;:\; \mathrm{Gr}(m,\mathbb{R}_{m+p-1}[t]) \;\longrightarrow\; \mathbb{RP}^{mp}\,.$$

This maps the Grassmannian of real subspaces of polynomials of degree $m+P-1$ to the space of real polynomials of degree mp. Recall that the topological degree (or mapping degree) of a map $f\colon X \to Y$ between two oriented manifolds X and Y of the same dimension is the number d such that $f_*[X] = d[Y]$, where $[X]$ and $[Y]$ are the fundamental cycles of X and Y in homology, respectively, and f_* is the functorial map in homology. When f is differentiable, this mapping degree may be computed as follows. Let $y \in Y$ be a regular value of f so that the derivative map on tangent spaces $df_x\colon T_xX \to T_yY$ is an isomorphism at any point x in the fiber $f^{-1}(y)$ above y. Since X and Y are oriented, the isomorphism df_x either preserves the orientation of the tangent spaces or it reverses the orientation. Let P be the number of points $x \in f^{-1}(y)$ at which df_x preserves the orientation and R be the number of points where the orientation is reversed. Then the mapping degree of f is the difference $P - R$.

There is a slight problem in computing the mapping degree of $\mathrm{Wr}_{\mathbb{R}}$, as neither the real Grassmannian $\mathrm{Gr}_{\mathbb{R}}$ nor the real projective space $\mathbb{RP}^{mp}$ are orientable when $m{+}p$ is odd, and thus the mapping degree of $\mathrm{Wr}_{\mathbb{R}}$ is not defined when $m{+}p$ is odd. Eremenko and Gabrielov get around this by computing the degree of the restriction of the Wronski map to open cells of $\mathrm{Gr}_{\mathbb{R}}$ and $\mathbb{RP}^{mp}$, where $\mathrm{Wr}_{\mathbb{R}}$ is a proper map. They also show that it is the degree of a lift of the Wronski map to oriented double covers of both spaces. The degree bears a resemblance to the Welschinger invariant as it has the form $|\sum \pm 1|$, the sum over all real points in $\mathrm{Wr}_{\mathbb{R}}^{-1}(\Phi)$, for Φ a regular value of the Wronski map. This resemblance is no accident. Solomon [**136**] showed how to orient a moduli space of rational curves with marked points so that the Welschinger invariant is indeed the degree of a map.

While both of these examples of geometric problems possessing a lower bound on their numbers of real solutions are quite interesting, they are rather special. The existence of lower bounds for more general geometric problems or for more general systems of polynomials would be quite important in applications, as these lower bounds guarantee the existence of real solutions.

With Soprunova, we [**138**] set out to develop a theory of lower bounds for sparse polynomial systems, using the approach of Eremenko and Gabrielov via mapping degree. This is a first step toward practical applications of these ideas. Chapters 7 and 8 will elaborate this theory. Here is an outline:

(*i*) Realize the solutions to a system of polynomials as the fibers of a map from a toric variety.
(*ii*) Characterize when a toric variety (or its double cover) is orientable, thus determining when the degree of this map (or a lift to double covers) exists.
(*iii*) Develop a method to compute the degree in some (admittedly special) cases.
(*iv*) Give a nice family of examples to which this theory applies.
(*v*) Use the sagbi degeneration of a Grassmannian to a toric variety [**154**, Ch. 11] and the systems of (*iv*) to reprove the result of Eremenko and Gabrielov.

Example 1.16. We close this overview with one example from this theory. Let t, x, y, z be indeterminates, and consider a sparse polynomial of the form

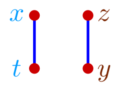
$$c_4\, txyz \;+\; c_3(txz + xyz) \;+\; c_2(tx + xz + yz) \;+\; c_1(x + z) \;+\; c_0\,, \tag{1.11}$$

where the coefficients $c_0, \ldots, c_4$ are real numbers.

Theorem 1.17. *A system involving four polynomials of the form* (1.11) *has six solutions, at least two of which are real.*

We make some remarks to illustrate the ingredients of this theory. First, the monomials in the sparse system (1.11) are the integer points in the order polytope of the poset P,

P :=



.

That is, each monomial corresponds to an order ideal of P (a subset which is closed upwards). The number of complex roots is the number of linear extensions of the poset P. There are six, as each is a permutation of the word $txyz$ where t precedes x and y precedes z.

One result (*ii*) characterizes polytopes whose associated polynomial systems will have a lower bound, and many order polytopes satisfy these conditions. Another result (*iv*) computes that lower bound for certain families of polynomials with support an order polytope. Polynomials in these families have the form (1.11) in that monomials with the same total degree have the same coefficient. For such polynomials, the lower bound is the absolute value of the sum of the signs of the permutations underlying the linear extensions. We list these for P.

permutation	$txyz$	$tyxz$	$ytxz$	$tyzx$	$ytzx$	$yztx$	sum
sign	$+$	$-$	$+$	$+$	$-$	$+$	2

This shows that the lower bound in Theorem 1.17 is two.

Table 1.1 records the frequency of the different numbers of real solutions in each of 10,000,000 instances of this polynomial system, where the coefficients were chosen uniformly from $[-200, 200]$. This computation took 13 gigahertz-hours.

Table 1.1. Observed frequencies of numbers of real solutions.

number of real solutions	0	2	4	6
frequency	0	9519429	0	480571

The apparent gap in the numbers of real solutions in Table (1.1) (four does not seem a possible number of real solutions) is proven for the system of Example 1.16 in Section 8.3. This is the first instance we have seen of this phenomena of gaps in the numbers of real solutions. More are found in [**138**], [**123**], and some are presented in Chapters 8, 13, and 14. Many examples of lower bounds continue to be found, e.g. [**3**].

CHAPTER 2

Real Solutions to Univariate Polynomials

Before we study the real solutions to systems of multivariate polynomials, we will review some of what is known for univariate polynomials. The strength and precision of results concerning real roots of univariate polynomials forms the gold standard in this subject of real solutions to systems of polynomials. We will discuss two results about univariate polynomials: Descartes's rule of signs and Sturm's Theorem. Descartes's rule of signs, or rather its generalization in the Budan-Fourier Theorem, gives a bound for the number of roots in an interval, counted with multiplicity. Sturm's theorem is topological—it simply counts the number of roots of a univariate polynomial in an interval without multiplicity. From Sturm's Theorem we obtain a simple and very useful symbolic algorithm to count the number of real solutions to a system of multivariate polynomials in many cases. We underscore the topological nature of Sturm's Theorem by presenting a new proof due to Burda and Khovanskii [**24**]. These and other fundamental results about real roots of univariate polynomials were established in the 19th century. In contrast, the main results about real solutions to multivariate polynomials have only been established in recent decades.

2.1. Descartes's rule of signs

Descartes's rule of signs [**34**] is fundamental for real algebraic geometry. Suppose that we write the terms of a univariate polynomial f in increasing order of their exponents,

$$f \;=\; c_0t^{a_0} + c_1t^{a_1} + \cdots + c_mt^{a_m}\,, \tag{2.1}$$

where $c_i \neq 0$ and $a_0 < a_1 < \cdots < a_m$.

Theorem 2.1 (Descartes's rule of signs). *The number, r, of positive roots of f, counted with multiplicity, is at most the variation in sign of its coefficients,*

$$r \;\leq\; \#\{i \mid 1 \leq i \leq m \text{ and } c_{i-1}c_i < 0\}\,,$$

and the difference between the variation and r is even.

This has an immediate corollary, giving a bound on the numbers of real roots of a univariate polynomial.

Corollary 2.2 (Descartes' bound). *A univariate polynomial* (2.1) *with $m{+}1$ terms has at most m positive roots, $2m$ nonzero roots, and $2m{+}1$ real roots.*

These bounds are sharp, and realized by the following polynomial,

$$x(x^2-1)(x^2-2)\cdots(x^2-m)\,. \tag{2.2}$$

We will prove a generalization of Descartes' rule of signs, the Budan-Fourier Theorem, which provides a similar estimate for any interval in $\mathbb{R}$. We first formalize this notion of variation in sign that appears in Descartes's rule.

The *variation* $\mathrm{var}(c)$ in a finite sequence c of real numbers is the number of times that consecutive elements of the sequence have opposite signs, after we remove any 0s in the sequence. For example, the first sequence below has variation four, while the second has variation three.

$$8, -4, -2, -1, 2, 3, -5, 7, 11, 12 \qquad\qquad -1, 0, 1, 0, 1, -1, 1, 1, 0, 1\,.$$

Given a sequence $F = (f_0, \dots, f_k)$ of polynomials and a real number $a \in \mathbb{R}$, $\mathrm{var}(F, a)$ is the variation in the sequence $f_0(a), f_1(a), \dots, f_k(a)$. This notion also makes sense when $a = \pm\infty$: We set $\mathrm{var}(F, \infty)$ to be the variation in the sequence of leading coefficients of the $f_i(t)$, which are the signs of $f_i(a)$ for $a \gg 0$, and set $\mathrm{var}(F, -\infty)$ to be the variation in the leading coefficients of $f_i(-t)$.

Let δf be the sequence of derivatives of a polynomial $f(t)$ of degree k,

$$\delta f \;:=\; (f(t), f'(t), f''(t), \dots, f^{(k)}(t))\,.$$

For $a, b \in \mathbb{R} \cup \{\pm\infty\}$, let $r(f, a, b)$ be the number of roots of f in the interval $(a, b]$, counted with multiplicity. Budan [**23**] and Fourier [**50**] generalized Descartes's rule.

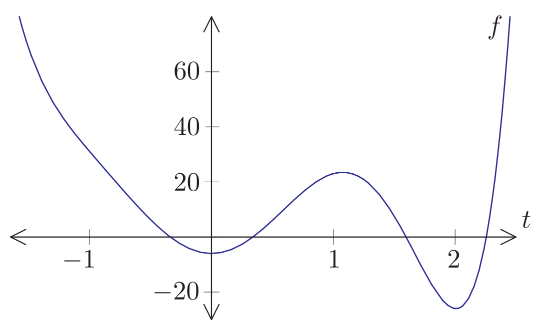
Theorem 2.3 (Budan-Fourier). *Let $f \in \mathbb{R}[t]$ be a univariate polynomial and $a < b$ two numbers in $\mathbb{R} \cup \{\pm\infty\}$. Then*

$$\mathrm{var}(\delta f, a) \;-\; \mathrm{var}(\delta f, b) \;\geq\; r(f, a, b)\,,$$

and the difference is even.

We deduce Descartes's rule of signs from the Budan-Fourier Theorem once we observe that for the polynomial $f(t)$ (2.1), $\mathrm{var}(\delta f, 0) = \mathrm{var}(c_0, c_1, \dots, c_m)$, while $\mathrm{var}(\delta f, \infty) = 0$, as the leading coefficients of δf all have the same sign.

Example 2.4. The the sextic $f = 5t^6 - 4t^5 - 27t^4 + 55t^2 - 6$ whose graph is displayed below



has four real zeroes at approximately -0.3393, 0.3404, 1.598, and 2.256. If we evaluate the derivatives of f at 0 we obtain

$$\delta f(0) \;=\; -6,\ 0,\ 110,\ 0,\ -648,\ -480,\ 3600\,,$$

which has 3 variations in sign. If we evaluate the derivatives of f at 2, we obtain

$$\delta f(2) \;=\; -26,\ -4,\ 574,\ 2544,\ 5592,\ 6720,\ 3600\,,$$

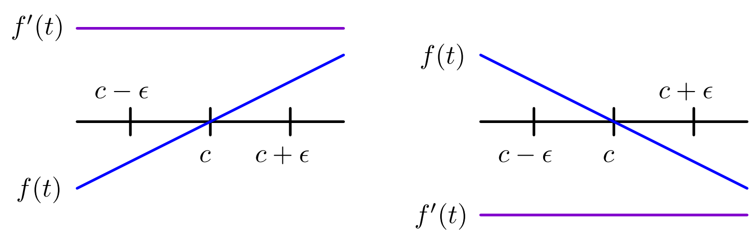
which has one sign variation. Thus, by the Budan-Fourier Theorem, f has either zero or two roots in the interval $(0, 2)$, counted with multiplicity. This agrees with our observation that f has two roots in the interval $[0, 2]$.

Proof of Budan-Fourier Theorem. The variation $\mathrm{var}(\delta f, t)$ is constant except possibly when t passes a root c of some polynomial in the sequence δf of derivatives of f. Suppose that c is a root of some derivative of f and let $\epsilon > 0$ be a positive number such that no derivative $f^{(i)}$ has a root in the interval $[c-\epsilon, c+\epsilon]$, except possibly at c. Let m be the order of vanishing of f at c. We will prove that

$$
\begin{array}{lll}
 & (1) & \mathrm{var}(\delta f, c) = \mathrm{var}(\delta f, c+\epsilon), \ \text{ and} \\
(2.3) & & \\
 & (2) & \mathrm{var}(\delta f, c-\epsilon) \geq \mathrm{var}(\delta f, c) + m, \quad \text{and the difference is even.}
\end{array}
$$

We deduce the Budan-Fourier theorem from these conditions. As t ranges from a to b, $r(f, a, t)$ and $\mathrm{var}(\delta f, t)$ can only change when t passes a root c of f or one of its derivatives. At such a point, $r(f, a, t)$ jumps by the multiplicity m of the point c as a root of f, while $\mathrm{var}(\delta f, t)$ drops by m, plus a nonnegative even integer. Thus the sum $r(f, a, t) + \mathrm{var}(\delta f, t)$ can only change at roots c of f or of its derivatives, where it drops by a nonnegative even integer. The Budan-Fourier Theorem follows, as this sum equals $\mathrm{var}(\delta f, a)$ when $t = a$.

We prove our claim about the behavior of $\mathrm{var}(\delta f, t)$ in a neighborhood of a root c of some derivative $f^{(i)}$ by induction on the degree of f. When f has degree 1, then we are in one of the following two cases, depending upon the sign of f'



In both cases, $\mathrm{var}(\delta f, c-\epsilon) = 1$, but $\mathrm{var}(\delta f, c) = \mathrm{var}(\delta f, c+\epsilon) = 0$, which proves the claim when f is linear.

Now suppose that the degree of f is greater than 1 and let m be the order of vanishing of f at c. We first treat the case when $f(c) = 0$, and hence $m > 0$ so that f' vanishes at c to order $m-1$. By our induction hypothesis for f',

$$
\begin{array}{rcl}
\mathrm{var}(\delta f', c) & = & \mathrm{var}(\delta f', c+\epsilon)\,, \qquad \text{and} \\
\mathrm{var}(\delta f', c-\epsilon) & \geq & \mathrm{var}(\delta f', c) \ + \ (m-1)\,,
\end{array}
$$

and the difference is even. By Lagrange's Mean Value Theorem applied to the intervals $[c-\epsilon, c]$ and $[c, c+\epsilon]$, f and f' must have opposite signs at $c-\epsilon$, but the same signs at $c+\epsilon$, and so

$$
\begin{array}{rcl}
\mathrm{var}(\delta f, c) & = & \mathrm{var}(\delta f', c) \ = \ \mathrm{var}(\delta f', c+\epsilon) \ = \ \mathrm{var}(\delta f, c+\epsilon)\,, \\
\mathrm{var}(\delta f, c-\epsilon) & = & \mathrm{var}(\delta f', c-\epsilon) + 1 \\
 & \geq & \mathrm{var}(\delta f', c) + (m-1) + 1 \ = \ \mathrm{var}(\delta f, c) + m\,,
\end{array}
$$

and the difference is even. This proves the claim when $f(c) = 0$.

Now suppose that $f(c) \neq 0$ so that $m = 0$. Let n be the order of vanishing of f' at c. We apply our induction hypothesis to f' to obtain that

$$\operatorname{var}(\delta f', c) \;=\; \operatorname{var}(\delta f', c+\epsilon), \qquad \text{and} \qquad \operatorname{var}(\delta f', c-\epsilon) \;\geq\; \operatorname{var}(\delta f', c) \;+\; n\,,$$

and the difference is even. We have $f(c) \neq 0$, but $f'(c) = \cdots = f^{(n)}(c) = 0$, and $f^{(n+1)}(c) \neq 0$. Multiplying f by -1 if necessary, we may assume that $f^{(n+1)}(c) > 0$. There are four cases: n even or odd, and $f(c)$ positive or negative.

Suppose that n is even. Then both $f'(c-\epsilon)$ and $f'(c+\epsilon)$ are positive and so for each $t \in \{c-\epsilon, c, c+\epsilon\}$ the first nonzero term in the sequence

$$f'(t),\; f''(t),\; \ldots,\; f^{(k)}(t) \tag{2.4}$$

is positive. When $f(c)$ is positive, this implies that $\operatorname{var}(\delta f, t) = \operatorname{var}(\delta f', t)$ and when $f(c)$ is negative, that $\operatorname{var}(\delta f, t) = \operatorname{var}(\delta f', t) + 1$. This proves the claim as it implies that $\operatorname{var}(\delta f, c) = \operatorname{var}(\delta f, c+\epsilon)$ and also that

$$\operatorname{var}(\delta f, c-\epsilon) \;-\; \operatorname{var}(\delta f, c) \;=\; \operatorname{var}(\delta f', c-\epsilon) \;-\; \operatorname{var}(\delta f', c)\,,$$

but this last difference exceeds n by an even number, and so is even as n is even.

Now suppose that n is odd. Then $f'(c-\epsilon) < 0 < f'(c+\epsilon)$ and so the first nonzero term in the sequence (2.4) has sign $-, +, +$ at $t = c-\epsilon, c, c+\epsilon$, respectively. If $f(c)$ is positive, then $\operatorname{var}(\delta f, c-\epsilon) = \operatorname{var}(\delta f', c-\epsilon)+1$ and the other two variations are unchanged, but if $f(c)$ is negative, then the variation at $t = c-\epsilon$ is unchanged, but it increases by 1 at $t = c$ and $t = c+\epsilon$. This again implies the claim, as $\operatorname{var}(\delta f, c) = \operatorname{var}(\delta f, c+\epsilon)$, but

$$\operatorname{var}(\delta f, c-\epsilon) - \operatorname{var}(\delta f, c) \;=\; \operatorname{var}(\delta f', c-\epsilon) - \operatorname{var}(\delta f', c) \;\pm\; 1\,.$$

The difference $\operatorname{var}(\delta f', c-\epsilon) - \operatorname{var}(\delta f', c)$ equals the order n of the vanishing of f' at c plus a nonnegative even number. Adding or subtracting 1 gives a nonnegative even number. This completes the proof of the Budan-Fourier Theorem. ∎

2.2. Sturm's Theorem

The *Sylvester sequence* of univariate polynomials f, g is

$$f_0 \;:=\; f,\quad f_1 \;:=\; g,\quad f_2,\; \ldots,\; f_k\,,$$

where f_k is a greatest common divisor of f and g, and

$$-f_{i+1} \;:=\; \operatorname{remainder}(f_{i-1}, f_i)\,,$$

the usual remainder from the Euclidean algorithm. Note the sign. We remark that we have polynomials $q_1, q_2, \ldots, q_{k-1}$ such that

$$f_{i-1}(t) \;=\; q_i(t) f_i(t) \;-\; f_{i+1}(t)\,, \tag{2.5}$$

and the degree of f_{i+1} is less than the degree of f_i. The *Sturm sequence* of a univariate polynomial f is the Sylvester sequence of the polynomials f, f'.

Theorem 2.5 (Sturm's Theorem). *Let f be a univariate polynomial and $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$ and $f(a), f(b) \neq 0$. Then the number of zeroes of f in the interval (a, b) is the difference*

$$\operatorname{var}(F, a) \;-\; \operatorname{var}(F, b)\,,$$

where F is the Sturm sequence of f.

EXAMPLE 2.6. The sextic f of Example 2.4 has Sturm sequence

$$\begin{aligned} f &= 5t^6 - 4t^5 - 27t^4 + 55t^2 - 6 \\ f_1 := f'(t) &= 30t^5 - 20t^4 - 108t^3 + 110t \\ f_2 &= \tfrac{84}{9}t^4 + \tfrac{12}{5}t^3 - \tfrac{110}{3}t^2 - \tfrac{22}{9}t + 6 \\ f_3 &= \tfrac{559584}{36125}t^3 + \tfrac{143748}{1445}t^2 - \tfrac{605394}{7225}t - \tfrac{126792}{7225} \\ f_4 &= \tfrac{229905821875}{724847808}t^2 + \tfrac{1540527685625}{4349086848}t + \tfrac{7904908625}{120807968} \\ f_5 &= -\tfrac{280364022223059296}{58526435357253125}t + \tfrac{174201756039315072}{292632176786265625} \\ f_6 &= -\tfrac{17007035533771824564661037625}{162663080627869030112013128}\,. \end{aligned}$$

The Sturm sequence at $t = 0$,

$$-6,\ 0,\ 6,\ -\tfrac{126792}{7225},\ \tfrac{174201756039315072}{292632176786265625},\ -\tfrac{17007035533771824564661037625}{162663080627869030112013128}\,,$$

has four variations in sign, while the Sturm sequence at $t = 2$,

$$-26\,,\ -4\,,\ \tfrac{1114}{45}\,,\ \tfrac{3210228}{36125}\,,\ -\tfrac{1076053821625}{2174543424}\,,\quad \text{and}$$
$$-\tfrac{2629438466191277888}{292632176786265625}\,,\ -\tfrac{17007035533771824564661037625}{162663080627869030112013128}\,,$$

has two variations in sign. Thus by Sturm's Theorem, we see that f has two roots in the interval $[0, 2]$, which we have already seen by other methods.

Sturm's Theorem may be used to isolate real solutions to a univariate polynomial f by finding intervals that contain a unique root of f. When $(a, b) = (-\infty, \infty)$, Sturm's Theorem gives the total number of real roots of a f. In this way, it leads to an algorithm to investigate the number of real roots of systems of polynomials. This algorithm has been used in an essential way to get information on real solutions which helped to formulate many results discussed in later chapters, and to count the numbers of real solutions reported in the tables of Chapters 13 and 14.

Suppose that we have a system of real multivariate polynomials

$$f_1(x_1, \ldots, x_n) = f_2(x_1, \ldots, x_n) = \cdots = f_N(x_1, \ldots, x_n) = 0\,, \tag{2.6}$$

whose number of real roots we wish to determine. Let $I \subset \mathbb{R}[x_1, \ldots, x_n]$ be the ideal generated by the polynomials $f_1, f_2, \ldots, f_N$. If (2.6) has finitely many complex zeroes, then the dimension of the quotient ring $\mathbb{R}[x_1, \ldots, x_n]/I$ (the *degree* of I) is finite. Thus, for each variable x_i, there is a univariate polynomial $g(x_i) \in I$ of minimal degree, called an *eliminant* for I. The significance of eliminants comes from the following observation.

LEMMA 2.7. *The roots of an eliminant $g(x_i) \in I$ form the set of ith coordinates of solutions to* (2.6).

The algorithm for counting the number of real solutions to (2.6) is a consequence of Sturm sequences and the Shape Lemma [**8**].

THEOREM 2.8 (Shape Lemma). *Suppose that I has an eliminant $g(x_i)$ whose degree is equal to the degree of I and is square-free. Then the number of real solutions to* (2.6) *is equal to the number of real roots of g.*

Suppose that the coefficients of the polynomials f_i in the system (2.6) lie in a computable subfield of $\mathbb{R}$, for example, $\mathbb{Q}$ (e.g. if the coefficients are integers). Then we may compute a Gröbner basis for I. From this, we may compute the degree of I, and we may also use the Gröbner basis to compute an eliminant $g(x_i)$. Since

Buchberger's algorithm does not enlarge the field of the coefficients, $g(x_i) \in \mathbb{Q}[x_i]$ has rational coefficients, and so we may use Sturm sequences to compute the number of its real roots. We state this more precisely.

ALGORITHM 2.9. GIVEN: $I = \langle f_1, \ldots, f_N \rangle \subset \mathbb{Q}[x_1, \ldots, x_n]$

(1) Compute a Gröbner basis G of I.
(2) Use G to compute the degree d of I.
(3) Use G to compute an eliminant $g(x_i) \in I \cap \mathbb{Q}[x_i]$ for I.
(4) If $\deg(g) = d$ and g is square-free, then use Sturm sequences to compute the number r of real roots of $g(x_i)$, and output "The ideal I has r real solutions."
(5) Otherwise output "The ideal I does not satisfy the hypotheses of the Shape Lemma for the variable x_i."

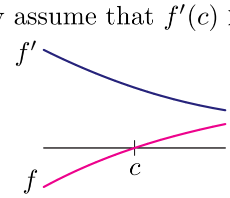
If this algorithm halts with a failure (step 5), it may be called again to compute an eliminant for a different variable. Another strategy is to apply a random linear transformation before eliminating. An even more sophisticated form of elimination is Roullier's rational univariate representation [**122**].

PROOF OF STURM'S THEOREM. Let $f(t)$ be a real univariate polynomial with Sturm sequence F. We prove Sturm's Theorem by looking at the variation $\mathrm{var}(F, t)$ as t increases from a to b. This variation can only change when t passes a number c where some member f_i of the Sturm sequence has a root, for then the sign of f_i could change. We will show that if $i > 0$, then this has no effect on the variation of the sequence, but when c is a root of $f = f_0$, then the variation decreases by exactly 1 as t passes c. Since multiplying a sequence by a nonzero number does not change its variation, we will at times make an assumption on the sign of some value $f_j(c)$ to reduce the number of cases to examine.

Observe first that by (2.5), if $f_i(c) = f_{i+1}(c) = 0$, then f_{i-1} also vanishes at c, as do all the other polynomials f_j. In particular $f(c) = f'(c) = 0$, so f has a multiple root at c. Suppose first that this does not happen, either that $f(c) \neq 0$ or that c is a simple root of f.

Suppose that $f_i(c) = 0$ for some $i > 0$. Together with (2.5), this implies that $f_{i-1}(c)$ and $f_{i+1}(c)$ have opposite signs. Then, whatever the sign of $f_i(t)$ for t near c, there is exactly one variation in sign coming from the subsequence $f_{i-1}(t), f_i(t), f_{i+1}(t)$, and so the vanishing of f_i at c has no effect on the variation as t passes c. This argument works equally well for any Sylvester sequence.

Now we consider the effect on the variation when c is a simple root of f. In this case $f'(c) \neq 0$, so we may assume that $f'(c) > 0$.



But then $f(t)$ is negative for t to the left of c and positive for t to the right of c. In particular, the variation $\mathrm{var}(F, t)$ decreases by exactly 1 when t passes a simple root of f and does not change when f does not vanish.

We are left with the case when c is a multiple root of f. Suppose that its multiplicity is $m+1$. Then $(t-c)^m$ divides every polynomial in the Sturm sequence of f. Consider the sequence of quotients,

$$G \;=\; (g_0,\ldots,g_k) \;:=\; \bigl(f/(t-c)^m,\; f'/(t-c)^m,\; f_2/(t-c)^m,\; \cdots,\; f_k/(t-c)^m\bigr)\,.$$

Note that $\operatorname{var}(G,t)=\operatorname{var}(F,t)$ when $t\neq c$, as multiplying a sequence by a nonzero number does not change its variation. Observe also that G is a Sylvester sequence. Since $g_1(c)\neq 0$, not all polynomials g_i vanish at c. But we showed in this case that there is no contribution to a change in the variation by any polynomial g_i with $i>0$.

It remains to examine the contribution of g_0 to the variation as t passes c. If we write $f(t)=(t-c)^{m+1}h(t)$ with $h(c)\neq 0$, then

$$f'(t) \;=\; (m+1)(t-c)^m h(t) \;+\; (t-c)^{m+1}h'(t)\,.$$

In particular,

$$g_0(t) \;=\; (t-c)h(t) \qquad\text{and}\qquad g_1(t) \;=\; (m+1)h(t) \;+\; (t-c)h'(t)\,.$$

If we assume that $h(c)>0$, then $g_1(c)>0$ and $g_0(t)$ changes from negative to positive as t passes c. Once again we see that the variation $\operatorname{var}(F,t)$ decreases by 1 when t passes a root of f. This completes the proof of Sturm's Theorem.

2.3. A topological proof of Sturm's Theorem

We present a second, very elementary, proof of Sturm's Theorem due to Burda and Khovanskii [**24**] whose virtue is in its tight connection to topology. We first recall the definition of the degree of a continuous function $\rho\colon \mathbb{RP}^1\to\mathbb{RP}^1$ from topology. Since $\mathbb{RP}^1$ is isomorphic to the quotient $\mathbb{R}/\mathbb{Z}$, we may pull ρ back to the interval $[0,1]$ to obtain a map $[0,1]\to\mathbb{RP}^1$. This map lifts to the universal cover of $\mathbb{RP}^1$ to obtain a map $\widetilde{\rho}\colon[0,1]\to\mathbb{R}$. Then the *mapping degree*, $\operatorname{mdeg}(\rho)$, of ρ is simply $\widetilde{\rho}(1)-\widetilde{\rho}(0)$, which is an integer. We call this the mapping degree to distinguish it from the usual algebraic degree of a polynomial or rational function.

The key ingredient in this proof is a formula to compute the mapping degree of a rational function $\rho\colon\mathbb{RP}^1\to\mathbb{RP}^1$. Any rational function $\rho=f/g$ where $f,g\in\mathbb{R}[t]$ are polynomials has a continued fraction expansion of the form

$$\rho \;=\; q_0 \;+\; \cfrac{1}{q_1+\cfrac{1}{q_2+\cfrac{1}{\ddots+\cfrac{1}{q_k}}}} \tag{2.7}$$

where $q_0,\ldots,q_k$ are polynomials. Indeed, this continued fraction is constructed recursively. If we divide f by g with remainder h, so that $f=q_0g+h$ with the degree of h less than the degree of g, then

$$\rho \;=\; q_0+\frac{h}{g} \;=\; q_0+\cfrac{1}{\cfrac{g}{h}}\,.$$

We may now divide g by h with remainder, $g = q_1 h + k$ and obtain

$$\rho \;=\; q_0 + \cfrac{1}{q_1 + \cfrac{1}{\cfrac{h}{k}}}\,.$$

As the degrees of the numerator and denominator drop with each step, this process terminates with an expansion (2.7) of ρ.

For example, if $f = 4t^4 - 18t^2 - 6t$ and $g = 4t^3 + 8t^2 - 1$, then

$$\frac{f}{g} \;=\; t-2 \;+\; \cfrac{1}{-2t+1 \;+\; \cfrac{1}{-2t-3 \;+\; \cfrac{1}{t+1}}}$$

This continued fraction expansion is just the Euclidean algorithm in disguise.

Suppose that $q = c_0 + c_1 t + \cdots + c_d t^d$ is a real polynomial of degree d. Define

$$[q(t)] \;:=\; \operatorname{sign}(c_d)\cdot(d \mod 2) \;\in\; \{\pm 1, 0\}\,.$$

THEOREM 2.10. *Suppose that ρ is a rational function with continued fraction expansion* (2.7)*. Then the mapping degree of ρ is*

$$[q_1(t)] - [q_2(t)] + \cdots + (-1)^{k-1}[q_k(t)]\,.$$

We may use this to count the roots of a real polynomial f by the following lemma.

LEMMA 2.11. *The number of roots of a polynomial f, counted without multiplicity, is the mapping degree of the rational function f/f'.*

PROOF OF STURM'S THEOREM FROM LEMMA 2.11. Let $f_0, f_1, \ldots, f_k$ be the Sturm sequence for f. Then $f_0 = f$, $f_1 = f'$, and for $i > 1$, we have $-f_{i+1} :=$ remainder(f_{i-1}, f_i). That is, $\deg(f_i) < \deg(f_{i-1})$ and there are univariate polynomials $g_1, g_2, \ldots, g_k$ with

$$f_{i-1} \;=\; g_i f_i - f_{i+1} \qquad \text{for} \quad i = 1, 2, \ldots, k{-}1\,.$$

We relate these polynomials to those obtained from the Euclidean algorithm applied to f, f' and thus to the continued fraction expansion of f/f'. It is clear that the f_i differ only by a sign from the remainders in the Euclidean algorithm. Set $r_0 := f$ and $r_1 = f'$, and for $i > 1$, $r_i :=$ remainder(r_{i-2}, r_{i-1}). Then $\deg(r_i) < \deg(r_{i-1})$, and there are univariate polynomials $q_1, q_2, \ldots, q_k$ with

$$r_{i-i} \;=\; q_i r_i + r_{i+1} \qquad \text{for} \quad i = 1, \ldots, k{-}1\,.$$

We leave the proof of the following lemma as an exercise for the reader.

LEMMA 2.12. *We have $g_i = (-1)^{i-1} q_i$ and $f_i = (-1)^{\lfloor \frac{i}{2} \rfloor} r_i$, for $i = 1, 2, \ldots, k$.*

Write F for the Sturm sequence $(f_0, f_1, f_2, \ldots, f_k)$ of f and f^{top} for the leading coefficient of f_i. Then var(F, ∞) is the variation in the leading coefficients $(f_0^{\text{top}}, f_1^{\text{top}}, \ldots, f_k^{\text{top}})$ of the polynomials in F. Similarly, var$(F, -\infty)$ is the variation in the sequence

$$((-1)^{\deg(f_0)} f_0^{\text{top}},\; (-1)^{\deg(f_1)} f_1^{\text{top}},\; \ldots,\; (-1)^{\deg(f_k)} f_k^{\text{top}})\,.$$

Note that the variation in a sequence $(c_0, c_1, \ldots, c_k)$ is just the sum of the variations in each length two subsequence (c_{i-1}, c_i) for $i = 1, \ldots, k$. Thus

$$\begin{aligned}(2.8)\quad &\mathrm{var}(F, -\infty) - \mathrm{var}(F, \infty)\\ &= \sum_{i=1}^{k} \Big(\mathrm{var}\big((-1)^{\deg(f_{i-1})} f_{i-1}^{\mathrm{top}},\, (-1)^{\deg(f_i)} f_i^{\mathrm{top}}\big) \;-\; \mathrm{var}(f_{i-1}^{\mathrm{top}},\, f_i^{\mathrm{top}}) \Big)\,.\end{aligned}$$

Since $f_{i-1} = g_i f_i - f_{i+1}$ and $\deg(f_{i+1}) < \deg(f_i) < \deg(f_{i-1})$, we have

$$f_{i-1}^{\mathrm{top}} \;=\; g_i^{\mathrm{top}} f_i^{\mathrm{top}} \qquad \text{and} \qquad \deg(f_{i-1}) \;=\; \deg(g_i) + \deg(f_i)\,.$$

Thus we have

$$\begin{aligned} \mathrm{var}(f_{i-1}^{\mathrm{top}},\, f_i^{\mathrm{top}}) \;&=\; \mathrm{var}(g_i^{\mathrm{top}}, 1)\,, \qquad \text{and}\\ \mathrm{var}\big((-1)^{\deg(f_{i-1})} f_{i-1}^{\mathrm{top}},\, (-1)^{\deg(f_i)} f_i^{\mathrm{top}}\big) \;&=\; \mathrm{var}((-1)^{\deg(g_i)} g_i^{\mathrm{top}}, 1)\,.\end{aligned}$$

Thus the summands in (2.8) are

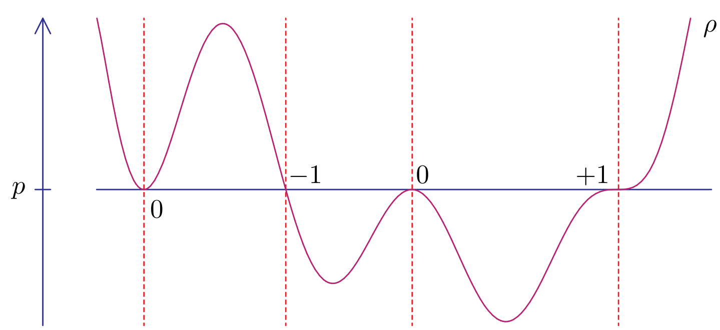
$$\begin{aligned} \mathrm{var}((-1)^{\deg(g_i)} g_i^{\mathrm{top}}, 1) \;-\; \mathrm{var}(g_i^{\mathrm{top}}, 1) \;&=\; \mathrm{sign}(g_i^{\mathrm{top}})(\deg(g_i) \mod 2)\\ &=\; [g_i(t)] \;=\; (-1)^{i-1}[q_i(t)]\,,\end{aligned}$$

This proves that

$$\begin{aligned} \mathrm{var}(F, -\infty) - \mathrm{var}(F, \infty) \;&=\; [g_1(t)] + [g_2(t)] + \cdots + [g_k(t)]\\ &=\; [q_1(t)] - [q_2(t)] + \cdots + (-1)^{k-1}[q_k(t)]\,.\end{aligned}$$

But this proves Sturm's Theorem, as this is the number of roots of f, by Theorem 2.10 and Lemma 2.11.

The key to the proof of Lemma 2.11 is an alternative formula for the mapping degree of a continuous function $\rho\colon \mathbb{RP}^1 \to \mathbb{RP}^1$. Suppose that $p \in \mathbb{RP}^1$ is a point with finitely many inverse images $\rho^{-1}(p)$. To each inverse image q of p we associate an index that records the behavior of $\rho(t)$ as t increases past q. The index is $+1$ if $\rho(t)$ increases when t passes q, it is -1 if $\rho(t)$ decreases when t passes q, and it is 0 if $\rho(t)$ stays on the same side of p as t passes q. Here, increase/decrease are taken with respect to the orientation of $\mathbb{RP}^1$.) For example, here is a graph of a function ρ in relation to the value p with the indices of inverse images indicated.



With this definition, the mapping degree of ρ is the sum of the indices of the points in a fiber $\rho^{-1}(p)$, whenever the fiber is finite. That is,

$$\mathrm{mdeg}(\rho) \;=\; \sum_{a \in \rho^{-1}(p)} \text{index of } a\,.$$

Proof of Lemma 2.11. The zeroes of the rational function $\rho := f/f'$ coincide with the zeroes of f. Suppose $f(a) = 0$ so that a lies in $\rho^{-1}(0)$. The lemma will follow once we show that a has index $+1$.

Since $f(a) = 0$, we may write $f(t) = (t-a)^d h(t)$, where h is a polynomial with $h(a) \neq 0$. We see that $f'(t) = d(t-a)^{d-1}h(t) + (t-a)^d h'(t)$, and so

$$\rho(t) \;=\; \frac{f(t)}{f'(t)} \;=\; \frac{(t-a)h(t)}{dh(t) + (t-a)h'(t)} \;\approx\; \frac{t-a}{d}\,,$$

the last approximation being valid for t near a as $h(t) \neq 0$. Since d is positive, we see that the index of the point a in the fiber $\rho^{-1}(0)$ is $+1$. ∎

Proof of Theorem 2.10. Suppose first that ρ and ρ' are rational functions with no common poles. Then

$$\mathrm{mdeg}(\rho + \rho') \;=\; \mathrm{mdeg}(\rho) + \mathrm{mdeg}(\rho')\,.$$

To see this, note that $(\rho+\rho')^{-1}(\infty)$ is just the union of the sets $\rho^{-1}(\infty)$ and $\rho'^{-1}(\infty)$, and the index of a pole of ρ equals the index of the same pole of $(\rho+\rho')$.

Next, observe that $\mathrm{mdeg}(\rho) = -\,\mathrm{mdeg}(1/\rho)$. For this, consider the behavior of ρ and $1/\rho$ near the level set 1. If $\rho > 1$ than $1/\rho < 1$ and vice-versa. The two functions have index 0 at the same points, and opposite index at the remaining points in the fiber $\rho^{-1}(1) = (1/\rho)^{-1}(1)$.

Now consider the mapping degree of $\rho = f/g$ as we construct its continued fraction expansion. At the first step $f = f_0 g + h$, so that $\rho = f_0 + h/g$. Since f_0 is a polynomial, its only pole is at ∞, but as the degree of h is less than the degree of g, h/g does not have a pole at ∞. Thus the mapping degree of ρ is

$$\mathrm{mdeg}(f_0 + h/g) \;=\; \mathrm{mdeg}(f_0) + \mathrm{mdeg}(h/g) \;=\; \mathrm{mdeg}(f_0) - \mathrm{mdeg}(g/h)\,.$$

The theorem follows by induction, as $\mathrm{mdeg}(f_0) = [f_0(t)]$. ∎

We close this chapter with an application of this method. Suppose that we are given two polynomials f and g, and we wish to count the zeroes a of f where $g(a) > 0$. If $g = (x-b)(x-c)$ with $b < c$, then this will count the zeroes of f in the interval $[b, c]$, which we may do with either of the main results of this chapter. If g has more roots, it is not *a priori* clear how to use the methods in the first two sections of this chapter to solve this problem.

A first step toward solving this problem is to compute the mapping degree of the rational function

$$\rho \;:=\; \frac{f}{gf'}\,.$$

We consider the indices of its zeroes. First, the zeroes of ρ are those zeroes of f that are not zeroes of g, together with a zero at infinity if $\deg(g) > 1$. If $f(a) = 0$ but $g(a) \neq 0$, then $f = (t-a)^d h(t)$ with $h(a) \neq 0$. For t near a,

$$\rho(t) \;\approx\; \frac{t-a}{d \cdot g(a)}\,,$$

and so the preimage $a \in \rho^{-1}(0)$ has index $\mathrm{sign}(g(a))$. If $\deg(g) = e > 1$ and $\deg(f) = d$ then the asymptotic expansion of ρ for t near infinity is

$$\rho(t) \;\approx\; \frac{1}{dg_e t^{e-1}}\,,$$

where g_e is the leading coefficient of g. Thus the index of $\infty \in \rho^{-1}(0)$ is $\operatorname{sign}(g_e)(e-1 \mod 2) \;=\; [g'(t)]$. We summarize this discussion.

LEMMA 2.13. *If* $\deg(g) > 1$*, then*

$$\sum_{\{a|f(a)=0\}} \operatorname{sign}(g(a)) \;=\; \operatorname{mdeg}(\rho) \;-\; [g'(t)]\,,$$

and if $\deg(g) = 1$*, the correction term* $-[g'(t)]$ *is omitted.*

Since $\operatorname{mdeg}(\rho) = -\operatorname{mdeg}(1/\rho)$, we have the alternative expression for this sum.

LEMMA 2.14. *Let* $q_1, q_2, \ldots, q_k$ *be the successive quotients in the Euclidean algorithm applied to the division of* $f'g$ *by* f*. Then*

$$\sum_{\{a|f(a)=0\}} \operatorname{sign}(g(a)) \;=\; [q_2(t)] - [q_3(t)] + \cdots + (-1)^k[q_k(t)]\,.$$

PROOF. We have

$$\operatorname{mdeg}\frac{f}{f'g} \;=\; -\operatorname{mdeg}\frac{f'g}{f} \;=\; -[q_1(t)] + [q_2(t)] - \cdots + (-1)^k[q_k(t)]\,,$$

by Theorem 2.10. Note that we have $f'g = q_1 f + r_1$. If we suppose that $\deg(f) = d$ and $\deg(g) = e$, then $\deg(q_1) = e-1$. Also, the leading term of q is dg_e, where g_e is the leading term of g, which shows that $[q_1(t)] = [g'(t)]$. Thus the lemma follows from Lemma 2.13, when $\deg(g) \geq 2$.

But it also follows when $\deg(g) < 2$ as $[q_1(t)] = 0$ in that case. ∎

Now we may solve our problem. For simplicity, suppose that $\deg g > 1$. Note that

$$\tfrac{1}{2}\big(\operatorname{sign}(g^2(a)) + \operatorname{sign}(g(a))\big) \;=\; \begin{cases} 1 & \text{if } g(a) > 0 \\ 0 & \text{otherwise} \end{cases}\,.$$

And thus

$$\#\{a \mid f(a) = 0,\; g(a) > 0\} \;=\; \frac{1}{2}\operatorname{mdeg}\left(\frac{f}{g^2 f'}\right) \;+\; \frac{1}{2}\operatorname{mdeg}\left(\frac{f}{gf'}\right)\,,$$

which solves the problem.

CHAPTER 3

Sparse Polynomial Systems

Consider a system of n polynomials in n variables

$$f_1(x_1,\ldots,x_n) \;=\; f_2(x_1,\ldots,x_n) \;=\; \cdots \;=\; f_n(x_1,\ldots,x_n) \;=\; 0\,, \tag{3.1}$$

where the polynomial f_i has total degree d_i. By Bézout's Theorem [**14**], this system has at most $d_1d_2\cdots d_n$ isolated complex solutions, and exactly that number if the polynomials are generic among all polynomials with the given degrees.

Polynomials in nature (e.g. from applications) are not necessarily generic—often they have additional structure which we would like our count of solutions to reflect. The goal of this chapter and the next is to explain and present the basic polyhedral bounds of Kushnirenko and Bernstein, which count the number of solutions to a system (3.1) when the extra structure comes from geometric combinatorics—the collection of monomials which appear in the polynomials.

Example 3.1. Consider the system of two polynomials in the variables (x,y),

$$f \;:=\; x^2y+2xy^2+xy-1 \;=\; 0 \quad\text{and}\quad g \;:=\; x^2y-xy^2-xy+2 \;=\; 0\,. \tag{3.2}$$

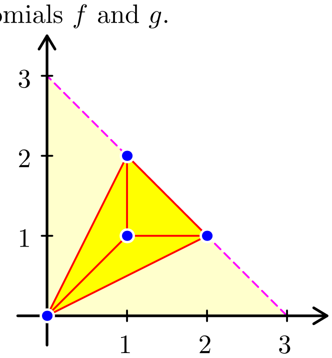
These equations have the algebraic consequences,

$$\begin{aligned} f\cdot(y-x+1)+g\cdot(x+2y+1) \;&=\; 3x+3y+1\,, \qquad \text{and}\\ f\cdot(3y^3-3xy^2+5y^2-2xy+2y-3)&\\ +\,g\cdot(6y^3+3xy^2+7y^2+2xy+2y+3) \;&=\; 9y^3+9y^2+2y+9\,. \end{aligned}$$

The reader may check that f and g are algebraic consequences of the linear polynomial and the cubic polynomial in y, which shows that the original system (3.1) has three solutions—the cubic in y has three solutions, and for each, the linear polynomial gives the corresponding x-coordinate of the solution to the system.

Both polynomials f and g have degree three, but they only have three common solutions, which is fewer than the nine predicted by Bézout's Theorem. The key idea behind this deficit of $6=9-3$ is illustrated by plotting exponent vectors of the monomials which occur in the polynomials f and g.

$$\begin{aligned} x^2y &\;\leftrightarrow\; (2,1)\\ xy^2 &\;\leftrightarrow\; (1,2)\\ xy &\;\leftrightarrow\; (1,1)\\ 1 &\;\leftrightarrow\; (0,0) \end{aligned} \tag{3.3}$$



The *Newton polytope* of f (and of g) is the convex hull of these exponent vectors. This triangle has area $\frac{3}{2}$ and is covered by three lattice triangles. Kushnirenko's Theorem implies that this number of lattice triangles equals the number of solutions to (3.2).

3.1. Polyhedral bounds

The polynomial system in Example 3.1 is a sparse system whose support is the set of integer points in the triangle of (3.3). More generally, let $\mathcal{A} \subset \mathbb{Z}^n$ be a finite set of exponent vectors that affinely spans $\mathbb{R}^n$. That is, the differences of the vectors in $\mathcal{A}$ linearly span $\mathbb{R}^n$. This is necessary for the system (3.1) to have finitely many solutions for general polynomials f_i.

A *sparse polynomial* f with *support* $\mathcal{A}$ is a linear combination

$$f \;=\; \sum_{a\in\mathcal{A}} c_a x^a \qquad c_a \in \mathbb{R} \tag{3.4}$$

of monomials with exponents from $\mathcal{A}$. While sparse polynomials occur naturally—multilinear or multihomogeneous polynomials are an example—they also occur due to human psychology. The difficulty of reasoning with polynomials having thousands of terms, leads us to seek compact problem formulations with fewer terms.

Before we recall Kushnirenko's Theorem from Chapter 1, we make some definitions. A sum of the form $\sum_{a\in\mathcal{A}} \lambda_a \cdot a$ where each $\lambda_a \geq 0$ and $\sum_{a\in\mathcal{A}} \lambda_a = 1$ is a *convex combination* of the points in $\mathcal{A}$. Write $\Delta_{\mathcal{A}} \subset \mathbb{R}^n$ for the convex hull of the vectors in $\mathcal{A}$. That is, is it the set of convex combinations of vectors in $\mathcal{A}$,

$$\Delta_{\mathcal{A}} \;:=\; \Big\{ \sum_{a\in\mathcal{A}} \lambda_a \cdot a \mid \lambda_a \geq 0 \quad \text{and} \quad \sum_{a\in\mathcal{A}} \lambda_a = 1 \Big\}\,.$$

Write $\text{volume}(\Delta)$ for the Euclidean volume of a polytope Δ, and recall that $\mathbb{T} := \mathbb{C}^* := \mathbb{C} \setminus \{0\}$. General principles in algebraic geometry imply that when the system (3.1) of sparse polynomials with support $\mathcal{A}$ are general, then the number $d(\mathcal{A})$ of its solutions in $\mathbb{T}^n$ is constant. There are at least two relevant notions of general. One implies that the number of solutions in $\mathbb{T}^n$ is $d(\mathcal{A})$, counted with multiplicity, and another is that the solutions occur without multiplicity.

THEOREM 3.2 (Kushnirenko's Theorem). *A system* (3.1) *of n polynomials in n variables with support $\mathcal{A}$ has at most $n!\,\text{volume}(\Delta_{\mathcal{A}})$ isolated solutions in $\mathbb{T}^n$, and exactly this number if the polynomials are generic given their support $\mathcal{A}$.*

Thus $d(\mathcal{A}) = n!\,\text{volume}(\Delta_{\mathcal{A}})$. This chapter gives an algebraic-geometric proof of this result due to Khovanskii and the next chapter will present a more algorithmic proof. These proofs introduce useful geometry of sparse systems of polynomials.

We may also consider a system of polynomials (3.1) in which each polynomial f_i may have a different support $\mathcal{A}_i$. We follow Bernstein's exposition [**11**] to motivate his bound for the number of solutions in $\mathbb{T}^n$ to such a *mixed system*.

When the coefficients of the polynomials $f_1, \ldots, f_n$ do not lie on some discriminant hypersurface in the space of all possible coefficients, the number $d(f_1, \ldots, f_n)$ of solutions in $\mathbb{T}^n$ (counted with multiplicity) to a mixed system $f_1 = \cdots = f_n = 0$ is constant and so it depends upon the supports. Write $d(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ for this number. This number is invariant under translating any support $\mathcal{A}_i$ by a vector $a \in \mathbb{Z}^n$, as that corresponds to multiplying f_i by the invertible monomial x^a. This number is also invariant under an invertible change of coordinates on $\mathbb{T}^n$—these are monomial substitutions coming from the automorphisms $GL(n,\mathbb{Z})$ of the lattice $\mathbb{Z}^n$ of characters of $\mathbb{T}^n$. If we replace f_1 in (3.1) by the product $f_1 \cdot f_1'$, then we have

$$d(f_1\cdot f_1', f_2,\dots,f_n) \;=\; d(f_1,f_2,\dots,f_n) \;+\; d(f_1',f_2,\dots,f_n)\,. \tag{3.5}$$

The support of $f_1 \cdot f_1'$ is a subset of the pointwise sum $\mathcal{A}_1 + \mathcal{A}_1'$ of their supports, and its convex hull equals the Minkowski sum of the convex hulls,

$$\Delta_{\mathcal{A}_1+\mathcal{A}_1'} \;=\; \Delta_{\mathcal{A}_1} + \Delta_{\mathcal{A}_1'} \;:=\; \{a+b \mid a\in\Delta_{\mathcal{A}_1}\,,\; b\in\Delta_{\mathcal{A}_1'}\}\,.$$

It is natural to expect that the function $d(\mathcal{A}_1,\dots,\mathcal{A}_n)$ is also multilinear,

$$d(\mathcal{A}_1+\mathcal{A}_1',\dots,\mathcal{A}_n) \;=\; d(\mathcal{A}_1,\dots,\mathcal{A}_n) \;+\; d(\mathcal{A}_1',\dots,\mathcal{A}_n)\,. \tag{3.6}$$

Lastly, if all the supports were equal, $\mathcal{A}_i = \mathcal{A}$, then this number should reduce to the number in Kushnirenko's Theorem, $d(\mathcal{A},\dots,\mathcal{A}) = d(\mathcal{A}) = n!\,\mathrm{volume}(\Delta_{\mathcal{A}})$.

There is a unique function of the convex hulls of the supports $(\mathcal{A}_1,\dots,\mathcal{A}_n)$ satisfying these properties, namely Minkowski's mixed volume. Minkowski (see [**48**]) showed that given convex bodies $K_1,\dots,K_n$ in $\mathbb{R}^n$ and positive numbers $\lambda_1,\dots,\lambda_n$, $\mathrm{volume}(\lambda_1K_1+\cdots+\lambda_nK_n)$ is a homogeneous polynomial of degree n in $\lambda_1\dots,\lambda_n$, so there exist coefficients $V(K_{i_1},\dots,K_{i_n})$ for $i_1,\dots,i_n\in[n]$ such that

$$\mathrm{volume}(\lambda_1K_1+\cdots+\lambda_nK_n) \;=\; \sum_{i_1,\dots,i_n\in[n]} V(K_{i_1},\dots,K_{i_n})\lambda_{i_1}\cdots\lambda_{i_n}\,. \tag{3.7}$$

The *mixed volume*, $\mathrm{MV}(K_1,\dots,K_n)$ of $K_1,\dots,K_n$ is is $n!V(K_1,\dots,K_n)$ (the $n!$ is due to the normalized volume in Kushnirenko's Theorem). Since (3.7) is a symmetric form in $\lambda_1,\dots,\lambda_n$, mixed volume is multilinear, an so the algebraic identity

$$n!a_1\cdots a_n \;=\; \sum_{k=1}^{n}(-1)^{n-k}\sum_{1\le i_1<i_2<\cdots<i_k\le n}(a_{i_1}+\cdots+a_{i_k})^n\,,$$

implies the following formula for mixed volume

$$\sum_{k=1}^{n}(-1)^{n-k}\sum_{1\le i_1<i_2<\cdots<i_k\le n}\mathrm{volume}(K_{i_1}+K_{i_2}+\cdots+K_{i_k})\,.$$

THEOREM 3.3 (Bernstein's Theorem). *A system of n polynomials in n variables where the polynomials have support $\mathcal{A}_1,\dots,\mathcal{A}_n$ has at most $\mathrm{MV}(\Delta_{\mathcal{A}_1},\dots,\Delta_{\mathcal{A}_n})$ isolated solutions in $\mathbb{T}^n$, and exactly this number when the polynomials are generic for their given support.*

The gap in the argument for Theorem 3.3 was that the function $d(\mathcal{A}_1,\dots,\mathcal{A}_n)$ is multilinear. In Section 3.4 we present Bernstein's elegant proof of this multilinearity, which also characterizes when the system has finitely many solutions.

3.2. Geometric interpretation of sparse polynomial systems

Consider the map

$$\varphi_{\mathcal{A}} \;:\; \mathbb{T}^n \ni x \;\longmapsto\; [x^a \mid a\in\mathcal{A}] \;\in\; \mathbb{P}^{\mathcal{A}}\,, \tag{3.8}$$

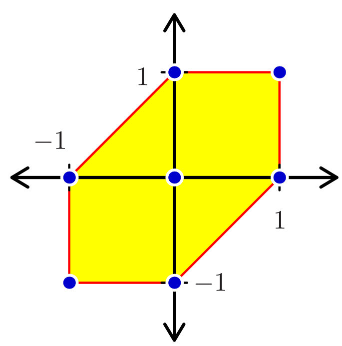
where $\mathbb{P}^{\mathcal{A}}$ is the projective space with *homogeneous coordinates* $[z_a \mid a\in\mathcal{A}]$ indexed by $\mathcal{A}$. That is, $\mathbb{P}^{\mathcal{A}}$ is the quotient $(\mathbb{C}^{\mathcal{A}}\setminus\{0\})/\mathbb{T}$, where $\mathbb{T}$ acts by scalars. Write $[y_a \mid a\in\mathcal{A}]$ for the image of the vector $(y_a \mid a\in\mathcal{A})\in\mathbb{C}^{\mathcal{A}}\setminus\{0\}$ and note that $[y_a \mid a\in\mathcal{A}] = [ty_a \mid a\in\mathcal{A}]$ for any $t\in\mathbb{T}$. We use a different notation for points $y\in\mathbb{P}^{\mathcal{A}}$ than for the coordinate functions z_a as they are elements of dual vector spaces. While this may seem to be a pedantic distinction, it will be important in our study of toric degenerations in Chapter 4, as the points y and coordinates z transform differently under group actions.

This parameterization map (3.8) factors

$$\mathbb{T}^n \longrightarrow \mathbb{T}^{\mathcal{A}} \longrightarrow\!\!\!\!\rightarrow \mathbb{T}^{\mathcal{A}}/\delta(\mathbb{T}) \subset \mathbb{P}^{\mathcal{A}},$$

where $\mathbb{T}^{\mathcal{A}} = (\mathbb{C}^*)^{|\mathcal{A}|}$ is the torus with coordinates indexed by $\mathcal{A}$ and $\delta(\mathbb{T}) \subset \mathbb{T}^{\mathcal{A}}$ is the diagonal torus. The quotient torus $\mathbb{T}^{\mathcal{A}}/\delta(\mathbb{T})$ consists of those points $[z_a \mid a \in \mathcal{A}] \in \mathbb{P}^{\mathcal{A}}$ with no coordinate equal to zero. Notice that $\varphi_{\mathcal{A}}$ is a homomorphism into this dense torus. Its kernel has the following description which we will explain in Section 3.3. As $\mathcal{A}$ affinely spans $\mathbb{R}^n$, its integer affine span $\mathbb{Z}\mathcal{A}$ (the subgroup of $\mathbb{Z}^n$ spanned by differences of vectors in $\mathcal{A}$) is a full rank subgroup of $\mathbb{Z}^n$, and so the quotient group $\mathbb{Z}^n/\mathbb{Z}\mathcal{A}$ is finite. The kernel of the map $\varphi_{\mathcal{A}}$ is identified with the group of homomorphisms $\mathbb{Z}^n/\mathbb{Z}\mathcal{A} \to \mathbb{T}$, whose order is equal to $|\mathbb{Z}^n/\mathbb{Z}\mathcal{A}|$, which is the index of $\mathbb{Z}\mathcal{A}$ in the lattice $\mathbb{Z}^n$, called its *lattice index* and written $[\mathbb{Z}^n : \mathbb{Z}\mathcal{A}]$.

EXAMPLE 3.4. Suppose that $\mathcal{A}$ consists of the seven exponent vectors $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$, $(-1,0)$, $(0,-1)$, and $(-1,-1)$. The convex hull $\Delta_{\mathcal{A}}$ of these points is the hexagon,



and the map $\varphi_{\mathcal{A}}$ is

$$\varphi_{\mathcal{A}} : (x,y) \in \mathbb{T}^2 \longmapsto [1, x, y, xy, x^{-1}, y^{-1}, x^{-1}y^{-1}] \in \mathbb{P}^{\mathcal{A}} \simeq \mathbb{P}^6 .$$

Given a linear form Λ on $\mathbb{P}^{\mathcal{A}}$,

$$\Lambda = \sum_{a \in \mathcal{A}} c_a z_a ,$$

its pullback $\varphi^*_{\mathcal{A}}(\Lambda)$ along $\varphi_{\mathcal{A}}$ is a polynomial with support $\mathcal{A}$,

$$\varphi^*_{\mathcal{A}}(\Lambda) = \sum_{a \in \mathcal{A}} c_a x^a .$$

This simple observation provides a bijective correspondence between linear forms on $\mathbb{P}^{\mathcal{A}}$ and sparse polynomials with support $\mathcal{A}$. The zero set of a sparse polynomial on $\mathbb{T}^n$ is mapped to a hyperplane section $H \cap \varphi_{\mathcal{A}}(\mathbb{T}^n)$ of $\varphi_{\mathcal{A}}(\mathbb{T}^n)$ (the hyperplane $H \subset \mathbb{P}^{\mathcal{A}}$ is where the corresponding linear form vanishes). This leads to the following geometric formulation of systems of polynomials with support $\mathcal{A}$.

LEMMA 3.5. *The map $\varphi_{\mathcal{A}}$ gives a bijective correspondence between zero sets of sparse polynomials with support $\mathcal{A}$ and pullbacks $\varphi^{-1}_{\mathcal{A}}(H \cap \varphi_{\mathcal{A}}(\mathbb{T}^n))$ of hyperplane sections. This extends to systems of polynomials. The solution set of a system of polynomials* (3.1) *with support $\mathcal{A}$ is the pullback $\varphi^{-1}_{\mathcal{A}}(L) = \varphi^{-1}_{\mathcal{A}}(L \cap \varphi_{\mathcal{A}}(\mathbb{T}^n))$ of a* *linear section of* $\varphi_{\mathcal{A}}(\mathbb{T}^n)$*, where* L *has codimension equal to the dimension of the linear span of the polynomials* f_i*. When* $\mathbb{Z}\mathcal{A} = \mathbb{Z}^n$ *this gives a bijection between solutions to the system and points in the linear section.*

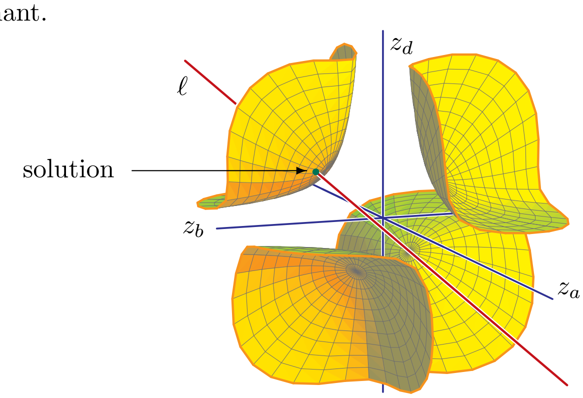
EXAMPLE 3.6. Consider the polynomial system of Example 3.1. Let a, b, c, d be the exponent vectors $(2,1)$, $(1,2)$, $(1,1)$, and $(0,0)$, respectively. Then the map $\varphi_{\mathcal{A}}$ is

$$(x, y) \longmapsto [x^2y\,,\ xy^2\,,\ xy\,,\ 1] \in \mathbb{P}^{\mathcal{A}} \simeq \mathbb{P}^3\,.$$

Its image consists of those points $[z_a, z_b, z_c, z_d]$ with $z_a z_b z_d = z_c^3 \neq 0$, which defines (part of) a cubic surface. The polynomial system (3.2) corresponds to the two linear forms

$$z_a + 2z_b + z_c - z_d \;=\; z_a - z_b - z_c + 2z_d \;=\; 0\,,$$

which defines a line ℓ in $\mathbb{P}^3$. We show ℓ and (part of) the cubic surface. This is in the affine part of $\mathbb{P}^{\mathcal{A}}$ where $z_c \neq 0$ in the box $[-4, 4]^3$. The best view is from the $+-+$-orthant.



From this, we see that there is one real solution to the system (3.2).

This description gives an interpretation for the number $d(\mathcal{A})$ of solutions to a general sparse system with support $\mathcal{A}$. The *degree,* $\deg(X)$, of a subvariety X of $\mathbb{P}^m$ dimension n is the number of points in a linear section $L \cap X$ of X by a general linear subspace L of codimension n. Define the *toric variety* $X_{\mathcal{A}}$ parameterized by the monomials $\mathcal{A}$ to be the closure of the image of $\varphi_{\mathcal{A}}$. Since $\varphi_{\mathcal{A}}$ is a homomorphism, we have the product

$$d(\mathcal{A}) \;=\; |\ker(\varphi_{\mathcal{A}})| \cdot \deg(X_{\mathcal{A}})\,. \tag{3.9}$$

Indeed, if L is a general linear subspace of $\mathbb{P}^{\mathcal{A}}$ of codimension n, then Bertini's theorem implies that $L \cap \varphi_{\mathcal{A}}(\mathbb{T}^n) = L \cap X_{\mathcal{A}}$ and this intersection is transverse. The number of points in such a linear section is the degree $\deg(X_{\mathcal{A}})$ of $X_{\mathcal{A}}$, and each point pulls back under $\varphi_{\mathcal{A}}$ to $|\ker(\varphi_{\mathcal{A}})|$ solutions to the sparse system corresponding to the linear section. When the intersection is zero-dimensional but not transverse, $d(\mathcal{A})$ will be the sum of the solutions counted with algebraic multiplicities.

3.3. Proof of Kushnirenko's Theorem

We prove Kushnirenko's Theorem by showing that

$$n! \cdot \mathrm{volume}(\Delta_{\mathcal{A}}) \;=\; |\ker(\varphi_{\mathcal{A}})| \cdot \deg(X_{\mathcal{A}}) \;=\; d(\mathcal{A})\,.$$

This proof is due to Khovanskii [**85**].

We first determine the kernel of the map $\varphi_{\mathcal{A}}$, which is the composition

$$\begin{array}{ccccccc} \mathbb{T}^n & \longrightarrow & \mathbb{T}^{\mathcal{A}} & \longrightarrow & \mathbb{T}^{\mathcal{A}}/\delta(\mathbb{T}) & \subset & \mathbb{P}^{\mathcal{A}} \\ x & \longmapsto & (x^a \mid a \in \mathcal{A}) & \longmapsto & [x^a \mid a \in \mathcal{A}]\,. & & \end{array}$$

To facilitate this computation, we assume that $0 \in \mathcal{A}$. This is no loss of generality, for if $0 \notin \mathcal{A}$, then we simply translate $\mathcal{A}$ so that one of its exponent vectors is the origin. This has the effect of multiplying each point in $\varphi_{\mathcal{A}}(\mathbb{T}^n)$ by a scalar, and so it does not change the projective variety $X_{\mathcal{A}}$. It also multiplies each polynomial in (3.1) by a common monomial, which affects neither the solutions in $\mathbb{T}^n$ nor their number $d(\mathcal{A})$. By the relation (3.9), this translation does not change the cardinality of the kernel of $\varphi_{\mathcal{A}}$.

Since $0 \in \mathcal{A}$, the z_0-coordinate of $\varphi_{\mathcal{A}}$ is constant ($x^0 = 1$) and so the map which sends $x \in \mathbb{T}^n$ to $(x^a \mid a \in \mathcal{A})$ maps $\mathbb{T}^n$ into $1 \times \mathbb{T}^N \subset \mathbb{T}^{\mathcal{A}}$, where $|\mathcal{A}| = N+1$. The composition of the two maps

$$1 \times \mathbb{T}^N \longrightarrow \mathbb{T}^{\mathcal{A}} \longrightarrow \mathbb{T}^{\mathcal{A}}/\delta(\mathbb{T}) \ \left(\subset \mathbb{P}^{\mathcal{A}}\right)$$

is an isomorphism. Thus it is sufficient to compute the kernel of the map

$$\begin{array}{ccc} \mathbb{T}^n & \longrightarrow & \mathbb{T}^{\mathcal{A}}\,, \\ x & \longmapsto & (x^a \mid a \in \mathcal{A})\,, \end{array}$$

which is $\{x \in \mathbb{T}^n \mid x^a = 1 \text{ for all } a \in \mathcal{A}\}$. Identifying $\mathbb{T}^n$ with $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{T})$, this is exactly those homomorphisms $\mathbb{Z}^n \to \mathbb{T}$ which are trivial on $\mathbb{Z}\mathcal{A}$, or simply the group $\mathrm{Hom}(\mathbb{Z}^n/\mathbb{Z}\mathcal{A}, \mathbb{T})$. Since $\mathbb{Z}\mathcal{A} \subset \mathbb{Z}^n$ has full rank n, the quotient group $\mathbb{Z}^n/\mathbb{Z}\mathcal{A}$ is a finite abelian group, and so its dual group $\mathrm{Hom}(\mathbb{Z}^n/\mathbb{Z}\mathcal{A}, \mathbb{T})$ is also finite with cardinality equal to that of $\mathbb{Z}^n/\mathbb{Z}\mathcal{A}$. Thus

$$|\ker(\varphi_{\mathcal{A}})| \;=\; [\mathbb{Z}^n : \mathbb{Z}\mathcal{A}] \;=\; |\mathbb{Z}^n/\mathbb{Z}\mathcal{A}|\,. \tag{3.10}$$

The *homogeneous coordinate ring* $\mathbb{C}[X]$ of a projective variety $X \subset \mathbb{P}^{\mathcal{A}}$ is the quotient of the homogeneous coordinate ring $\mathbb{C}[z_a \mid a \in \mathcal{A}]$ of $\mathbb{P}^{\mathcal{A}}$ by the homogeneous ideal I_X of polynomials vanishing on X, which is naturally graded by the degree of the polynomials. Write $\mathbb{C}_d[X]$ for the dth graded piece of $\mathbb{C}[X]$.

The *Hilbert polynomial* $h_X(d)$ of a projective variety X is the polynomial which is eventually equal to the dimension of the dth graded piece $\mathbb{C}_d[X]$ of the homogeneous coordinate ring of X,

$$h_X(d) \;=\; \dim(\mathbb{C}_d[X])\,, \text{ for all } d \text{ sufficiently large.}$$

The Hilbert polynomial encodes many numerical invariants of X. For example, the degree of the Hilbert polynomial is the dimension n of X and its leading coefficient is $\deg(X)/n!$. For a discussion of Hilbert polynomials, see Section 9.3 of [**31**].

We determine the Hilbert polynomial of the toric variety $X_{\mathcal{A}}$. For this, it is helpful to consider a homogeneous version of the parameterization map $\varphi_{\mathcal{A}}$. We lift $\mathcal{A} \subset \mathbb{Z}^n$ to a homogenized set of exponent vectors $\mathcal{A}^+ \subset 1 \times \mathbb{Z}^n$ by prepending a component of 1 to each vector in $\mathcal{A}$. That is,

$$\mathcal{A}^+ \;:=\; \{(1, a) \mid a \in \mathcal{A}\}\,.$$

Figure 3.1 shows lifted hexagon, where the first coordinate is vertical.

The map $\varphi_{\mathcal{A}^+}$ on $\mathbb{T}^{1+n}$ has the same image in $\mathbb{P}^{\mathcal{A}}$ as does $\varphi_{\mathcal{A}}$. The advantage of $\varphi_{\mathcal{A}^+}$ is that its image in $\mathbb{C}^{\mathcal{A}}$ is stable under multiplication by scalars—this is due to the new first coordinate of $\mathcal{A}^+$. If t is the first coordinate of $\mathbb{T}^{1+n}$, then

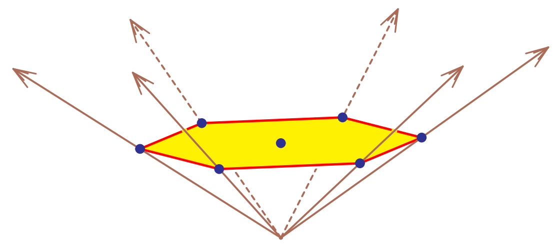


FIGURE 3.1. Lifted hexagon.

the pullback of the coordinate ring of $\mathbb{P}^{\mathcal{A}}$ to the ring of Laurent polynomials (the coordinate ring of $\mathbb{T}^{1+n}$) is

$$S_{\mathcal{A}} \ := \ \mathbb{C}[tx^a \mid a \in \mathcal{A}] \ \simeq \ \mathbb{C}[\mathbb{N}\mathcal{A}^+]\,.$$

This is also the homogeneous coordinate ring of the toric variety $X_{\mathcal{A}} = \overline{\varphi_{\mathcal{A}^+}(\mathbb{T}^{1+n})}$.

The grading on $S_{\mathcal{A}}$ is given by the exponent of the variable t. It follows that the dth graded piece of $S_{\mathcal{A}}$ has a basis of monomials

$$\{t^d x^a \mid (d,a) \in \mathbb{N}\mathcal{A}^+\}\,.$$

This index set is $\mathbb{N}\mathcal{A}^+ \cap d\Delta_{\mathcal{A}^+}$, which is equal to $d\mathcal{A}^+$, the set of d-fold sums of vectors in $\mathcal{A}^+$. If we set $t = 1$, which amounts to projecting the set $d\mathcal{A}^+$ to the last n coordinates, we see that this set is in bijection with the set $d\mathcal{A}$ of d-fold sums of vectors in $\mathcal{A}$. If we let $H_{\mathcal{A}}(d)$ be the dimension of the dth graded piece of the homogeneous coordinate ring of $X_{\mathcal{A}}$ (also called the *Hilbert function* of $X_{\mathcal{A}}$), then these arguments show that

$$H_{\mathcal{A}}(d) \ = \ |d\mathcal{A}|\,.$$

We will estimate this Hilbert function, which will enable us to determine the leading coefficient of the Hilbert polynomial, as the Hilbert function and Hilbert polynomial agree for d sufficiently large.

An upper bound on the values of the Hilbert function is given by the observation that $d\mathcal{A}$ is a subset of the set of integer points in the dth dilation $d\Delta_{\mathcal{A}}$ of the polytope $\Delta_{\mathcal{A}}$, that is, $d\mathcal{A} \subset \mathbb{Z}\mathcal{A} \cap d\Delta_{\mathcal{A}}$. More generally, let $M \simeq \mathbb{Z}^n$ be a lattice in $\mathbb{R}^n$ and Δ be a polytope with vertices in M. Ehrhart [**38**] (see also the book [**7**]) showed that the counting function

$$P_{\Delta} \ : \ \mathbb{N} \ni d \ \longmapsto \ |d\Delta \cap M|$$

for the points of M contained in positive integer multiples of the polytope Δ is a polynomial in d. This polynomial is called the *Ehrhart polynomial* of the polytope Δ, and its degree is the dimension of the affine span of Δ. When Δ has dimension n, its leading coefficient is the volume of Δ, normalized so that a fundamental parallelepiped of the lattice M has volume 1. That is, it is the Euclidean volume divided by the lattice index $[\mathbb{Z}^n : \mathbb{Z}\mathcal{A}]$. When $M = \mathbb{Z}^n$, this is the ordinary Euclidean volume of Δ.

Now suppose that $\Delta = \Delta_{\mathcal{A}}$, the convex hull of $\mathcal{A}$. Since $d\mathcal{A} \subset d\Delta_{\mathcal{A}} \cap \mathbb{Z}\mathcal{A}$, if $M = \mathbb{Z}\mathcal{A}$, we have the upper bound for $H_{\mathcal{A}}(d)$,

$$P_{\Delta_{\mathcal{A}}}(d) \ \geq \ H_{\mathcal{A}}(d)\,. \tag{3.11}$$

We now give a lower bound for $H_{\mathcal{A}}(d)$. Let $\mathcal{B}$ be the set of points b in $\mathbb{Z}\mathcal{A}$ which may be written as

$$b \;=\; \sum_{a\in\mathcal{A}} \beta_a a\,,$$

where β_a is a rational number in $[0,1)$. Fix an expression for each $b\in\mathcal{B}$ as an integer linear combination of elements of $\mathcal{A}$, and let $-\nu$ with $\nu\geq 0$ be an integer lower bound for the coefficients in these expressions for the finitely many elements of $\mathcal{B}$. Since $0\in\mathcal{A}$, we may further assume that there is an integer $\mu\geq 0$ such that for each such expression $b=\sum_{a\in\mathcal{A}} b_a a$, we have $\mu=\sum_{a\in\mathcal{A}} b_a$, and thus

$$(\mu,b) \;=\; \sum_{a\in\mathcal{A}} b_a(1,a)\,, \qquad \text{where} \quad -\nu\leq b_a\in\mathbb{Z} \quad \text{and} \quad \mu,\nu\in\mathbb{N}\,. \tag{3.12}$$

For $d\geq \nu|\mathcal{A}|+\mu$ we claim that translation by the vector $\nu\sum_{a\in\mathcal{A}}(1,a)+(\mu,0)$ defines a map

$$\mathbb{Z}\mathcal{A}^+\cap(d-\nu|\mathcal{A}|-\mu)\Delta_{\mathcal{A}^+} \;\longrightarrow\; \mathbb{N}\mathcal{A}^+\cap d\Delta_{\mathcal{A}^+}\,.$$

Indeed, a point $v\in\mathbb{Z}\mathcal{A}^+\cap(d-\nu|\mathcal{A}|-\mu)\Delta_{\mathcal{A}^+}$ is a nonnegative rational combination of the vectors in $\mathcal{A}^+$,

$$v \;=\; \sum_{a\in\mathcal{A}} \alpha_a(1,a)\,, \quad \alpha_a\in\mathbb{Q}_{\geq} \qquad \text{with} \quad d-\nu|\mathcal{A}|-\mu=\sum_{a\in\mathcal{A}}\alpha_a\,.$$

Writing $\alpha_a=\beta_a+\gamma_a$, where $\beta_a\in[0,1)$ and $\gamma_a\in\mathbb{N}$, we have

$$v \;=\; \sum_{a\in\mathcal{A}}\beta_a(1,a) \;+\; \sum_{a\in\mathcal{A}}\gamma_a(1,a) \;=\; (\beta,b) \;+\; \sum_{a\in\mathcal{A}}\gamma_a(1,a)\,,$$

where $(\beta,b)=\sum_{a\in\mathcal{A}}\beta_a(1,a)$. Then $b\in\mathcal{B}$ and $\beta\in\mathbb{N}$, as $\beta=d-\sum_a\gamma_a$. Using the fixed expression (3.12), we have

$$\begin{aligned} v \;&=\; (\beta,0) \;+\; (0,b) \;+\; \sum_{a\in\mathcal{A}}\gamma_a(1,a)\\ &=\; \beta(1,0) \;+\; -\mu(1,0) \;+\; (\mu,b) \;+\; \sum_{a\in\mathcal{A}}\gamma_a(1,a)\\ &=\; \beta(1,0) \;+\; -\mu(1,0) \;+\; \sum_{a\in\mathcal{A}}b_a(1,a) \;+\; \sum_{a\in\mathcal{A}}\gamma_a(1,a)\,. \end{aligned}$$

Thus

$$v \;+\; \nu\sum_{a\in\mathcal{A}}(1,a)+(\mu,0) \;=\; \beta(1,0) \;+\; \sum_{a\in\mathcal{A}}(b_a+\nu)\cdot(1,a) \;+\; \sum_{a\in\mathcal{A}}\gamma_a(1,a)\,,$$

which is a nonnegative integer linear combination of vectors $(1,a)\in\mathcal{A}^+$ that lies in $\mathbb{N}\mathcal{A}^+\cap d\Delta_{\mathcal{A}^+}$. This proves the claim.

The claim shows that

$$H_{\mathcal{A}}(d) \;\geq\; P_{\Delta_{\mathcal{A}}}(d-\nu|\mathcal{A}|-\mu)\,.$$

If we combine this estimate with (3.11), and use that the Hilbert function equals the Hilbert polynomial for d sufficiently large, then we have shown that the Hilbert polynomial $h_{\mathcal{A}}$ of $X_{\mathcal{A}}$ has the same degree and leading coefficient as the Ehrhart polynomial $P_{\Delta_{\mathcal{A}}}$.

Thus the Hilbert polynomial has degree n and its leading coefficient is the normalized volume of the polytope $\Delta_{\mathcal{A}}$ with respect to the lattice $\mathbb{Z}\mathcal{A}$, which is volume$(\Delta_{\mathcal{A}})/[\mathbb{Z}^n\colon \mathbb{Z}\mathcal{A}]$. Since the degree of $X_{\mathcal{A}}$ is $n!$ times this leading coefficient, we conclude that the degree of $X_{\mathcal{A}}$ is

$$n!\,\frac{\mathrm{volume}(\Delta_{\mathcal{A}})}{[\mathbb{Z}^n\colon \mathbb{Z}\mathcal{A}]}\,.$$

Recall (3.10) that the kernel of $\varphi_{\mathcal{A}}$ has order $[\mathbb{Z}^n\colon \mathbb{Z}\mathcal{A}]$. Then the formula (3.9) for the number $d(\mathcal{A})$ of solutions to a sparse system (3.1) with support $\mathcal{A}$ becomes

$$d(\mathcal{A}) \;=\; |\ker(\varphi_{\mathcal{A}})|\cdot\deg(X_{\mathcal{A}}) \;=\; [\mathbb{Z}^n\colon \mathbb{Z}\mathcal{A}]\cdot n!\,\frac{\mathrm{volume}(\Delta_{\mathcal{A}})}{[\mathbb{Z}^n\colon \mathbb{Z}\mathcal{A}]} \;=\; n!\,\mathrm{volume}(\Delta_{\mathcal{A}})\,,$$

which proves Kushnirenko's Theorem.

3.4. Facial systems and degeneracies

An element $\omega\in\mathbb{Z}^n$ gives a linear function $\langle\omega,\cdot\rangle$ on $\mathbb{Z}^n$,

$$\mathbb{Z}^n \;\ni\; a \;\longmapsto\; \langle\omega,a\rangle \;:=\; \omega^T a \;\in\; \mathbb{Z}\,.$$

Let $m(\omega,\mathcal{A})$ be the minimum value of $\langle\omega,\cdot\rangle$ on a finite set $\mathcal{A}\subset\mathbb{Z}^n$, and set

$$\mathcal{A}_\omega \;:=\; \{a\in\mathcal{A}\mid \langle\omega,a\rangle = m(\omega,\mathcal{A})\}$$

to be the subset of $\mathcal{A}$ where $\langle\omega,\cdot\rangle$ achieves its minimum value. This consists of all elements of $\mathcal{A}$ lying in the face of its convex hull along which the linear function $\langle\omega,\cdot\rangle$ is minimized.

The *initial form* with respect to ω of a polynomial f with support $\mathcal{A}$ (3.4) is

$$\mathrm{in}_\omega(f) \;:=\; \sum_{a\in\mathcal{A}_\omega} c_a x^a\,.$$

This is essentially a polynomial in fewer than n variables. To see this, note that the element $\omega\in\mathbb{Z}^n$ defines a map $\omega\colon\mathbb{T}\to\mathbb{T}^n$ via

$$\mathbb{T}\;\ni\; t\;\longmapsto\; t^\omega \;:=\; (t^{\omega_1},t^{\omega_2},\dots,t^{\omega_n})\;\in\;\mathbb{T}^n\,.$$

We compute $\mathrm{in}_\omega(f)(t^\omega\cdot x)$, which is

$$\sum_{a\in\mathcal{A}_\omega} c_a(t^\omega\cdot x)^a \;=\; \sum_{a\in\mathcal{A}_\omega} c_a t^{\langle\omega,a\rangle}\cdot x^a \;=\; t^{m(\omega,\mathcal{A})}\sum_{a\in\mathcal{A}_\omega} c_a x^a \;=\; t^{m(\omega,\mathcal{A})}\,\mathrm{in}_\omega(f)(x)\,.$$

Thus $\mathrm{in}_\omega(f)$ is a semi-invariant of the subgroup $\omega(\mathbb{T})\subset\mathbb{T}^n$.

Multiplying f by x^{-a} for any $a\in\mathcal{A}_\omega$ does not change the zero set of f, but it translates $\mathcal{A}$ to the set $\mathcal{A}':=\mathcal{A}-a$. Since $m(\omega,\mathcal{A}')=m(\omega,\mathcal{A})-\langle\omega,a\rangle=0$, the new initial form

$$\mathrm{in}_\omega(x^{-a}f) \;=\; x^{-a}\,\mathrm{in}_\omega(f)$$

is an invariant of the subgroup $\omega(\mathbb{T})\subset\mathbb{T}^n$, and thus it induces a well-defined Laurent polynomial on the quotient torus $\mathbb{T}^n/\omega(\mathbb{T})\simeq\mathbb{T}^{n-1}$.

Given a system F (3.1) of polynomials and a vector $\omega\in\mathbb{Z}^n$, the corresponding initial system, or *facial system* $\mathrm{in}_\omega(F)$ is

$$\mathrm{in}_\omega(f_1) \;=\; \mathrm{in}_\omega(f_2) \;=\; \cdots \;=\; \mathrm{in}_\omega(f_n) \;=\; 0\,.$$

After possibly multiplying each polynomial by a monomial, this becomes a system of n polynomials on $\mathbb{T}^n/\omega(\mathbb{T})\simeq\mathbb{T}^{n-1}$, which we would expect to have no solutions.

These definitions make sense for any $\omega\in\mathbb{Q}^n$, with the exception of the interpretation of $\mathrm{in}_\omega(F)$ as a system on the quotient $\mathbb{T}^n/\omega(\mathbb{T})$. For that, we must replace $\omega\in\mathbb{Q}^n$ by the shortest integer vector in the ray $\mathbb{R}_>\omega$ generated by ω.

We now state Bernstein's characterization of which systems (3.1) do not have the expected number of solutions in $\mathbb{T}^n$, even when counted with multiplicity.

THEOREM 3.7. *Let F be a system of polynomials*

$$f_1(x_1,x_2,\ldots,x_n) \;=\; f_2(x_1,x_2,\ldots,x_n) \;=\; \cdots \;=\; f_n(x_1,x_2,\ldots,x_n) \;=\; 0\,, \tag{3.13}$$

where the support of f_i is $\mathcal{A}_i \in \mathbb{Z}^n$. If no facial system $\mathrm{in}_\omega(F)$ for $\omega \neq 0$ has a solution, then all solutions are isolated and $d(f_1,\ldots,f_n) = d(\mathcal{A}_1,\ldots,\mathcal{A}_n)$. If some facial system $\mathrm{in}_\omega(F)$ for $\omega \neq 0$ has a solution, then $d(f_1,\ldots,f_n)$ is strictly smaller than $d(\mathcal{A}_1,\ldots,\mathcal{A}_n)$, unless $d(\mathcal{A}_1,\ldots,\mathcal{A}_n) = 0$, in which case $d(f_1,\ldots,f_n) = 0$.

The multilinearity (3.6) of $d(\mathcal{A}_1,\ldots,\mathcal{A}_n)$ follows from this and the additivity (3.5) of $d(f_1,f_2,\ldots,f_n)$. This is enough to imply Bernstein's Theorem.

Our proof of Theorem 3.7 uses the field $\mathbb{C}\{t\}$ of Puiseaux series. Elements of $\mathbb{C}\{t\}$ are formal Laurent series

$$\sum_{p \geq p_0} c_p t^{p/q}\,,$$

in fractional powers of the indeterminate t (here, $p \in \mathbb{Z}$ and q is a positive integer). The Newton-Puiseaux Theorem (see [**131**, Prop. II.8]) asserts that $\mathbb{C}\{t\}$ is algebraically closed. Since it contains the ring $\mathbb{C}[t]$ of univariate polynomials, it contains the algebraic closure of the field $\mathbb{C}(t)$ of rational functions in t.

The value of $\mathbb{C}\{t\}$ for us is this last fact. Given an algebraic curve $C \subset \mathbb{T}^n$ with a dominant map to $\mathbb{C}^*$, each component not mapping to a point gives an algebraic function of $t \in \mathbb{C}^*$. This function may be expanded as a vector-valued Puiseaux series in t,

$$x(t) \;=\; c_\omega t^\omega \;+\; \text{higher order terms in } t\,, \tag{3.14}$$

where $c_\omega \in \mathbb{T}^n$ and $t^\omega = (t^{\omega_1},\ldots,t^{\omega_n})$ with $\omega \in \mathbb{Q}^n$.

PROOF OF THEOREM 3.7. Suppose first that the system (3.13) has a positive-dimensional set of solutions. Choose a curve C in this set of solutions. This projects dominantly onto some coordinate, t, and so we may expand C as a vector Puiseaux series in t (3.14). Since t is a coordinate, the exponent ω is nonzero. A polynomial of the system evaluated on this series (3.14) must vanish identically. In particular, the coefficient of the lowest power of t must vanish. Since the term involving the lowest power of t in $f_i(x(t))$ is

$$t^{m(\omega,\mathcal{A}_i)} \cdot \mathrm{in}_\omega(f_i)(c_\omega)\,,$$

we see that the initial term $\mathrm{in}_\omega(f_i)$ vanishes at c_ω, and so c_ω is a common solution to the facial system given by ω.

If the solutions to the system (3.13) are isolated, but there are fewer than expected, $d(f_1,\ldots,f_n) < d(\mathcal{A}_1,\ldots,\mathcal{A}_n)$, then we consider a family of systems depending upon a parameter t. Let $g_1,\ldots,g_n$ be general polynomials with supports $\mathcal{A}_1,\ldots,\mathcal{A}_n$, so that $d(g_1,\ldots,g_n) = d(\mathcal{A}_1,\ldots,\mathcal{A}_n)$. Then the system F_t involving the polynomials $(1-t)f_1 + tg_1,\ldots,(1-t)f_n + tg_n$ defines an algebraic function of t. Expand each branch as a vector Puiseaux series in t (3.14).

If $\omega = 0$ for a branch, then $x(0) = c_\omega$ is a solution to the system (3.13). There must be branches with $\omega \neq 0$, for otherwise $d(f_1,\ldots,f_n) = d(g_1,\ldots,g_n) = d(\mathcal{A}_1,\ldots,\mathcal{A}_n)$ as when $t = 1$ there are $d(g_1,\ldots,g_n) = d(\mathcal{A}_1,\ldots,\mathcal{A}_n)$ solutions.

For a branch with $\omega \neq 0$, the coefficient c_ω will be a solution to the facial system corresponding to ω.

For the second statement, suppose that $c_\omega \in \mathbb{T}^n/\omega(\mathbb{T})$ is a solution to a facial system $\text{in}_\omega(F) = 0$ for ω nonzero, and suppose that $\omega \in \mathbb{Z}^n$ is primitive. Then we may identify $\mathbb{T} \times (\mathbb{T}^n/\omega(\mathbb{T}))$ with $\mathbb{T}^n$, and consider its compactification $\mathbb{C} \times (\mathbb{T}^n/\omega(\mathbb{T}))$. Translating the support of the polynomial f_i if necessary, we may assume that $m(\omega, \mathcal{A}_i) = 0$, and therefore F is a system on $\mathbb{C} \times (\mathbb{T}^n/\omega(\mathbb{T}))$, with $\text{in}_\omega(F)$ equal to the restriction of F to $\{0\} \times (\mathbb{T}^n/\omega(\mathbb{T}))$, and therefore $(0, c_\omega)$ is a solution to this extended system.

If we perturb the system F by a general system with support $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ to obtain F_t as before, then for t general, F_t has $d(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ solutions. Each of these lie on a curve of solutions to $F_t = 0$, as does the solution $(0, c_\omega)$ to $F_0 = 0$. This shows that in the limit as $t \to 0$, at least one curve of solutions to $F_t = 0$ leads to $(0, c_\omega)$, and therefore $F_0 = 0$ has fewer than $d(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ solutions in $\mathbb{T}^n = \mathbb{T} \times (\mathbb{T}^n/\omega(\mathbb{T}))$.

When $d(\mathcal{A}_1, \ldots, \mathcal{A}_n) = 0$, there will be no curves of solutions to F_t, and therefore no solutions in $\mathbb{T}^n$ to $F_0 = 0$. ∎

CHAPTER 4

Toric Degenerations and Kushnirenko's Theorem

Our study of bounds for the number of real solutions to systems of sparse polynomial equations will make use of geometric constructions involving toric varieties. This chapter introduces the important toric degenerations (sometimes called Gröbner degenerations) on the way to a second proof of Kushnirenko's Theorem for the number of solutions to a system

$$f_1(x_1,\ldots,x_n) \;=\; f_2(x_1,\ldots,x_n) \;=\; \cdots \;=\; f_n(x_1,\ldots,x_n) \;=\; 0\,, \tag{4.1}$$

of polynomials with support $\mathcal{A}$. The fundamental case of Kushnirenko's Theorem is when the monomials $\mathcal{A}$ form a simplex ($|\mathcal{A}| = n{+}1$)—this is also fundamental in our study of real solutions. The general case uses toric degenerations. The idea is to replace the toric variety $X_\mathcal{A}$ by a simpler variety whose degree is evident, and then argue that the passage from $X_\mathcal{A}$ to this simpler variety preserves the degree. We close with an application of toric degenerations to prove Theorem 1.8, showing that some polynomial systems may have all their solutions be real.

4.1. Kushnirenko's Theorem for a simplex

Suppose that $|\mathcal{A}| = n+1$. Since $\mathcal{A}$ affinely spans $\mathbb{R}^n$, its convex hull $\Delta_\mathcal{A}$ is a simplex with vertices $\mathcal{A}$. Then $X_\mathcal{A} = \mathbb{P}^n = \mathbb{P}^\mathcal{A}$. By Lemma 3.5, the solutions to a sparse system (4.1) with support $\mathcal{A}$ have the form $\varphi_\mathcal{A}^{-1}(L)$, where $L \subset \mathbb{P}^n$ is the codimension n plane cut out by the linear forms which define the polynomials of the system. Thus L is a point $\beta \in \mathbb{P}^n$ (which lies in the dense torus as the equations are general) and the solutions have the form $\varphi_\mathcal{A}^{-1}(\beta)$. Since $\varphi_\mathcal{A}$ is a homomorphism to the dense torus of $\mathbb{P}^\mathcal{A}$, these solutions form a single coset of $\ker(\varphi_\mathcal{A})$.

We may determine these solutions explicitly. Assume that $0 \in \mathcal{A}$. Since $|\mathcal{A}| = n+1$, we may write the sparse system (4.1) as

$$C \cdot (x^{a_1}, x^{a_2}, \ldots, x^{a_n})^T \;=\; b\,, \tag{4.2}$$

where $\mathcal{A} - \{0\} = \{a_1, a_2, \ldots, a_n\}$, C is the n by n matrix of coefficients, and $b \in \mathbb{C}^n$. If our system is generic, then C is invertible and row operations on C and hence on the system (4.2) lead to an equivalent system of binomials

$$x^{a_i} \;=\; \beta_i \qquad \text{for } i = 1, \ldots, n\,. \tag{4.2$'$}$$

where $\beta_1, \ldots, \beta_n \in \mathbb{T}$, and so the system has the form $\varphi_\mathcal{A}^{-1}(\beta)$. (In fact, the requirements that C be invertible and that the resulting constants $\beta_i \in \mathbb{T}$ are the conditions for genericity of this system.)

Let A be the n by n matrix whose columns are the nonzero exponent vectors in $\mathcal{A}$. We use the integer linear algebra of the matrix A to solve the system (4.2)$'$.

EXAMPLE 4.1. Consider the system of equations whose support is the simplex $\mathcal{A} = \{(16,14),(22,18),(0,0)\}$,

$$\begin{aligned} 23x^{16}y^{14} \;-\; x^{22}y^{18} \;&=\; -27\,, \qquad \text{and}\\ 35x^{16}y^{14} \;-\; x^{22}y^{18} \;&=\; 9\,. \end{aligned} \tag{4.3}$$

The difference of the two equations is the binomial,

$$12x^{16}y^{14} \;=\; 36 \qquad \text{or} \qquad x^{16}y^{14} \;=\; 3\,.$$

Back substitution gives $x^{22}y^{18} = 96$. Thus we have the equivalent binomial system

$$x^{16}y^{14} \;=\; 3 \qquad \text{and} \qquad x^{22}y^{18} \;=\; 96\,. \tag{4.4}$$

Under the invertible substitution (the inverse is given by $u = x^8y^7$ and $v = (xy)^{-1}$),

$$x \;=\; uv^7 \qquad \text{and} \qquad y \;=\; u^{-1}v^{-8}\,, \tag{4.5}$$

our equations become triangular in u and v,

$$\begin{aligned} (uv^7)^{16}(u^{-1}v^{-8})^{14} \;&=\; u^{16}v^{112}u^{-14}v^{-112} \;=\; u^2 \;=\; 3\,, \qquad \text{and}\\ (uv^7)^{22}(u^{-1}v^{-8})^{18} \;&=\; u^{22}v^{154}u^{-18}v^{-144} \;=\; u^4v^{10} \;=\; 96\,. \end{aligned} \tag{4.6}$$

While the solution is now immediate, we make one further simplifying substitution. Write f for the first equation and g for the second. Replacing g by gf^{-2} yields a diagonal system which is now completely trivial to solve

$$\begin{aligned} f \;&:\; u^2 \;=\; 3\,,\\ gf^{-2} \;&:\; v^{10} \;=\; 32/3\,. \end{aligned} \tag{4.7}$$

The solutions are $u = \pm\sqrt{3}$ and $v = \zeta\sqrt{2}$, where ζ runs over all 10th roots of $\frac{1}{3}$. Substituting these into (4.5), gives the 20 solutions to our original system of equations (4.3).

Underlying these simplifications is the relation between the integer linear algebra of n by n matrices and (multiplicative) coordinate changes in $\mathbb{T}^n$. Since $\mathbb{T}^n = \mathrm{Hom}(\mathbb{Z}^n, \mathbb{T})$, its automorphism group is $GL(n,\mathbb{Z})$, the group of invertible n by n integer matrices, which is the source of that relation.

REMARK 4.2. The monomials in (4.4) correspond to the columns of the matrix

$$A \;=\; \begin{pmatrix} 16 & 22\\ 14 & 18 \end{pmatrix}\,,$$

and the coordinate change (4.5) corresponds to left multiplication (hence row operations) by the matrix

$$\begin{pmatrix} 1 & -1\\ 7 & -8 \end{pmatrix}\,.$$

Indeed,

$$\begin{pmatrix} 1 & -1\\ 7 & -8 \end{pmatrix}\begin{pmatrix} 16 & 22\\ 14 & 18 \end{pmatrix} \;=\; \begin{pmatrix} 2 & 4\\ 0 & 10 \end{pmatrix}\,,$$

which corresponds to the exponent vectors in the triangular system (4.6). This matrix is the *Hermite normal form* of the matrix A—the row reduced echelon form over $\mathbb{Z}$. This notion makes sense for matrices whose entries lie in any principal ideal domain, such as the integers $\mathbb{Z}$ or univariate polynomials over a field.

Multiplicative reductions using the equations (4.7) correspond to multiplicative coordinate changes in the target torus $\mathbb{T}^2$ and are represented by column operations, or right multiplication by integer matrices. Indeed

$$\begin{pmatrix}2 & 4\\ 0 & 10\end{pmatrix}\begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix} \;=\; \begin{pmatrix}2 & 0\\ 0 & 10\end{pmatrix}\,,$$

which is the Smith normal form of the integer matrix A. ◆

This discussion leads to a determination of the solutions to a general sparse system supported on a simplex.

THEOREM 4.3 (Kushnirenko's Theorem for a simplex). *A general system* (4.1) *of polynomials supported on a simplex* $\mathcal{A}$ *has* $n!\,\mathrm{volume}(\Delta_{\mathcal{A}})$ *distinct nonzero complex solutions which may be computed by Algorithm* 4.4 *using Gaussian elimination, Smith normal form, and extraction of roots.*

ALGORITHM 4.4. Given a general system (4.1) of polynomials supported on a simplex $\mathcal{A}$, first multiply each polynomial by a monomial x^{-a} for some $a \in \mathcal{A}$, replacing $\mathcal{A}$ by its translation $\mathcal{A} - a$ so that $0 \in \mathcal{A}$.

Next, write the sparse system in the form (4.2), use Gaussian elimination on the coefficient matrix C to convert it into diagonal form $(4.2)'$, written as

$$x^A \;=\; \beta \;:=\; (\beta_1, \ldots, \beta_n) \in \mathbb{T}^n\,, \tag{4.8}$$

where A is the matrix whose columns are the nonzero vectors in $\mathcal{A}$.

Then, compute the Smith normal form of the matrix A, obtaining invertible integer n by n matrices P, Q and integers $d_1, \ldots, d_n$ such that

$$PAQ \;=\; D \;:=\; \mathrm{diag}(d_1, d_2, \ldots, d_n)\,.$$

The solutions to (4.8) and thus to (4.1) are obtained as follows. Let $q_1, \ldots, q_n$ be the columns of Q and $\xi_1, \ldots, \xi_n$ be any of the $d_1 d_2 \cdots d_n$ choices of roots

$$\xi_j \;:=\; \sqrt[d_j]{\beta^{q_j}} \qquad \text{for } j = 1, \ldots, n\,. \tag{4.9}$$

Letting $p_1, \ldots, p_n$ be the columns of P, then $x = (\xi^{p_1}, \ldots, \xi^{p_n}) =: \xi^P$ is a solution to (4.8), and all solutions are obtained in this way.

CORRECTNESS OF ALGORITHM 4.4. Rewrite (4.9) as $\xi^D = \beta^Q$. Since $\xi^D = \xi^{PAQ}$, the invertible monomial substitution $x = \xi^P$ gives $x^{AQ} = \beta^Q$. Acting on the vector of equations by the invertible multiplicative substitutions given by Q^{-1}, we obtain $x^A = \beta$, which shows that the algorithm gives solutions to (4.8). The correctness and root count follow as these transformations are invertible. ◆

PROOF OF THEOREM 4.3. By Algorithm 4.4, the equation (4.9) has $d_1 d_2 \cdots d_n$ distinct solutions. This number is $|\det A|$, as $D = PAQ$ with P and Q invertible integer matrices. But $|\det A|$ is the Euclidean volume of the parallelepiped spanned by the vectors $a_1, \ldots, a_n$, which is $n!$ times the volume of the simplex spanned by 0 and these vectors. ◆

4.2. Regular subdivisions and toric degenerations

Let $\mathcal{A} \subset \mathbb{Z}^n$ be a finite set that spans $\mathbb{R}^n$. Lemma 3.5 and the definition of toric variety provide a correspondence between solutions to systems of polynomials with support $\mathcal{A}$ and linear sections $L \cap X_{\mathcal{A}}$ of the toric variety $X_{\mathcal{A}}$ by complementary dimensional linear spaces L. In Section 3.3 we computed the degree of $X_{\mathcal{A}}$. Here, we replace $X_{\mathcal{A}}$ by a simpler variety related to it, $\mathrm{in}_\omega(X_{\mathcal{A}})$, and in the next section show how the passage $X_{\mathcal{A}} \rightsquigarrow \mathrm{in}_\omega(X_{\mathcal{A}})$ affects the linear section.

This passage to a simpler variety relies upon the choice of an auxiliary function $\omega\colon \mathcal{A} \to \mathbb{Z}$, which gives weights for an action of $\mathbb{T}$ on the space $\mathbb{C}^{\mathcal{A}}$,

$$t.(y_a \mid a \in \mathcal{A}) \;=\; (t^{\omega(a)} y_a \mid a \in \mathcal{A}),$$

inducing an action on $\mathbb{P}^{\mathcal{A}}$. Consider the family of varieties,

$$\mathcal{X}_{\mathcal{A}} \;:=\; \overline{\{(t,y) \in \mathbb{T} \times \mathbb{P}^{\mathcal{A}} \mid t.y \in X_{\mathcal{A}}\}} \;\subset\; \mathbb{C} \times \mathbb{P}^{\mathcal{A}}.$$

This has a natural projection $\pi\colon \mathcal{X}_{\mathcal{A}} \to \mathbb{C}$, with the fiber $\pi^{-1}(t)$ over a point $t \in \mathbb{T}$ the translated toric variety $t^{-1}.X_{\mathcal{A}}$. The fiber $\pi^{-1}(0)$ of $\mathcal{X}_{\mathcal{A}}$ over $0 \in \mathbb{C}$ is called the *scheme-theoretic limit* of the family $t^{-1}.X_{\mathcal{A}}$, and we write

$$\lim_{t \to 0} t^{-1}.X_{\mathcal{A}} \;:=\; \pi^{-1}(0).$$

This passage to a scheme-theoretic limit of an action of $\mathbb{T}$ is a *toric degeneration*.

We identify this scheme-theoretic limit. Let $I_{\mathcal{A}}$ be the ideal of the toric variety $X_{\mathcal{A}}$, called a *toric ideal*. The function ω determines a linear function $\langle \omega, \cdot \rangle$ on $\mathbb{N}^{\mathcal{A}}$,

$$\mathbb{N}^{\mathcal{A}} \;\ni\; \alpha \;\longmapsto\; \langle \omega, \alpha \rangle \;:=\; \sum_{a \in \mathcal{A}} \omega(a)\alpha_a \;\in\; \mathbb{Z}.$$

As in Section 3.4, this leads to the notion of the initial form $\mathrm{in}_\omega(f)$ of a polynomial $f \in \mathbb{C}[z_a \mid a \in \mathcal{A}]$, which is the sum of the terms in f whose exponents achieve the minimum value under $\langle \omega, \cdot \rangle$. The *initial ideal* $\mathrm{in}_\omega(I_{\mathcal{A}})$ is

$$\mathrm{in}_\omega(I_{\mathcal{A}}) \;:=\; \{\mathrm{in}_\omega(f) \mid f \in I_{\mathcal{A}}\},$$

and the *initial scheme* $\mathrm{in}_\omega(X_{\mathcal{A}})$ is the scheme associated to the initial ideal. We state the fundamental result about toric degenerations.

THEOREM 4.5. *The initial scheme equals the toric degeneration,*

$$\mathrm{in}_\omega(X_{\mathcal{A}}) \;=\; \pi^{-1}(0) \;=\; \lim_{t \to 0} t^{-1}.X_{\mathcal{A}}.$$

This is an example of a flat family over $\mathbb{C}$ [**39**, Ch. 15]. (This technical fact implies that all fibers have the same Hilbert polynomial.) This result that the toric degeneration is the initial scheme is true for any variety.

EXAMPLE 4.6. Consider this for the toric variety of Example 3.6 associated to the lattice triangle of Example 3.1. This cubic hypersurface has equation

$$f \;:=\; z_a z_b z_d - z_c^3,$$

where a, b, c, d are the lattice points $(2,1)$, $(1,2)$, $(1,1)$, and $(0,0)$, respectively. This equation comes from the vector identity

$$1 \cdot (2,1) \;+\; 1 \cdot (1,2) \;+\; 1 \cdot (0,0) \;=\; (3,3) \;=\; 3 \cdot (1,1),$$

so the polynomial f is $z^\alpha - z^\beta$ where $\alpha = (1,1,0,1)$ and $\beta = (0,0,3,0)$. Let ω take the value 2 on $c = (1,1)$ and the value 1 on the remaining lattice points, which are vertices of the triangle. Then

$$\langle \omega, \alpha \rangle \;=\; 1+1+1 = 3 \;<\; 6 = 2 \cdot 3 \;=\; \langle \omega, \beta \rangle,$$

and so $\mathrm{in}_\omega(f)$ is the monomial $z_a z_b z_d$. The ideal of $\mathcal{X}_{\mathcal{A}}$ is generated by

$$t^{-3} t^{\langle \omega, \alpha \rangle} z_a z_b z_d \;-\; t^{-3} t^{\langle \omega, \beta \rangle} z_c^3 \;=\; z_a z_b z_d \;-\; t^3 z_c^3.$$

Specializing this at $t = 0$ gives $z_a z_b z_d = \mathrm{in}_\omega(f)$.

We first determine the toric ideal $I_{\mathcal{A}}$ and then study the passage to the limit. As in Section 3.3, replace $\mathcal{A}$ by its lift $\mathcal{A}^+$ and consider the cone over $X_{\mathcal{A}}$, which is the closure of the image of the map from $\mathbb{T}^{1+n}$,

$$\varphi_{\mathcal{A}^+}(x_0, x) \ := \ (x_0 x^a \mid a \in \mathcal{A})\,.$$

(Unlike in Section 3.3, we use x_0 rather than t for the homogenizing variable.) This induces an algebra map,

$$\mathbb{C}[z_a \mid a \in \mathcal{A}] \ \xrightarrow{\varphi^*_{\mathcal{A}^+}} \ \mathbb{C}[x_0^{\pm}, x_1^{\pm}, \ldots, x_n^{\pm}]$$

whose image is isomorphic to the homogeneous coordinate ring $\mathbb{C}[X_{\mathcal{A}}]$ of the toric variety $X_{\mathcal{A}}$. This identifies the toric ideal $I_{\mathcal{A}}$ as the kernel of the map $\varphi^*_{\mathcal{A}^+}$.

Let $\alpha = (\alpha_a \mid a \in \mathcal{A}) \in \mathbb{N}^{\mathcal{A}}$, which is an exponent vector of a monomial z^α in $\mathbb{C}[z_a \mid a \in \mathcal{A}]$. The image of z^α under $\varphi^*_{\mathcal{A}^+}$ is

$$\varphi^*_{\mathcal{A}^+}(z^\alpha) \ = \ \varphi^*_{\mathcal{A}^+}\big(\prod_{a\in\mathcal{A}} z_a^{\alpha_a}\big) \ = \ \prod_{a\in\mathcal{A}} (x_0, x^a)^{\alpha_a} \ = \ x_0^{\sum \alpha_a} x^{\mathcal{A}\alpha}\,,$$

where $\mathcal{A}\alpha$ is the product of the column vector α with the matrix $\mathcal{A}$ whose columns are the exponents in $\mathcal{A}$.

This computation determines some binomials that lie in $I_{\mathcal{A}}$, namely

$$z^\alpha - z^\beta \qquad \text{where} \quad \sum_{a\in\mathcal{A}} \alpha_a \ = \ \sum_{a\in\mathcal{A}} \beta_a \quad \text{and} \quad \mathcal{A}\alpha \ = \ \mathcal{A}\beta\,. \tag{4.10}$$

Writing $\mathcal{A}^+$ for the $(1{+}n)$ by $|\mathcal{A}|$ matrix whose columns are the lifted exponents, the condition in (4.10) becomes $\mathcal{A}^+\alpha = \mathcal{A}^+\beta$. These binomials span the ideal.

LEMMA 4.7 ([**154**, Lemma 4.1]). *The toric ideal $I_{\mathcal{A}}$ is spanned as a complex vector space by the homogeneous binomials* (4.10).

A point (t, y) with $t \in \mathbb{T}$ lies in $\mathcal{X}_{\mathcal{A}}$ if and only if $t.y \in X_{\mathcal{A}}$. Then, for any $\alpha, \beta \in \mathbb{N}^{\mathcal{A}}$ with $\mathcal{A}^+\alpha = \mathcal{A}^+\beta$, we have

$$0 \ = \ (t.y)^\alpha - (t.y)^\beta \ = \ t^{\langle\omega,\alpha\rangle} y^\alpha - t^{\langle\omega,\beta\rangle} y^\beta\,,$$

and so we obtain a binomial $t^{\langle\omega,\alpha\rangle} z^\alpha - t^{\langle\omega,\beta\rangle} z^\beta$ vanishing on $\mathcal{X}_{\mathcal{A}} \cap (\mathbb{T} \times \mathbb{P}^{\mathcal{A}})$. Since t is invertible on $\mathbb{T}$, we may multiply this by t^c for any $c \in \mathbb{Z}$ to obtain another vanishing binomial. Thus the ideal of $\mathcal{X}_{\mathcal{A}} \cap (\mathbb{T} \times \mathbb{P}^{\mathcal{A}})$ is spanned by the binomials

$$t^c t^{\langle\omega,\alpha\rangle} z^\alpha - t^c t^{\langle\omega,\beta\rangle} z^\beta \qquad \text{for} \quad c \in \mathbb{Z}\,,\ \alpha, \beta \in \mathbb{N}^{\mathcal{A}} \text{ with } \mathcal{A}^+\alpha = \mathcal{A}^+\beta\,. \tag{4.11}$$

The ideal of $\mathcal{X}_{\mathcal{A}}$ is the intersection of this ideal with the ring $\mathbb{C}[t, z_a \mid a \in \mathcal{A}]$, so that it is spanned by those binomials (4.11) having nonnegative exponents of t.

COROLLARY 4.8. *The ideal of $\mathcal{X}_{\mathcal{A}}$ is the linear span of binomials*

$$t^c t^{\langle\omega,\alpha\rangle} z^\alpha \ - \ t^c t^{\langle\omega,\beta\rangle} z^\beta$$

for $c \geq -\min(\langle\omega,\alpha\rangle, \langle\omega,\beta\rangle)$ where $\mathcal{A}^+\alpha = \mathcal{A}^+\beta$.

The ideal of the fiber $\pi^{-1}(0)$ of $\mathcal{X}_{\mathcal{A}}$ is obtained by setting $t = 0$ in every polynomial in the ideal of $\mathcal{X}_{\mathcal{A}}$. The binomials in Corollary 4.8 will vanish unless $c = -\min(\langle\omega,\alpha\rangle, \langle\omega,\beta\rangle)$. In that case there are two possibilities. Suppose that $\mathcal{A}^+\alpha = \mathcal{A}^+\beta$, and we have $\langle\omega,\alpha\rangle \leq \langle\omega,\beta\rangle$. Then this specialization gives

$$z^\alpha \quad \text{if } \langle\omega,\alpha\rangle < \langle\omega,\beta\rangle \qquad \text{and} \qquad z^\alpha - z^\beta \quad \text{if } \langle\omega,\alpha\rangle = \langle\omega,\beta\rangle\,. \tag{4.12}$$

In either case, this specialization is the initial form $\mathrm{in}_\omega(z^\alpha - z^\beta)$ of the corresponding binomial. This completes the proof of Theorem 4.5.

We interpret the binomials in $I_{\mathcal{A}}$ (4.10) in terms of the convex geometry of the set $\mathcal{A}$ of exponents. This will facilitate our partial description (Theorem 4.9) of the initial ideal $\mathrm{in}_\omega(I_{\mathcal{A}})$ in terms of the convex geometry of the function $\omega\colon \mathcal{A} \to \mathbb{Z}$.

Suppose that $\alpha, \beta \in \mathbb{N}^{\mathcal{A}}$ are exponent vectors with $\mathcal{A}^+\alpha = \mathcal{A}^+\beta$ so that $z^\alpha - z^\beta \in I_{\mathcal{A}}$. This product $\mathcal{A}^+\alpha$ is a nonnegative integer combination of the vectors in $\mathcal{A}^+$. Thus the binomial $z^\alpha - z^\beta$ corresponds to a vector in the cone $\mathbb{N}\mathcal{A}^+$ with two distinct representations as a nonnegative integer sum of vectors in $\mathcal{A}^+$. Dividing the vector $\mathcal{A}^+\alpha$ by its first coordinate $|\alpha| := \sum_{a\in\mathcal{A}} \alpha_a$, we obtain a vector $(1, x)$ and a representation

$$x \;=\; \sum_{a\in\mathcal{A}} \frac{\alpha_a}{|\alpha|} a$$

of x as a rational convex combination of vectors in $\mathcal{A}$. Since $A^+\alpha = A^+\beta$, this point x is a rational convex combination of elements of $\mathcal{A}$ in two distinct ways. Thus x is a rational point lying in the convex hull of two different subsets of $\mathcal{A}$. We explain below how any such point x gives rise to a binomial $z^\alpha - z^\beta \in I_{\mathcal{A}}$.

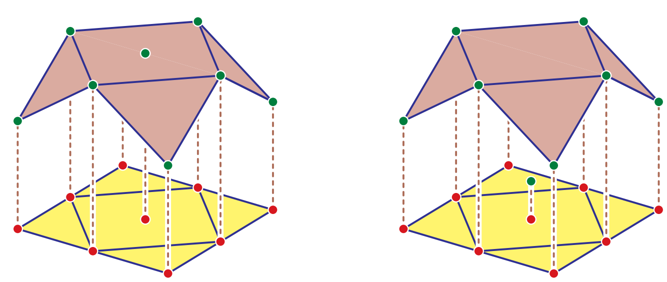
We remark that if $\mathcal{A}$ is a simplex, then points in its convex hull have unique representations as convex combinations of points of $\mathcal{A}$, so $I_{\mathcal{A}} = 0$ (which we already saw as $X_{\mathcal{A}} = \mathbb{P}^{\mathcal{A}}$ is the full projective space).

The graph of the map $\omega\colon \mathcal{A} \to \mathbb{Z}$ is a configuration of lattice points in $\mathbb{R}^{1+n}$. Let P_ω be the convex hull of this configuration,

$$P_\omega \;:=\; \mathrm{conv}\{(\omega(a), a) \mid a \in \mathcal{A}\}\,. \tag{4.13}$$

The *upper facets* of the polytope P_ω are those facets whose inward-pointing normal vector has a negative first coordinate. Projecting these upper facets back to $\mathbb{R}^n$ gives the facets in the *regular polyhedral subdivision* Δ_ω of the convex hull $\Delta_{\mathcal{A}}$ of $\mathcal{A}$ induced by the function ω. Faces of these facets give *upper faces* of P_ω and faces of the polyhedral subdivision Δ_ω.

This construction induces a *regular subdivision* $\mathcal{S}_\omega$ of $\mathcal{A}$, which is a collection of subsets $\mathcal{F}$ of $\mathcal{A}$, called faces, indexed by the faces of Δ_ω. The subset $\mathcal{F}$ corresponding to a face F of Δ_ω consists of those points in $\mathcal{A}$ whose lift to P_ω lies in the upper face corresponding to F. The convex hull of $\mathcal{F}$ is the face F, but not every element of $\mathcal{A} \cap F$ need be contained in $\mathcal{F}$. We show two different functions that induce the same regular polyhedral subdivision of the 2×2 square, but different regular subdivisions of $\mathcal{A}$.



The difference is that the center point of $\mathcal{A}$ does not lie in any face of the subdivision on the right as its lift does not lie on any upper face.

We use the polytope P_ω and the corresponding regular subdivision $\mathcal{S}_\omega$ to understand the elements (4.12) of the initial ideal of $I_\mathcal{A}$. Let x be a point of $\mathrm{conv}(\mathcal{A})$ with more than one representation as a rational convex combination of points of $\mathcal{A}$, and consider two such representations. We may assume that one involves only those points of $\mathcal{A}$ lying in some face $\mathcal{F}$ of the regular subdivision $\mathcal{S}_\omega$. Thus we have rational vectors $\lambda = (\lambda_a \mid a \in \mathcal{A})$ and $\mu = (\mu_b \mid b \in \mathcal{F})$ with

$$\sum_{a\in\mathcal{A}} \lambda_a \;=\; \sum_{b\in\mathcal{F}} \mu_b \;=\; 1 \qquad \text{and} \qquad \sum_{a\in\mathcal{A}} \lambda_a \cdot a \;=\; x \;=\; \sum_{b\in\mathcal{F}} \mu_b \cdot b\,. \tag{4.14}$$

Extending μ by zero to get a vector in $\mathbb{Q}^\mathcal{A}$ and multiplying λ and μ by a common denominator, we obtain integer vectors $\alpha \in \mathbb{N}^\mathcal{A}$ and $\beta \in \mathbb{N}^\mathcal{F} \subset \mathbb{N}^\mathcal{A}$ with $\mathcal{A}^+\alpha = \mathcal{A}^+\beta (= \mathcal{F}^+\beta)$ and so $z^\alpha - z^\beta \in I_\mathcal{A}$.

We may lift the expression (4.14) using the function ω to obtain points

$$\Bigl(\sum_{a\in\mathcal{A}} \lambda_a\omega(a)\,,\, \sum_{a\in\mathcal{A}} \lambda_a \cdot a\Bigr) \;=\; (\langle\omega,\lambda\rangle\,,\, x) \qquad \text{and} \qquad (\langle\omega,\mu\rangle\,,\, x)$$

of P_ω. Since the lifted points $(\omega(b), b)$ for $b \in \mathcal{F}$ lie on an upper face of P_ω, we have the inequality $\langle\omega,\lambda\rangle \leq \langle\omega,\mu\rangle$ with equality if and only if the support $\{a \mid \lambda_a \neq 0\}$ of λ is a subset of $\mathcal{F}$. We obtain the same inequalities between $\langle\omega,\alpha\rangle$ and $\langle\omega,\beta\rangle$, and thus $\mathrm{in}_\omega(z^\alpha - z^\beta)$ is

$$\begin{array}{rl} z^\alpha & \text{if the support of } \alpha \text{ is not contained in a face of } \mathcal{S}_\omega, \\ z^\alpha - z^\beta & \text{if the support of } \alpha \text{ is contained in the face } \mathcal{F} \text{ of } \mathcal{S}_\omega. \end{array}$$

In fact, all such monomials and binomials span the initial ideal. The complete description of the initial ideal depends upon the arithmetic of rational convex coincidences (4.14) involving $\mathcal{A}$. We offer a simpler partial description.

THEOREM 4.9. *The initial ideal* $\mathrm{in}_\omega(I_\mathcal{A})$ *is generated by monomials* z^α *coming from coincident rational convex representations* (4.14) *in which the support of* α *is not a subset of any face of the subdivision* $\mathcal{S}_\omega$*, and binomials* $z^\alpha - z^\beta$ *in toric ideals* $I_\mathcal{F}$ *of faces* $\mathcal{F}$ *of the subdivision* $\mathcal{S}_\omega$*. Given any set* $\mathcal{B} \subset \mathcal{A}$ *which is not a subset of a face of* $\mathcal{S}_\omega$*, there is some* $\alpha \in \mathbb{N}^\mathcal{A}$ *with support* $\mathcal{B}$ *such that* $z^\alpha \in \mathrm{in}_\omega(I_\mathcal{A})$.

PROOF. We need only show the last statement. The convex hull of $\mathcal{B}$ meets some face F of Δ_ω, and $\mathcal{B}$ is not a subset of the corresponding set $\mathcal{F}$ of $\mathcal{S}_\omega$. If x is a rational point common to F and to the convex hull of $\mathcal{B}$, there is an expression (4.14) with the support of λ contained in $\mathcal{B}$ (but not in $\mathcal{F}$) and therefore a binomial $z^\alpha - z^\beta$ in $I_\mathcal{A}$ with initial term z^α where the support of α is a subset of $\mathcal{B}$. Multiplying this monomial by $\prod_{b\in\mathcal{B}} z_b$ gives a monomial in $\mathrm{in}_\omega(I_\mathcal{A})$ with support $\mathcal{B}$.

EXAMPLE 4.10. Consider the polytope P_ω and initial ideal for the lattice triangle and function ω of Example 4.6. This involved the lattice points $(2,1)$, $(1,2)$, $(1,1)$, and $(0,0)$, which we refer to as a, b, c, and d, respectively, and the function ω took the value 1 at the vertices a, b, d and 2 at the center point c. Figure 4.1 shows two views of the upper hull of P_ω and the corresponding subdivision. The only subsets of $\{a, b, c, d\}$ which are not faces of the subdivision $\mathcal{S}_\omega$ are the set of vertices $\{a, b, d\}$ and the whole set. Since the subdivision is a triangulation and $I_\mathcal{F} = 0$ for a triangle $\mathcal{F}$, Theorem 4.9 predicts that $\mathrm{in}_\omega(I_\mathcal{A})$ is a monomial ideal and that it contains a monomial $z_a^{\alpha_a} z_b^{\alpha_b} z_d^{\alpha_d}$ with $\alpha_a, \alpha_b, \alpha_d > 0$. Indeed, we saw that $\mathrm{in}_\omega(I_\mathcal{A})$ is generated by $z_a z_b z_d$.

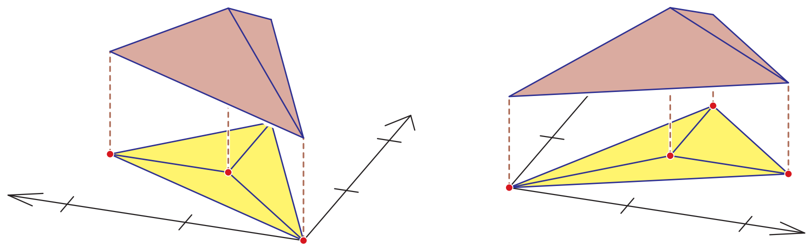


Figure 4.1. Two views of an upper hull

4.3. Kushnirenko's Theorem via toric degenerations

Since $\pi\colon \mathcal{X}_{\mathcal{A}} \to \mathbb{C}$ is a flat family of projective varieties, every fiber has the same degree, and so Kushnirenko's Theorem follows if we can prove that the degree of the limit scheme $\text{in}_\omega(X_{\mathcal{A}}) = \pi^{-1}(0)$ is $n!\,\text{volume}(\Delta_{\mathcal{A}})$ divided by the degree $[\mathbb{Z}^n : \mathbb{Z}\mathcal{A}]$ of $\varphi_{\mathcal{A}}$. We will prove this in the case when every facet $\mathcal{F}$ of the regular subdivision $\mathcal{S}_\omega$ consists of $n{+}1$ vectors forming an n-dimensional simplex. Such a regular subdivision is called a *regular triangulation*. This is not a restrictive assumption, for every finite set $\mathcal{A}$ has at least one regular triangulation. One way to ensure that $\mathcal{S}_\omega$ is a triangulation is to require that there are no affine dependencies among subsets of $n{+}2$ lifted points in $\{(\omega(a), a) \mid a \in \mathcal{A}\}$. Figure 4.2 shows the upper facets and triangulation Δ_ω induced by a function ω on $\mathcal{A} =$

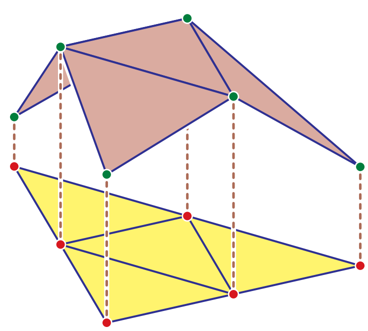


Figure 4.2. Upper facets and a regular triangulation.

$\{(0,0),(1,0),(0,1),(2,0),(1,1),(0,2)\}$. The decomposition of Figure 4.1 was also a triangulation.

If ω induces a triangulation of $\mathcal{A}$, then Theorem 4.9 implies that the initial ideal $\text{in}_\omega(I_{\mathcal{A}})$ is generated by monomials. Still, it can have complicated nonreduced structure. For the function $-\omega$ of Example 4.6, $\text{in}_{-\omega}(z_a z_b z_d - z_c^3) = z_c^3$, so that the initial ideal is not reduced. While the scheme structure depends upon the arithmetic of rational convex coincidences (4.14) involving $\mathcal{A}$, the radical of the initial ideal is easy to understand from the triangulation.

Theorem 4.11. *If $\mathcal{S}_\omega$ is a triangulation of $\mathcal{A}$ then the radical of the initial ideal is the intersection of ideals of coordinate subspaces, one for each facet simplex $\mathcal{F}$ of* *the triangulation* $\mathcal{S}_\omega$,

$$\sqrt{\mathrm{in}_\omega(I_{\mathcal{A}})} \;=\; \bigcap_{\mathcal{F}} \langle z_a \;\mid\; a \notin \mathcal{F}\rangle\,.$$

EXAMPLE 4.12. If ω is the function of Example 4.6, then the initial ideal of $z_az_bz_d - z_c^3$ is generated by $z_az_bz_d$ and we have

$$\langle z_az_bz_d\rangle \;=\; \langle z_a\rangle \;\cap\; \langle z_b\rangle \;\cap\; \langle z_d\rangle\,.$$

We saw in Example 4.10 that the triangulation induced by ω has three facet triangles, $\{b,c,d\}$, $\{a,c,d\}$, and $\{a,b,c\}$, which give the three ideals $\langle z_a\rangle$, $\langle z_b\rangle$, and $\langle z_d\rangle$, respectively. Each of these cuts out a coordinate subspace of $\mathbb{P}^3$ that corresponds to the appropriate triangle.

The ideal $\langle z_a \mid a \notin \mathcal{F}\rangle$ defines the coordinate subspace $\mathbb{P}^{\mathcal{F}}$ of $\mathbb{P}^{\mathcal{A}}$ which is spanned by the $n{+}1$ coordinates indexed by elements of $\mathcal{F}$.

COROLLARY 4.13. *The limit scheme* $\mathrm{in}_\omega(X_{\mathcal{A}})$ *is supported on the union of the coordinate subspaces* $\mathbb{P}^{\mathcal{F}}$ *for* $\mathcal{F}$ *a facet of the triangulation* $\mathcal{S}_\omega$.

PROOF OF THEOREM 4.11. If the set $\{a,b\}$ for $a,b\in\mathcal{A}$ is not a face of the triangulation $\mathcal{S}_\omega$, then its convex hull $\overline{a,b}$ crosses a minimal face $\Delta_{\mathcal{F}} = \operatorname{conv}\mathcal{F}$ of the triangulation Δ_ω. Any point in this intersection gives coincident convex combinations (4.14) with one involving $\{a,b\}$ and the other involving $\mathcal{F}$. This implies that there is a binomial $z_a^Mz_b^N - z^\gamma$, where M,N are positive integers and the monomial z^γ involves the variables in $\mathcal{F}$. Since $\mathcal{F}$ is a face and $\{a,b\}$ is not, the initial term is $z_a^Mz_b^N$, and so z_az_b lies in the radical $\sqrt{\mathrm{in}_\omega(I_{\mathcal{A}})}$ of the initial ideal. This argument extends to any other nonface of the triangulation. Since

$$\Bigl\langle \prod_{b\in\mathcal{B}} z_b \mid \mathcal{B} \text{ is not a face of } \mathcal{S}_\omega\Bigr\rangle \;=\; \bigcap_{\mathcal{F}} \langle z_a \;\mid\; a\notin\mathcal{F}\rangle\,,$$

this completes the proof.

By Corollary 4.13, the degree of the initial scheme is the sum of contributions from each coordinate subspace $\mathbb{P}^{\mathcal{F}}$ of a facet $\mathcal{F}$ of the triangulation,

$$\deg(\mathrm{in}_\omega(X_{\mathcal{A}})) \;=\; \sum_{\mathcal{F}} \mathrm{mult}_{\mathbb{P}^{\mathcal{F}}}(\mathrm{in}_\omega(X_{\mathcal{A}}))\,,$$

where $\mathrm{mult}_{\mathbb{P}^{\mathcal{F}}}(\mathrm{in}_\omega(X_{\mathcal{A}}))$ is the algebraic multiplicity of $\mathrm{in}_\omega(X_{\mathcal{A}})$ along $\mathbb{P}^{\mathcal{F}}$.

In [**154**, Chapter 8], and under the (mild) assumption that $\mathcal{A}$ is primitive ($\mathbb{Z}\mathcal{A} = \mathbb{Z}^n$), Sturmfels shows that this multiplicity is $n!\,\mathrm{volume}(\Delta_{\mathcal{F}})$. Since these facets cover $\Delta_{\mathcal{A}}$, the degree of the limit scheme $\mathrm{in}_\omega(X_{\mathcal{A}})$ is $n!\,\mathrm{volume}(\Delta_{\mathcal{A}})$. As the family $\mathcal{X}_{\mathcal{A}}$ is flat, this is the degree of $X_{\mathcal{A}}$, which implies Kushnirenko's Theorem.

Sturmfels' result enables the determination of the limit scheme in an important special case. A triangulation of a polytope in $\mathbb{R}^n$ is *unimodular* is every facet has minimal volume $1/n!$. Unimodular triangulations necessarily involve all of the points of $\mathcal{A}$, with $\mathcal{F} = \Delta_{\mathcal{F}}\cap\mathcal{A}$ for every face $\mathcal{F}$, so there is essentially no difference between the polyhedral subdivision Δ_ω and the subdivision $\mathcal{S}_\omega$.

COROLLARY 4.14. *Suppose that* Δ_ω *is a regular unimodular triangulation. Then the limit scheme of the corresponding toric degeneration is a union of coordinate* n*-planes, one for every facet* $\mathcal{F}$ *of* $\mathcal{S}_\omega$,

$$\mathrm{in}_\omega(X_{\mathcal{A}}) \;=\; \lim_{t\to 0} t^{-1}.X_{\mathcal{A}} \;=\; \bigcup_{\mathcal{F}} \mathbb{P}^{\mathcal{F}}\,.$$

Example 4.15. Consider the cubic hypersurface $z_a z_b z_d - z_c^3$ and the function ω of Example 4.6. Figure 4.3 shows the torus translates $t^{-1}.X_{\mathcal{A}}$ for $t = 1$ and $t = 1/7$ in the affine chart of $\mathbb{P}^{\mathcal{A}}$ where $z_c \neq 0$ (where $z_c = 1$) and in the box $[-2.2, 2.2]^3$, viewed from the $+-+$-orthant. This illustrates Corollary 4.14 as $t^{-1}.X_{\mathcal{A}}$ visibly

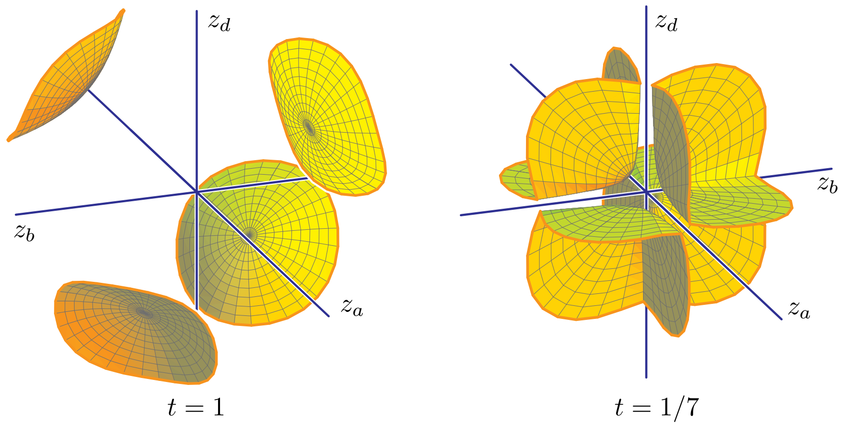


Figure 4.3. Toric degenerations of cubic surface

approaches the union of the coordinate planes as $t \to 0$, which is the variety of the initial ideal $\mathrm{in}_\omega(I_{\mathcal{A}}) = \langle z_a z_b z_d \rangle$.

Proof of Kushnirenko's Theorem. This is adapted from [**138**] and does not assume unimodularity. When $\mathcal{A}$ is primitive, it also determines that the algebraic multiplicity of $\mathrm{in}_\omega X_{\mathcal{A}}$ along $P^{\mathcal{F}}$ is $n!\,\mathrm{volume}(\Delta_{\mathcal{F}})$. This proof only works over $\mathbb{C}$ as it uses metric properties of $\mathbb{C}$.

The main idea is to fix a general linear subspace L of codimension n in $\mathbb{P}^{\mathcal{A}}$ and consider the linear sections $L \cap t^{-1}.X_{\mathcal{A}}$ for t near 0 as in Figure 4.4. The subspace L

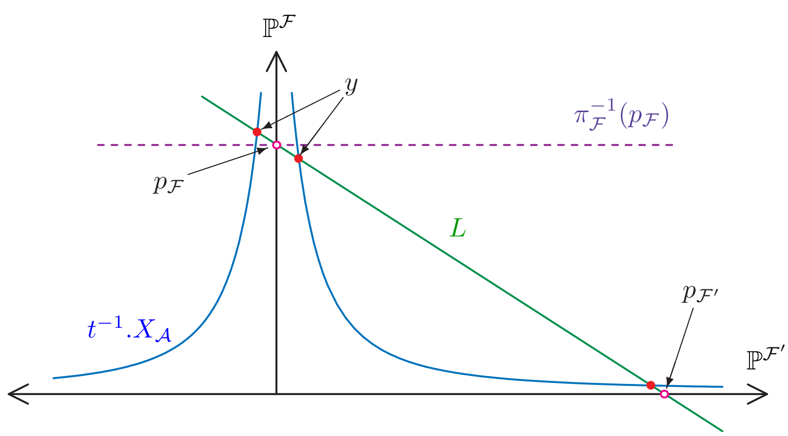


Figure 4.4. Points in $L \cap t^{-1}.X_{\mathcal{A}}$ near $\mathbb{P}^{\mathcal{F}}$ for small t.

will meet each facet n-plane $\mathbb{P}^{\mathcal{F}}$ in a single point $p_{\mathcal{F}}$, and the points y of $L \cap t^{-1}.X_{\mathcal{A}}$ for t small will be clustered near the different points $p_{\mathcal{F}}$.

We could determine the number of points y clustered near one of the points $p_{\mathcal{F}}$, which is the algebraic multiplicity $\mathrm{mult}_{\mathbb{P}^{\mathcal{F}}}(\mathrm{in}_\omega(X_{\mathcal{A}}))$. It is in fact easier to determine the total number of points in $\mathbb{T}^n$ of the form $\varphi_{\mathcal{A}}^{-1}(t^{-1}.y)$, for y some point of L near a particular $p_{\mathcal{F}}$ when t is small, and then sum over all facets $\mathcal{F}$. This is more direct, and it bypasses computing this algebraic multiplicity. This is also where we avoid the assumption of primitivity, but must work over $\mathbb{C}$.

In a neighborhood of $\mathbb{P}^{\mathcal{F}}$ the linear space L is isotopic to $\pi_{\mathcal{F}}^{-1}(p_{\mathcal{F}})$, which is a fiber of the coordinate projection $\pi_{\mathcal{F}}\colon \mathbb{P}^{\mathcal{A}} -\to \mathbb{P}^{\mathcal{F}}$. In Figure 4.4, this isotopy amounts to rotating the line L about $p_{\mathcal{F}}$ until it becomes horizontal. Thus the number of points in $\mathbb{T}^n$ coming from points in the linear section $L\cap t^{-1}.X_{\mathcal{A}}$ near $p_{\mathcal{F}}$ is equal to the number of points in $\mathbb{T}^n$ coming from points in the linear section $\pi_{\mathcal{F}}^{-1}(p_{\mathcal{F}})\cap t^{-1}.X_{\mathcal{A}}$ near $p_{\mathcal{F}}$. But every point in this second linear section is near $p_{\mathcal{F}}$, so this number is simply the degree of the map which is the composition of the parameterization $\varphi_{\mathcal{A}}$ of $X_{\mathcal{A}}$, the map $y\mapsto t^{-1}.y$ on $\mathbb{P}^{\mathcal{A}}$, and this projection $\pi_{\mathcal{F}}$,

$$\mathbb{T}^n \xrightarrow{\varphi_{\mathcal{A}}} X_{\mathcal{A}} \xrightarrow{t} t^{-1}.X_{\mathcal{A}} \xrightarrow{\pi_{\mathcal{F}}} \mathbb{P}^{\mathcal{F}}\,.$$

Since multiplication by t is isotopic to the identity and it commutes with the projection $\pi_{\mathcal{F}}$, we may assume now that $t=1$, and so this composition is just the parameterization $\varphi_{\mathcal{F}}$ of $\mathbb{P}^{\mathcal{F}}$ by the monomials corresponding to integer points of $\mathcal{F}$. The degree is the order of the kernel of $\varphi_{\mathcal{F}}$, which is $n!\,\mathrm{volume}(\Delta_{\mathcal{F}})$. Summing this quantity over all facets $\mathcal{F}$ of the triangulation Δ_ω shows that there are

$$\sum_{\mathcal{F}} n!\,\mathrm{volume}(\Delta_{\mathcal{F}}) \;=\; n!\,\mathrm{volume}(\Delta_{\mathcal{A}})$$

points in $\mathbb{T}^n$ which are pullbacks under $\varphi_{\mathcal{A}}$ of the linear section $L\cap X_{\mathcal{A}}$. This completes our proof of Kushnirenko's Theorem via toric degenerations.

This proof is algorithmic in that it (more-or-less) counts the solutions to the system $L\cap t^{-1}.X_{\mathcal{A}}$, for t small, while also giving enough information on their location and structure to determine them numerically. This proof also shows that the intersection $L\cap X_{\mathcal{A}}$ is transverse when L is general, and thus gives a proof of Bertini's Theorem in this context. To see this, note that when t is small, the intersection near $p_{\mathcal{F}}$ may be deformed to the intersection of $X_{\mathcal{A}}$ with the horizontal subspaces $\pi_{\mathcal{F}}^{-1}(p_{\mathcal{F}})$, and this is deformed to the system $\varphi_{\mathcal{F}}^{-1}(p_{\mathcal{F}})$, which consists of $n!\,\mathrm{volume}(\Delta_{\mathcal{F}})$ distinct points, so the general such intersection is transverse.

4.4. Polynomial systems with only real solutions

In the proof of Kushnirenko's Theorem in Section 4.3, we studied the toric degeneration for t small, using the arithmetic and geometry of its limit scheme to compute the number of points in $\varphi_{\mathcal{A}}^{-1}(t.L\cap X_{\mathcal{A}}))=\varphi_{\mathcal{A}}^{-1}(t.(L\cap t^{-1}.X_{\mathcal{A}}))$. This method, using the structure of the limit scheme to obtain information about $X_{\mathcal{A}}$ has a long pedigree in constructions in real algebraic geometry. We give one example which is a result of Sturmfels about the existence of real solutions to systems of equations.

If the triangulation Δ_ω is unimodular, in that each facet has minimal volume $1/n!$, then near each point $p_{\mathcal{F}}$ there will be exactly one point of $L\cap t^{-1}.X_{\mathcal{A}}$ and one corresponding solution in $\mathbb{T}^n$. If both L and t are real, then each $p_{\mathcal{F}}$ and each nearby point in $t^{-1}.X_{\mathcal{A}}$ will be real. Since

$$t.(L\cap t^{-1}.X_{\mathcal{A}}) \;=\; \big(t.L\big)\cap X_{\mathcal{A}}\,,$$

and the points in the left hand side are all real, so are the points in the right hand side. This right hand side corresponds to a system of real polynomials with support $\mathcal{A}$. This proves a theorem of Sturmfels [**153**], and gives his argument in a nutshell.

THEOREM 4.16. *If a lattice polytope $\Delta \subset \mathbb{Z}^n$ admits a regular unimodular triangulation, then there exist real polynomial systems with support $\Delta \cap \mathbb{Z}^n$ having all solutions real.*

A more careful analysis, which begins by examining real solutions when $|\mathcal{A}| = n+1$, leads to the more refined result for not necessarily unimodular triangulations that appears in Sturmfels's paper.

CHAPTER 5

Fewnomial Upper Bounds

The oldest—by far—nontrivial bound on the number of real solutions to a system of polynomials is Descartes' bound (Corollary 2.2) for univariate polynomials. That is, a univariate polynomial

$$c_0x^{a_0} + c_1x^{a_1} + \cdots + c_mx^{a_m}$$

with $m{+}1$ terms has at most m positive roots. This bound is tight as the polynomial

$$(x-1)(x-2)\cdots(x-m) \tag{5.1}$$

has $m+1$ distinct terms and m positive roots.

This chapter and the next will discuss fewnomial bounds, which are multivariate generalizations of Descartes' bound for systems of multivariate polynomials. Unlike Descartes' bound, these fewnomial bounds are not sharp and it is an important problem to improve our understanding of them. There is also no satisfactory generalization of Descartes' rule of signs to multivariate polynomials.

5.1. Khovanskii's fewnomial bound

An optimistic reading of Descartes's bound suggests that the number of real solutions to a system of polynomials depends not on the degree of the system, but rather on the complexity of its description. Bernstein and Kushnirenko formulated the principle that the topological complexity of a set in $\mathbb{R}^n$ defined by real polynomials is controlled by the complexity of the description of the polynomials, rather than by their degrees or Newton polytopes. This is exactly what Khovanskii proved for systems of equations in 1980 with his celebrated *fewnomial bound*.

Theorem 5.1 (Khovanskii [**83**]). *A system of n real polynomials in n variables involving $1{+}l{+}n$ distinct monomials has fewer than*

$$2^{\binom{l+n}{2}} \cdot (n+1)^{l+n} \tag{5.2}$$

nondegenerate positive solutions.

A solution $x \in \mathbb{R}^n$ to a system of polynomials is *nondegenerate* if the differentials of the polynomials at x span $\mathbb{R}^n$. Nondegenerate solutions are isolated, occur with multiplicity 1, and there are finitely many of them. This bound, like other bounds in this part of the subject, considers solutions in the *positive orthant* $\mathbb{R}^n_> := \{x \in \mathbb{R}^n \mid x_i > 0,\ i = 1, \ldots, n\}$. The existence of a bound that is independent of the degrees of the polynomials was revolutionary and is the main point of Khovanskii's result.

A consequence of Khovanskii's fewnomial bound is that for each positive integer l and n, there is a number $X(l, n)$ which is equal to the maximum number of positive solutions to a system of n polynomials in n variables having $1{+}l{+}n$ distinct monomials. A central question in this area is to determine this *Khovanskii number* $X(l,n)$ exactly, or give good bounds. Khovanskii's Theorem shows that $X(l,n)$ is bounded above by the quantity (5.2).

We also have the bound of $2^n X(l,n)$ for the number of nondegenerate nonzero real solutions to a system of n polynomials in n variables having $1+l+n$ distinct monomials. Given a system, replacing each variable by its square creates a new system with one real solution in each orthant for each positive solution to the original system. This occurs because the kernel $\ker \varphi_{\mathcal{A}}$ of the parameterization map given by the exponents $\mathcal{A}$ of the system of polynomials contains the real roots of unity $\{\pm 1\}^n$. More interesting is $X_{\mathbb{R}}(l,n)$, which is the maximum number of nondegenerate nonzero real solutions to a system of n polynomials in n variables having $1+l+n$ distinct monomials, where the exponents $\mathcal{A}$ of the monomials affinely span $\mathbb{Z}^n$ (are primitive) or span a sublattice of odd index, so that $\ker(\varphi_{\mathcal{A}}) \cap \mathbb{R}^n = \{1\}$. We have $X(l,n) \leq X_{\mathbb{R}}(l,n) \leq 2^n X(l,n)$, and the problem is to find good bounds for $X_{\mathbb{R}}(l,n)$. In particular, is $X_{\mathbb{R}}(l,n)$ closer to $X(l,n)$ or to $2^n X(l,n)$?

A complete proof of Theorem 5.1 may be found in Khovanskii's book [**84**], where much else is also developed. Chapter 1 of that book contains an accessible sketch. Benedetti and Risler [**9**, §4.1] have a careful and self-contained exposition of Khovanskii's fewnomial bound. We give a sketch of the main ideas in the exposition of Benedetti and Risler, to which we refer for further details (this is also faithful to Khovanskii's sketch). We remark that Sturmfels has also sketched ([**156**, pp. 39–40] and in [**155**]) a version of the proof. This omits some contributions to the root count and is therefore regrettably incorrect.

Khovanskii looks for solutions in the positive orthant $\mathbb{R}^n_>$, proving a far more general result involving solutions in $\mathbb{R}^n$ of polynomial functions in logarithms of the coordinates and monomials. Set

$$z_i \;:=\; \log(x_i) \qquad \text{and} \qquad y_j \;:=\; e^{z\cdot a_j} \;=\; x^{a_j}\,, \tag{5.3}$$

where $i = 1,\ldots,n$, and the exponents $a_j \in \mathbb{R}^n$ for $j = 1,\ldots,k$, can be real. Consider a system of functions of the form

$$F_i(z_1,\ldots,z_n,\, y_1,\ldots,y_k) \;=\; 0 \qquad i=1,\ldots,n\,, \tag{5.4}$$

where each $y_j = y_j(z)$ is an exponential function $e^{z\cdot a_j}$ and the F_i are polynomials in $n+k$ indeterminates.

Theorem 5.2 (Khovanskii's Theorem). *The number of nondegenerate real solutions to the system* (5.4) *is at most*

$$\Bigl(\prod_{i=1}^{n} \deg F_i\Bigr) \cdot \Bigl(1 + \sum_{i=1}^{n} \deg F_i\Bigr)^{k} \cdot 2^{\binom{k}{2}}\,, \tag{5.5}$$

and strictly less than this number if $k > 0$.

Proof of Theorem 5.1. Given a system of n real polynomials in n variables involving $1+l+n$ distinct monomials, we may assume that one of the monomials is 1. Under the substitution (5.3), this becomes a system of the form (5.4), where each F_i is a degree one polynomial whose z_i coefficients are zero and which involves $k = l+n$ exponential functions y_j. Then $\deg F_i = 1$ and the bound (5.5) reduces to (5.2). ∎

Sketch of proof of Theorem 5.2. We proceed by induction on k, skipping important technicalities involving Sard's Theorem. When $k = 0$, there are no exponential functions, and the system is just a system of n polynomials in n variables, whose number of nondegenerate isolated solutions is bounded above by the Bézout number,

$$\prod_{i=1}^{n} \deg F_i \,,$$

which is the bound (5.5) when $k = 0$.

Suppose that we have the bound (5.5) for systems of the form (5.4) with k exponential functions, and consider a system with $k+1$ exponential functions,

$$F_i(z_1, \ldots, z_n,\ y_1, \ldots, y_k,\ y_{k+1}) \;=\; 0 \qquad i = 1, \ldots, n\,.$$

Consider the (equivalent) new system with one added variable t.

$$\begin{aligned} G_i(z,t) \;:=\; F_i(z_1, \ldots, z_n,\ y_1, \ldots, y_k,\ t\cdot y_{k+1}) \;&=\; 0 \qquad i = 1, \ldots, n \\ t \;&=\; 1 \end{aligned} \tag{5.6}$$

The subsystem (5.6) defines an analytic curve C in $\mathbb{R}^{n+1}$, which we assume is smooth and transverse to the hyperplane at $t = 1$.

Write z_{n+1} for t and consider the vector field ξ in $\mathbb{R}^{n+1}$ with rth component

$$\xi_r \;:=\; (-1)^{n+1-r} \det \left(\frac{\partial G_i}{\partial z_j} \right)_{j=1,\ldots,\hat{r},\ldots,n+1}^{i=1,\ldots,n} . \tag{5.7}$$

This vector field is tangent to the curve C, and we write $\xi_t = \xi_{n+1}$ for its component in the t-direction. An important ingredient in the proof of Theorem 5.2 is a special case of the Khovanskii-Rolle Theorem [**84**, pp. 42–51].

Theorem 5.3 (Khovanskii-Rolle Theorem). *The number of points of C where $t = 1$ is bounded above by*

$$N \;+\; q\,,$$

where N is the number of points of C where $\xi_t = 0$ and q is the number of unbounded components of C.

Proof. Note that ξ_t varies continuously along C. Suppose that a and b are consecutive points along an arc of C where $t = 1$. Since C is transverse to the hyperplane $t = 1$, we have $\xi_t(a) \cdot \xi_t(b) < 0$, and so there is a point c of C on the arc between a and b with $\xi_t(c) = 0$.

The hyperplane $t = 1$ cuts a compact connected component of C into the same number of arcs as points where $t = 1$. Since the endpoints of each arc lie on the hyperplane $t = 1$, there is at least one point c on each arc with $\xi_t(c) = 0$. Similarly, the hyperplane $t = 1$ cuts a noncompact component into arcs, and each arc with two endpoints in the hyperplane $t = 1$ gives a point c with $\xi_t(c) = 0$. However, there will be one more point with $t = 1$ on this component than such arcs.

Figure 5.1 illustrates the argument in the proof.

The key to the induction in the proof of Khovanskii's Theorem is to replace the last exponential function by a new variable. This substitution is omitted in

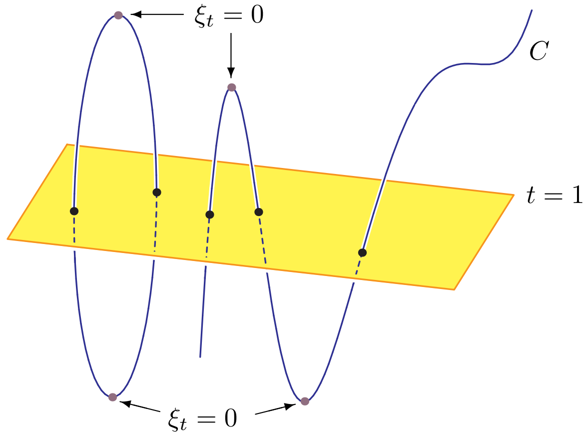


FIGURE 5.1. Idea of the Khovanskii-Rolle Theorem.

Sturmfels's argument, which also does not use the Khovanskii-Rolle Theorem 5.3. Recall that $y_j = e^{z\cdot a_j}$, and so

$$\begin{aligned}\frac{\partial G_i}{\partial z_r} \;&=\; \frac{\partial F_i}{\partial z_r}(z_1,\ldots,z_n,\ y_1,\ldots,y_k, t\cdot y_{k+1})\\ &\quad + \sum_{j=1}^{k}\frac{\partial F_i}{\partial y_j}(z_1,\ldots,z_n,\ y_1,\ldots,y_k, t\cdot y_{k+1})\cdot a_{j,r}\, y_j\\ &\quad + \frac{\partial F_i}{\partial y_{k+1}}(z_1,\ldots,z_n,\ y_1,\ldots,y_k, t\cdot y_{k+1})\cdot a_{k+1,r}\ t\, y_{k+1}\,.\end{aligned}$$

If we set $u := t\cdot y_{k+1}$ and define $\phi_t(z,u)$ to be the expression for $\xi_t = \xi_{n+1}$ (5.7) considered as a function of z and u, then the total degree (in $z_1,\ldots,z_n,y_1,\ldots,y_k,u$) of $\phi_t(z,u)$ is at most $\sum_{i=1}^n \deg F_i$. Since $y_{k+1} = e^{z\cdot a_{k+1}}$ is never zero, this coordinate change $(z,t)\to(z,u)$ is a homeomorphism on $\mathbb{R}^{n+1}$ and therefore does not change the topology of the curve C. That is, its numbers of bounded and unbounded components are preserved by this transformation.

This number N of Theorem 5.3 is the number of solutions to the system

$$\begin{aligned} F_i(z_1,\ldots,z_n,\ y_1,\ldots,y_k,\ u) \;&=\; 0 \qquad i=1,\ldots,n\\ \phi_t(z,u)\;&=\;0\,.\end{aligned}\tag{5.8}$$

This has the form (5.4) with k exponential functions. Given any solution to the system (5.8), we use the substitution $u = t\cdot y_{k+1}$ to solve for t and get a corresponding point c on the curve C with $\xi_t(c)=0$. We apply our induction hypothesis to the system (5.8) (which has k exponential functions and $n{+}1$ equations in $n{+}1$ variables) to obtain

$$N \;\leq\; \prod_{i=1}^n \deg F_i\cdot\Big(\sum_{i=1}^n \deg F_i\Big)\cdot\Big(1+2\sum_{i=1}^n\deg F_i\Big)^k\cdot 2^{\binom{k}{2}}\,.$$

We similarly estimate the number q of noncompact components of C. We claim that this is bounded above by the maximum number of points of intersection of C with a hyperplane. Indeed, since each noncompact component has two infinite branches, there are $2q$ points (counted with multiplicity) on the sphere $\mathbb{S}^n$ corresponding to directions of accumulation points of the branches of C at infinity. Any hyperplane through the origin not meeting these points will have at least q of these points in one of the hemispheres into which it divides the sphere. If we translate this hyperplane sufficiently far toward infinity, it will meet the branches giving these accumulation points, and thus will meet C in at least q points.

Thus q is bounded by the number of solutions to a system of the form

$$
\begin{aligned}
F_i(z_1,\ldots,z_n,\ y_1,\ldots,y_k,\ u) &= 0 \qquad i=1,\ldots,n \\
l_0 + l_1 z_1 + l_2 z_2 + \cdots + l_n z_n + l_u u &= 0\,,
\end{aligned}
\tag{5.9}
$$

where $l_0,\ldots,l_n,\ l_u$ are real numbers, as the second equation defines a hyperplane in our original $\mathbb{R}^{n+1}$ when $l_u = 0$. This again involves only k exponential functions, and the last equation has degree 1, so our induction hypothesis gives

$$q \;\leq\; \prod_{i=1}^{n} \deg F_i \cdot 1 \cdot \Bigl(1 + \sum_{i=1}^{n} \deg F_i \;+1\Bigr)^k \cdot 2^{\binom{k}{2}}\,.$$

Combining these estimates bounds $N{+}q$ by the expression

$$\prod_{i=1}^{n} \deg F_i \cdot 2^{\binom{k}{2}} \cdot \Bigl[\Bigl(\sum_{i=1}^{n} \deg F_i\Bigr) \cdot \Bigl(1 + 2\sum_{i=1}^{n} \deg F_i\Bigr)^k \;+\; \Bigl(2 + \sum_{i=1}^{n} \deg F_i\Bigr)^k\Bigr]\,.$$

This is complicated and not amenable to induction. We obtain a larger estimate that bounds $N+q$ by the number of solutions to the single system of equations,

$$
\begin{aligned}
F_i(z_1,\ldots,z_n,\ y_1,\ldots,y_k,\ u) &= 0 \qquad i=1,\ldots,n \\
F_{n+1} \;:=\; \phi_t(z,u)\cdot(l_0 + l_1 z_1 + l_2 z_2 + \cdots + l_n z_n + l_u u) &= 0\,.
\end{aligned}
$$

By our induction hypothesis, this gives

$$N + q \;\leq\; \prod_{i=1}^{n+1} \deg F_i \cdot \Bigl(1 + \sum_{i=1}^{n+1} \deg F_i\Bigr)^k \cdot 2^{\binom{k}{2}}\,.$$

But F_{n+1} has degree at most $1 + \sum_{i=1}^{n} \deg F_i$, and so the number of solutions to the system with $k+1$ exponential functions is bounded by $N+q$ which is at most

$$
\begin{aligned}
&\prod_{i=1}^{n} \deg F_i \cdot \Bigl(1 + \sum_{i=1}^{n} \deg F_i\Bigr) \cdot \Bigl(1 + \sum_{i=1}^{n} \deg F_i + 1 + \sum_{i=1}^{n} \deg F_i\Bigr)^k \cdot 2^{\binom{k}{2}} \\
&\qquad\qquad = \prod_{i=1}^{n} \deg F_i \cdot \Bigl(1 + \sum_{i=1}^{n} \deg F_i\Bigr)^{k+1} \cdot 2^{\binom{k+1}{2}}\,.
\end{aligned}
$$

This completes the proof of Theorem 5.2 and shows that if $k > 0$, then the bound is not sharp.

We see that the result of Theorem 5.2 is much more general than the statement of Theorem 5.1. Also, the bound is not sharp. While no one believed that Khovanskii's bound (5.2) was anywhere near the actual upper bound $X(l,n)$, it was extremely hard to improve it. We discuss the first steps in this direction.

5.2. Kushnirenko's Conjecture

One of the first proposals of a more reasonable bound than Khovanskii's for the number of positive solutions to a system of polynomials is attributed to Kushnirenko, and for many years experts believed that this conjecture may indeed be the truth.

CONJECTURE 5.4 (Kushnirenko). *A system $f_1 = f_2 = \cdots = f_n = 0$ of real polynomials where each f_i has m_i+1 terms has at most $m_1 m_2 \cdots m_n$ nondegenerate positive solutions.*

This generalizes the bound given by Descartes's rule of signs. It easy to use the example (5.1) for the sharpness of Descartes's rule to construct systems of the form

$$f_1(x_1) \;=\; f_2(x_2) \;=\; \cdots \;=\; f_n(x_n) \;=\; 0\,,$$

which achieve the bound of Conjecture 5.4.

Soon after Kushnirenko made this conjecture, Sevostyanov found a counterexample which was unfortunately lost. Nevertheless, this conjecture passed into folklore until Haas [**64**] found an example of two trinomials $(3 = 2+1)$ in variables x and y with $5\,(> 4 = 2\cdot 2)$ isolated nondegenerate positive solutions.

$$10x^{106} \;+\; 11y^{53} \;-\; 11y \;=\; 10y^{106} \;+\; 11x^{53} \;-\; 11x \;=\; 0\,. \tag{5.10}$$

There have been other attempts to find better bounds than the Khovanskii bound. Sturmfels [**153**] used a more sophisticated version of the toric degenerations introduced in Chapter 3 to show how to construct systems with many real roots (the root count depends upon a mixture the geometry of the Newton polytopes and some combinatorics of signs associated to lattice points). This inspired Itenberg and Roy [**76**] to propose a multivariate version of Descartes's rule of signs, which was later found to be too optimistic [**95**]. An interesting part of this story is told in the cheeky paper of Lagarias and Richardson [**90**].

More recently, Li, Rojas, and Wang considered Haas's counterexample to Kushnirenko's Conjecture, seeking to obtain realistic bounds for the number of positive solutions which depended only on the number of monomials in the different polynomials. For example, they showed that Haas's counterexample was the best possible.

THEOREM 5.5 (Li, Rojas, and Wang [**94**]). *A system consisting of two trinomials in two variables has at most five nondegenerate positive solutions.*

Dickenstein, Rojas, Rusek, and Shih [**35**] used exact formulas for $\mathcal{A}$-discriminants [**36**] to study systems of two trinomials in two variables which achieve this bound of five positive solutions. They gave the following example and analysis, which indicates how difficult it is to find systems with many real solutions.

EXAMPLE 5.6. Consider the family of systems of bivariate sextics,

$$x^6 \;+\; ay^3 \;-\; y \;=\; y^6 \;+\; bx^3 \;-\; x \;=\; 0\,, \tag{5.11}$$

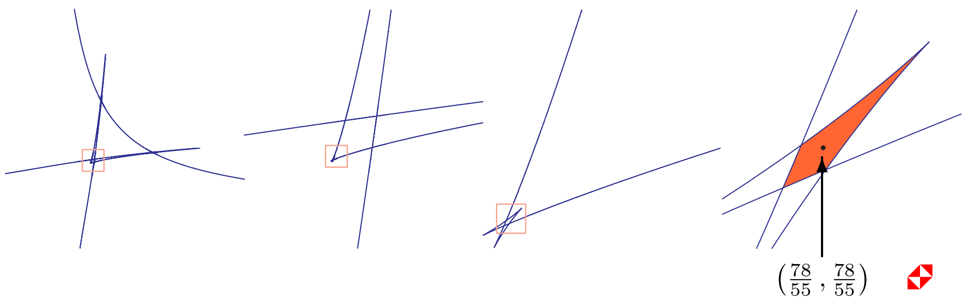
where a, b are real numbers. When $a = b = 78/55$, this has five positive real solutions

$$(0.814, 0.607)\,,\;\; (0.789, 0.673)\,,\;\; (0.740, 0.740)\,,\;\; (0.673, 0.789)\,,\;\; (0.607, 0.814)\,.$$

We now investigate the set of parameters (a, b) for which this achieves the trinomial bound of five positive solutions. This turns out to be a single connected component in the complement of the discriminant for this family of systems. This discriminant is a polynomial of degree 90 in a, b with 58 terms whose leading and trailing terms are

$$1816274895843482705708030487016037960921088a^{45}b^{45} \ + \ \cdots \ 56 \text{ terms } \cdots \ + \\ 110250749935414869595178643341350834816694259643554687 5.$$

We display this discriminant in the square $[0, 4] \times [0, 4]$, as well as three successive magnifications, each by a factor of 11. The shaded region in the last picture is the set of pairs (a, b) for which (5.11) achieves the trinomial bound of five positive real solutions.



To compare the trinomial bound in [**94**] to the fewnomial bound (5.2), note that we may multiply one of the polynomials by a monomial so that the two trinomials share a monomial. Then there are at most five distinct monomials occurring in the two trinomials. As $n = 2$, we then also have $l = 2$. The fewnomial bound for $l = n = 2$ is

$$X(2,2) \ \leq \ 2^{\binom{2+2}{2}} \cdot (2+1)^{2+2} \ = \ 5184\,.$$

We remark that a trinomial system such as (5.10) or (5.11) is not quite a general fewnomial system with $l = n = 2$. Still, the bound of five real solutions lent credence to the belief $X(2, 2)$ is closer to five than to 5184 and that Khovanskii's fewnomial bound (5.2) could be improved.

In addition to providing a counterexample to Kushnirenko's Conjecture, Sevostyanov also established the first result of fewnomial-type. He showed the existence of an absolute bound $c(d, m)$ for the number of real solutions to a system

$$f(x, y) \ = \ g(x, y) \ = \ 0\,,$$

where f is a polynomial of degree d and g has m terms. The proof of this result, like his counterexample, has unfortunately been lost. This result however, was the inspiration for Khovanskii to develop his theory of fewnomials.

Avendaño [**2**] established a precise version of a special case of Sevostyanov's theorem.

Theorem 5.7. *Suppose that $f(x, y)$ is linear and $g(x, y)$ has m terms. Then*

$$f(x, y) \ = \ g(x, y) = \ 0\,,$$

has at most $6m - 4$ real solutions.

5.3. Systems supported on a circuit

Restricting the analysis of Section 4.1 to real solutions shows that $X(0,n)=1$. Bihan [**15**] showed that $X(1,n)=1+n$. We discuss this here.

A collection $\mathcal{A}$ of $n+2$ vectors in $\mathbb{Z}^n$ which affinely spans $\mathbb{R}^n$ is called a *circuit*. The circuit is *primitive* if its $\mathbb{Z}$-affine span is all of $\mathbb{Z}^n$, so that $\varphi_{\mathcal{A}}$ is injective and there are no trivial extra zeroes coming from $\ker(\varphi_{\mathcal{A}})\cap\mathbb{T}^n_{\mathbb{R}}\subset\{\pm1\}^n$. When $0\in\mathcal{A}$, this means that $\mathbb{Z}^n=\mathbb{Z}\mathcal{A}$. (We could also assume that the affine span has odd index in $\mathbb{Z}^n$.)

THEOREM 5.8 (Bertrand, et al. [**13**]). *A polynomial system supported on a primitive circuit has at most* $2n+1$ *nondegenerate nonzero real solutions.*

THEOREM 5.9 (Bihan [**15**]). *A polynomial system supported on a circuit has at most* $n+1$ *nondegenerate positive solutions, and there exist systems supported on a circuit having* $n+1$ *positive solutions.*

This can be used to construct fewnomial systems with relatively many positive solutions.

COROLLARY 5.10 (Bihan, et al. [**16**]). *There exist systems of* n *polynomials in* n *variables having* $1+l+n$ *monomials and at least* $\lfloor\frac{l+n}{l}\rfloor^l$ *positive solutions.*

This gives a lower bound for $X(l,n)$ of $\lfloor\frac{l+n}{l}\rfloor^l$, and is the best construction when l is fixed and n is large. It remains an open problem to give constructions with more solutions, or constructions with many solutions when l is not fixed.

Proof. Suppose that $n=kl$ is a multiple of l, and let

$$f_1(x_1,\ldots,x_k) \;=\; f_2(x_1,\ldots,x_k) \;=\; \cdots \;=\; f_k(x_1,\ldots,x_k) \;=\; 0\,,$$

be a system with $k+2$ monomials and $k+1$ positive solutions. Such systems exist, by Theorem 5.9. Write $F(x)=0$ for this system and assume that one of its monomials is a constant. For each $i=1,\ldots,l$, let $y^{(i)}=(y^{(i)}_1,\ldots,y^{(i)}_k)$ be a set of k variables. Then the system

$$F(y^{(1)}) \;=\; F(y^{(2)}) \;=\; \cdots \;=\; F(y^{(l)}) \;=\; 0\,,$$

has $(k+1)^l$ solutions, kl variables, and $1+l+kl$ monomials. When $n=kl+r$ with $r<l$, adding extra variables y_i and equations $y_i=1$ for $i=1,\ldots,r$ gives a system with $\lfloor\frac{l+n}{l}\rfloor^l$ positive solutions and $1+l+n$ monomials. ∎

When $l=1$ the fewnomial bound (5.2) becomes

$$2^{\binom{1+n}{2}}\cdot(n+1)^{1+n}\,,$$

which is considerably larger than Bihan's bound of $n+1$. Replacing $l+n$ by n in the fewnomial bound, it becomes equal to Bihan's bound when $l=1$. When $l=0$, this same substitution in (5.2) yields 1, which is the sharp bound when $l=0$. This suggests that this substitution should yield a correct bound. Indeed that is nearly the case. In Chapter 6, we will outline generalizations of Theorems 5.8 and 5.9 to arbitrary l, giving the bound

$$X(l,n) \;<\; \frac{e^2+3}{4}2^{\binom{l}{2}}n^l \tag{5.12}$$

for positive solutions, and when $\mathcal{A}$ is primitive, a bound for all real solutions,

$$\frac{e^4+3}{4} 2^{\binom{l}{2}} n^l \,.$$

This is only slightly larger than (5.12)—the difference is in the exponents 2 and 4 of e. These bounds are proven in [**5, 17**]. It is instructive to look at the bound (5.12) when $n = l = 2$, which is $\frac{e^2+3}{4}2^{\binom{2}{2}}2^2 \approx 20.78$. Thus $X(2,2) \leq 20$, which is considerably less than 5184.

By Corollary 5.10 and the bound (5.12),

$$l^{-l}n^l \;<\; \left\lfloor \frac{l+n}{l} \right\rfloor^l \;<\; X(n,l) \;<\; \frac{e^2+3}{4} 2^{\binom{l}{2}} n^l \,.$$

This reveals the correct asymptotic information for $X(n,l)$ when l is fixed, namely $X(n,l) = \Theta(n^l)$.

Theorems 5.8 and 5.9 are related and we will outline their proofs following the papers in which they appear, where more details may be found. To begin, let

$$f_1(x_1,x_2,\ldots,x_n) \;=\; f_2(x_1,x_2,\ldots,x_n) \;=\; \cdots \;=\; f_n(x_1,x_2,\ldots,x_n) \;=\; 0 \tag{5.13}$$

be a system with support a circuit $\mathcal{A}$. Suppose that $0 \in \mathcal{A}$ and list the elements of the circuit $\mathcal{A} = \{0, a_0, a_1, \ldots, a_n\}$. After a multiplicative change of coordinates (if necessary), we may assume that $a_0 = \ell \mathbf{e}_n$, where $\mathbf{e}_n$ is the nth standard basis vector. Since the system (5.13) is generic, row operations on the equations put it into diagonal form

$$x^{a_i} \;=\; w_i \;+\; v_i x_n^{\ell} \qquad \text{for } i = 1,\ldots,n\,. \tag{5.14}$$

When $\mathcal{A}$ was a simplex we used integer linear algebra to reduce the equations to a very simple system in Section 4.1. We use (different) integer linear algebra to simplify this system supported on a circuit.

Suppose that $\{0, a_0, a_1, \ldots, a_n\} \subset \mathbb{Z}^n$ is a primitive circuit. We assume here that it is nondegenerate—there is no affine dependency involving a subset. (Bounds in the degenerate case are lower, replacing n by the size of this smaller circuit.) After possibly making a coordinate change, we may assume that $a_0 = \ell \cdot \mathbf{e}_n$, where $\mathbf{e}_n$ is the nth standard basis vector.

For each i, we may write $a_i = b_i + k_i \cdot \mathbf{e}_n$, where $b_i \in \mathbb{Z}^{n-1}$. Removing common factors from a nontrivial integer linear relation among the n vectors $\{b_1, \ldots, b_n\} \subset \mathbb{Z}^{n-1}$ gives us the *primitive relation* among them (which is well-defined up to multiplication by -1),

$$\sum_{i=1}^{p} \beta_i b_i \;=\; \sum_{i=p+1}^{n} \beta_i b_i \,.$$

Here, each $\beta_i > 0$, and we assume that the vectors are ordered so that the relation has this form. We further assume that

$$N \;:=\; \sum_{i=p+1}^{n} \beta_i k_i \;-\; \sum_{i=1}^{p} \beta_i k_i \;>\; 0\,.$$

Then we have

$$N\mathbf{e}_n \;+\; \sum_{i=1}^{p} \beta_i a_i \;-\; \sum_{i=p+1}^{n} \beta_i a_i \;=\; 0\,,$$

and so

$$x_n^N \cdot \prod_{i=1}^{p} (x^{a_i})^{\beta_i} \; - \; \prod_{i=p+1}^{n} (x^{a_i})^{\beta_i} \; = \; 0\,. \tag{5.15}$$

Using (5.14) to substitute for x^{a_i} in (5.15) gives the univariate consequence of (5.14)

$$f(x_n) \; := \; x_n^N \prod_{i=1}^{p} (w_i + v_i x_n^{\ell})^{\beta_i} \; - \; \prod_{i=p+1}^{n} (w_i + v_i x_n^{\ell})^{\beta_i} \; . \tag{5.16}$$

Some further arithmetic of circuits (which may be found in [**13**]) shows that $f(x_n)$ has degree equal to $n!\,\mathrm{volume}(\Delta_{\mathcal{A}})$. This is an eliminant of the system as in the Shape Lemma 2.8.

LEMMA 5.11. *The association of a solution x of* (5.14) *to its nth coordinate x_n gives a bijection between the solutions of* (5.14) *and the roots of f* (5.16) *which restricts to a bijection between their real solutions/roots.*

While f is the eliminant of the system, we do not have a Gröbner basis or even a triangular system to witness this fact, and the proof proceeds by explicitly constructing a solution to (5.14) from a root x_n of f.

For the upper bound, write $f = F - G$ as in (5.16) and then perturb it,

$$f_t(y) \; = \; t \cdot F(y) \; - \; G(y)\,.$$

We simply estimate the number of changes in the the real roots of f_t as t passes from $-\infty$ to 1, which can occur only at the singular roots of f_t. While similar to the proof of Khovanskii's theorem, this is not inductive, but relies on the form of the Wronskian $F'G - G'F$ whose roots are the singular roots of f_t. This may also be seen as an application of Rolle's Theorem. We note that this estimation also uses Viro's method for t near 0 and ∞.

These estimates prove the bounds in Theorems 5.8 and 5.9. Sharpness comes from a construction. In [**13**] Viro's method for univariate polynomials is used to construct polynomials f with $2n+1$ real solutions, for special primitive circuits. Bihan [**15**] constructs a system with $n+1$ positive solutions using Grothendieck's *dessins d'enfants*.

We close this chapter, giving a family of systems that illustrates the result of Theorem 5.8 (actually of an extension of it) and which may be treated by hand. These systems come from a family of polytopes $\Delta \subset \mathbb{Z}^n$ for which we prove a nontrivial upper bound on the number of real solutions to polynomial systems with primitive support $\mathcal{A} := \Delta \cap \mathbb{Z}^n$. That is, the integer points $\mathcal{A}$ in Δ affinely span $\mathbb{Z}^n$, so that general systems supported on Δ have $n!\,\mathrm{volume}(\Delta)$ complex solutions, but there are fewer than $n!\,\mathrm{volume}(\Delta)$ real solutions to polynomial systems with support $\mathcal{A}$. This is intended to not only give a glimpse of the more general results in [**13**], but also possible extensions which are not treated in [**17**]. In fact, such an extension has recently been found [**124**] (see Section 6.3).

Let $l > k > 0$ and $n \geq 3$ be integers and $\epsilon = (\epsilon_1, \ldots, \epsilon_{n-1}) \in \{0,1\}^{n-1}$ have at least one nonzero coordinate. The polytope $\Delta^{\epsilon}_{k,l} \subset \mathbb{R}^n$ is the convex hull of the points

$$(0,\ldots,0),\; (1,0,\ldots,0),\ldots,(0,\ldots,0,1,0),\; (0,\ldots,0,k),\; (\epsilon_1,\ldots,\epsilon_{n-1},l)\,.$$

The configuration $\mathcal{A}^{\epsilon}_{k,l} = \Delta^{\epsilon}_{k,l} \cap \mathbb{Z}^n$ also includes the points along the last axis

$$(0,\ldots,0,1),\ (0,\ldots,0,2),\ \ldots,\ (0,\ldots,0,k-1)\,.$$

These points include the standard basis and the origin, so $\mathcal{A}^{\epsilon}_{k,l}$ is primitive.

Set $|\epsilon| := \sum_i \epsilon_i$. Then the volume of $\Delta^{\epsilon}_{k,l}$ is $(l + k|\epsilon|)/n!$. Indeed, the configuration $\mathcal{A}^{\epsilon}_{k,l}$ can be triangulated into two simplices $\Delta^{\epsilon}_{k,l} \setminus \{(\epsilon_1,\ldots,\epsilon_{n-1},l)\}$ and $\Delta^{\epsilon}_{k,l} \setminus \{0\}$ with volumes $k/n!$ and $(l-k+k|\epsilon|)/n!$, respectively. One way to see this is to apply the affine transformation

$$(x_1,\ldots,x_n) \ \longmapsto\ (x_1,\ldots,x_{n-1},x_n - k + k\sum_{i=1}^{n-1} x_i)\,.$$

Theorem 5.12. *The number, r, of real solutions to a generic system of n real polynomials with support $\mathcal{A}^{\epsilon}_{k,l}$ lies in the interval*

$$0 \ \leq\ r\ \leq\ k + k|\epsilon| + 2\,,$$

and every number in this interval with the same parity as $l + k|\epsilon|$ occurs.

This upper bound does not depend on l and, since $k < l$, it is smaller than or equal to the number $l + k|\epsilon|$ of complex solutions. We use elimination to prove this result.

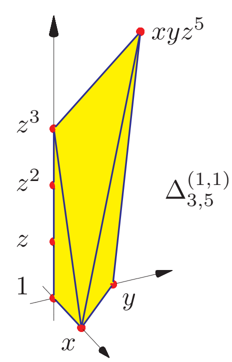
Example 5.13. Suppose that $n = k = 3$, $l = 5$, and $\epsilon = (1,1)$.

Then the system

$$\begin{aligned} x + y + xyz^5 + 1 + z + z^2 + z^3 &= 0\\ x + 2y + 3xyz^5 + 5 + 7z + 11z^2 + 13z^3 &= 0\\ 2x + 2y + xyz^5 + 4 + 8z + 16z^2 + 32z^3 &= 0 \end{aligned}$$

is equivalent to

$$\begin{aligned} x - (5 + 11z + 23z^2 + 41z^3) &= 0\\ y - (8 + 18z + 38z^2 + 72z^3) &= 0\\ xyz^5 - (2 + 6z + 14z^2 + 30z^3) &= 0 \end{aligned}$$



And thus its number of real roots equals the number of real roots of

$$z^5(5 + 11z + 23z^2 + 41z^3)(8 + 18z + 38z^2 + 72z^3)\ -\ (2 + 6z + 14z^2 + 30z^3)\,,$$

which, as we invite the reader to check, is 3.

Proof of Theorem 5.12. A generic real polynomial system with support $\mathcal{A}^{\epsilon}_{k,l}$ has the form

$$\sum_{j=1}^{n-1} c_{ij}x_j \ + c_{in}x^{\epsilon}x_n^l \ +\ f_i(x_n)\ =\ 0 \quad \text{for } i = 1,\ldots,n\,,$$

where each polynomial f_i has degree k and x^{ϵ} is the monomial $x_1^{\epsilon_1}\cdots x_{n-1}^{\epsilon_{n-1}}$.

Since all solutions to our system are simple, we may perturb the coefficient matrix $(c_{ij})_{i,j=1}^n$ if necessary and then use Gaussian elimination to obtain an equivalent system

$$x_1 - g_1(x_n) \;=\; \cdots \;=\; x_{n-1} - g_{n-1}(x_n) \;=\; x^\epsilon x_n^l - g_n(x_n) \;=\; 0\,, \tag{5.17}$$

where each polynomial g_i has degree k. Using the first $n-1$ polynomials to eliminate the variables $x_1, \ldots, x_{n-1}$ gives the univariate polynomial

$$x_n^l \cdot g_1(x_n)^{\epsilon_1} \cdots g_{n-1}(x_n)^{\epsilon_{n-1}} \;-\; g_n(x_n)\,, \tag{5.18}$$

which has degree $l + k|\epsilon| = v(\Delta_{k,l}^\epsilon)$. Any zero of this polynomial leads to a solution of the original system (5.17) by back substitution. This implies that the number of real roots of the polynomial (5.18) is equal to the number of real solutions to our original system (5.17).

The eliminant (5.18) has no terms of degree m for $k < m < l$, and so it has at most $k + k|\epsilon| + 2$ nonzero real roots, by Descartes's rule of signs. This proves the upper bound.

We omit the construction which shows that this bound is sharp. ∎

CHAPTER 6

Fewnomial Upper Bounds from Gale Dual Polynomial Systems

The fewnomial upper bounds that improve Khovanskii's bound arise from a construction that replaces a system of polynomials by an equivalent (Gale dual) system of rational functions defined on a different space. Using an induction based on the Khovanskii-Rolle Theorem, a bound is obtained for the number of solutions to the Gale dual system, which then implies the fewnomial bound.

We first give an example of this transformation to a Gale dual system. Suppose that we have the system of polynomials in either $\mathbb{T}^3_{\mathbb{R}}$ or $\mathbb{T}^3$,

$$\begin{array}{rcl} v^2w^3 - 11uvw^3 - 33uv^2w + 4v^2w + 15u^2v + 7 &=& 0\,,\\ v^2w^3 \qquad\qquad\quad + \ 5uv^2w - 4v^2w - \ 3u^2v + 1 &=& 0\,,\\ v^2w^3 - 11uvw^3 - 31uv^2w + 2v^2w + 13u^2v + 8 &=& 0\,. \end{array} \tag{6.1}$$

If we solve this for the monomials v^2w^3, v^2w, and uvw^3, we obtain

$$\begin{array}{rcl} v^2w^3 &=& 1 - u^2v - uv^2w\,,\\ v^2w &=& \frac{1}{2} - u^2v + uv^2w\,, \qquad \text{and}\\ uvw^3 &=& \frac{10}{11}(1 + u^2v - 3uv^2w)\,. \end{array} \tag{6.2}$$

Since

$$\begin{array}{rcl} \left(uv^2w\right)^3 \cdot \left(v^2w^3\right) &=& u^3v^8w^6 \ = \ \left(u^2v\right) \cdot \left(v^2w\right)^3 \cdot \left(uvw^3\right) \qquad \text{and}\\ \left(u^2v\right)^2 \cdot \left(v^2w^3\right)^3 &=& u^4v^8w^9 \ = \ \left(uv^2w\right)^2 \cdot \left(v^2w\right) \cdot \left(uvw^3\right)^2\,, \end{array}$$

we may substitute the expressions on the right hand sides of (6.2) for the monomials v^2w^3, v^2w, and uvw^3 in these expressions to obtain the system

$$\begin{array}{rcl} \left(uv^2w\right)^3 \left(1{-}u^2v{-}uv^2w\right) &=& \left(u^2v\right) \left(\frac{1}{2}{-}u^2v{+}uv^2w\right)^3 \left(\frac{10}{11}(1{+}u^2v{-}3uv^2w)\right)\,,\\ \left(u^2v\right)^2 \left(1{-}u^2v{-}uv^2w\right)^3 &=& \left(uv^2w\right)^2 \left(\frac{1}{2}{-}u^2v{+}uv^2w\right) \left(\frac{10}{11}(1{+}u^2v{-}3uv^2w)\right)^2\,. \end{array}$$

Writing x for u^2v and y for uv^2w and solving for 0, these become

$$\begin{array}{rcrcrcl} f &:=& y^3(1-x-y) &-& x(\frac{1}{2}-x+y)^3\left(\frac{10}{11}(1+x-3y)\right) &=& 0\,,\\ g &:=& x^2(1-x-y)^3 &-& y^2(\frac{1}{2}-x+y)\left(\frac{10}{11}(1+x-3y)\right)^2 &=& 0\,. \end{array} \tag{6.3}$$

Figure 6.1 shows the curves these define and the lines given by the linear factors in (6.3).

It is clear that the solutions to (6.3) in the complement of the lines are consequences of solutions to (6.1). More, however, it true. The two systems define isomorphic schemes as complex or as real varieties, with the positive solutions to (6.1)

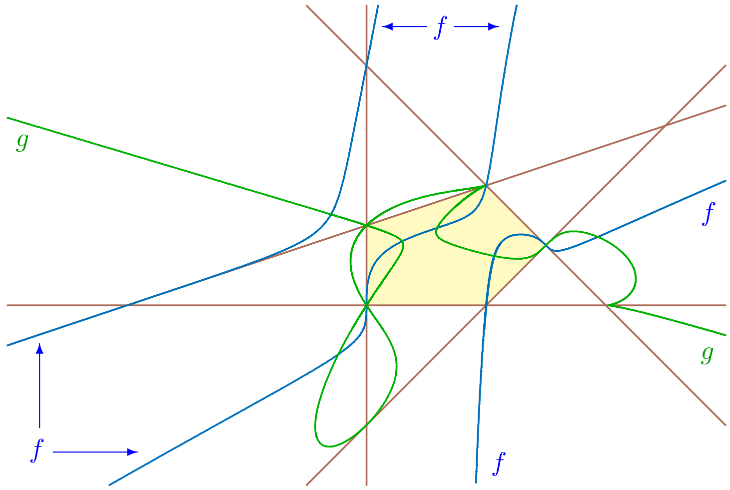


FIGURE 6.1. Curves and lines.

corresponding to the solutions of (6.3) lying in the central pentagon. Gale duality, which generalizes this isomorphism, is a first step towards the new fewnomial bounds of [**5, 17**].

The system (6.3) is equivalent to the vanishing of the logarithmic functions,

$$\begin{aligned}
&\log(x) + 3\log(\tfrac{1}{2}-x+y) + \log(\tfrac{10}{11}(1+x-3y)) - 3\log(y) - \log(1-x-y) \\
&2\log(y) + \log(\tfrac{1}{2}-x+y) + 2\log(\tfrac{10}{11}(1+x-3y)) - 2\log(x) - 3\log(1-x-y)
\end{aligned} \tag{6.4}$$

which makes sense only in the interior of the central pentagon of Figure 6.1—the region where the arguments of the logarithms are positive. The solutions to (6.4) lying in this central pentagon are exactly the points corresponding to positive solutions to (6.1). This transformation, into a logarithmic system in which the exponents are coefficients, and hence do not affect the structure of the system, is one reason that the fewnomial bounds do not depend upon the exponents of the monomials.

These new bounds are derived using the general method that Khovanskii developed in [**83**]. However, they take advantage of special geometry (encoded in Gale duality) available to systems of polynomials in a way that the proof of Khovanski's bound (Theorem 5.1) did not. Their main value is that they are sharp, in the asymptotic sense described after Corollary 5.10. These bounds are also only for unmixed systems of equations, such as those covered by Kushnirenko's Theorem. It remains open finding such good bounds for mixed systems, such as those covered by Bernstein's Theorem. For two different first steps in that direction see [**19, 94**].

6.1. Gale duality for polynomial systems

Gale duality is an alternative way to view a system of sparse polynomials. It was developed in [**18**] in more generality than we will treat here, and extended in [**124**]. Let us work over the complex numbers. Let $\mathcal{A} = \{0, a_1, \ldots, a_{l+n}\} \subset \mathbb{Z}^n$ be integer vectors which span $\mathbb{R}^n$ and suppose that we have a system

$$f_1(x_n, \ldots, x_n) \;=\; f_2(x_1, \ldots, x_n) \;=\; \cdots \;=\; f_n(x_1, \ldots, x_n) \;=\; 0 \tag{6.5}$$

of polynomials with support $\mathcal{A}$. As in Section 3.2, the solutions may be interpreted geometrically as $\varphi_{\mathcal{A}}^{-1}(L)$, where $\varphi_{\mathcal{A}}$ is the map

$$\begin{array}{rcl} \varphi_{\mathcal{A}} : \mathbb{T}^n & \longrightarrow & \mathbb{T}^{l+n} \subset \mathbb{C}^{l+n} \\ x & \longmapsto & (x^{a_1}, x^{a_2}, \ldots, x^{a_{l+n}}) , \end{array}$$

and $L \subset \mathbb{C}^{l+n}$ is a codimension n plane defined by degree one polynomials corresponding to the polynomials f_i. While we work here in $\mathbb{C}^{l+n}$, but used $\mathbb{P}^{\mathcal{A}} = \mathbb{P}^{l+n}$ in Section 3.2, there is no essential difference as the image of $\mathbb{T}^n$ under $\varphi_{\mathcal{A}}$ lies in the principal affine open subset of $\mathbb{P}^{\mathcal{A}}$ where $z_0 = 1$ is the coordinate corresponding to $0 \in \mathcal{A}$.

Suppose that $\mathcal{A}$ is primitive in that $\mathbb{Z}\mathcal{A} = \mathbb{Z}^n$, so that the homomorphism $\varphi_{\mathcal{A}}$ is injective. Then the subscheme[1] of $\mathbb{T}^n$ defined by the equations (6.5) is isomorphic to the subscheme $X := \varphi(\mathbb{T}^n) \cap L$ of $\mathbb{T}^{l+n}$ or $\mathbb{C}^{l+n}$. If we change our perspective and view X as the basic object, then the bijective parameterization $\varphi_{\mathcal{A}}$ of $\varphi_{\mathcal{A}}(\mathbb{T}^n)$ realizes X as the subscheme of $\mathbb{T}^n$ defined by (6.5).

The main idea behind Gale duality for polynomial systems is to instead parameterize L with a map $\psi_p \colon \mathbb{C}^l \to L$ and then consider the subscheme $\psi_p^{-1}(X)$ of $\mathbb{C}^l$, which is isomorphic to X. This is in fact what we did in transforming (6.1) into (6.3). We show that $\psi_p^{-1}(X)$ is defined in $\mathbb{C}^l$ by a system of rational functions.

Let $p_1(y), \ldots, p_{l+n}(y)$ be pairwise nonproportional degree one polynomials on $\mathbb{C}^l$. Their product $\prod_i p_i(y) = 0$ defines a hyperplane arrangement $\mathcal{H}$ in $\mathbb{C}^l$. Let $\beta \in \mathbb{Z}^{l+n}$ be an integer vector, called a *weight* for the arrangement $\mathcal{H}$. We use this to define a rational function p^β,

$$p^\beta \;=\; p(y)^\beta \;:=\; p_1(y)^{b_1} p_2(y)^{b_2} \cdots p_{l+n}(y)^{b_{l+n}} ,$$

where $\beta = (b_1, \ldots, b_{l+n})$. This rational function $p(y)^\beta$ is a *master function* for the weighted arrangement $\mathcal{H}$. As the components of β may be negative, its natural domain of definition is the complement $M_{\mathcal{H}} := \mathbb{C}^l \setminus \mathcal{H}$ of the arrangement.

A *system of master functions* in $M_{\mathcal{H}}$ with weights $\mathcal{B} = \{\beta_1, \ldots, \beta_l\}$ is the system of equations in $M_{\mathcal{H}}$,

$$p(y)^{\beta_1} \;=\; p(y)^{\beta_2} \;=\; \cdots \;=\; p(y)^{\beta_l} \;=\; 1 . \qquad (6.6)$$

More generally, we could instead consider equations of the form $p(y)^\beta = \alpha$, where $\alpha \in \mathbb{T}$ is an arbitrary nonzero complex number. We may however absorb such constants into the polynomials $p_i(y)$, as there are $l+n$ such polynomials but only l constants in a system of master functions. This is the source of the factor $\frac{10}{11}$ in (6.3). We further assume that the system (6.6) defines a zero-dimensional scheme in $M_{\mathcal{H}}$. This implies that the weights $\mathcal{B}$ are linearly independent, and that the suppressed constants multiplying the $p_i(y)$ are sufficiently general.

As with sparse systems, a system of master functions may be realized geometrically through an appropriate map. The degree one polynomials $p_1(y), \ldots, p_{l+n}(y)$ define an affine-linear map

$$\begin{array}{rcl} \psi_p : \mathbb{C}^l & \longrightarrow & \mathbb{C}^{l+n} \\ y & \longmapsto & \big(p_1(y), p_2(y), \ldots, p_{l+n}(y)\big) . \end{array}$$

[1] As it is zero-dimensional, this is simply the collection of points defined by (6.5), together with multiplicities encoded in the local rings at each point. See [**112**] for an elementary treatment.

This map is injective if and only if the polynomials $\{1, p_1(y), \ldots, p_{l+n}(y)\}$ span the space of degree one polynomials on $\mathbb{C}^l$, in which case the hyperplane arrangement $\mathcal{H}$ is called *essential*. Equivalently, the hyperplane arrangement is essential if normal vectors to the hyperplanes span $\mathbb{C}^l$. The hyperplane arrangement $\mathcal{H}$ is the pullback along ψ_p of the coordinate hyperplanes $z_i = 0$ in $\mathbb{C}^{l+n}$, and its complement $M_{\mathcal{H}}$ is the pullback of the torus $\mathbb{T}^{l+n}$ which is the complement of the coordinate hyperplanes in $\mathbb{C}^{l+n}$.

The weights $\mathcal{B}$ are *saturated* if they are linearly independent and span a saturated subgroup of $\mathbb{Z}^{l+n}$, that is, if $\mathbb{Z}\mathcal{B}$ equals its saturation, $\mathbb{Q}\mathcal{B} \cap \mathbb{Z}^{l+n}$, which consists of all integer points in the linear span of $\mathcal{B}$. Linear independence of $\mathcal{B}$ is equivalent to the subgroup $\mathbb{G}$ of the torus $\mathbb{T}^{l+n}$ defined by the equations

$$z^{\beta_1} = z^{\beta_2} = \cdots = z^{\beta_l} = 1 \tag{6.7}$$

having dimension n and saturation is equivalent to $\mathbb{G}$ being connected. (Here, $z_1, \ldots, z_{l+n}$ are the coordinates for $\mathbb{C}^{l+n}$.) In this way, we see that the equations (6.6) describe the pullback $\psi_p^{-1}(\mathbb{G})$ of this subgroup $\mathbb{G}$. We summarize this discussion.

THEOREM 6.1. *A system of master functions* (6.6) *in $M_{\mathcal{H}}$ is the pullback along ψ_p of the intersection of the linear space $\psi_p(\mathbb{C}^l)$ with a subgroup $\mathbb{G}$ of $\mathbb{T}^{l+n}$ of dimension n, and any such pullback defines a system of master functions in $M_{\mathcal{H}}$. When ψ_p is injective, it gives a scheme-theoretic isomorphism between the solutions to the system of master functions and the intersection $\mathbb{G} \cap \psi_p(\mathbb{C}^l)$.*

Theorem 6.1 is the new ingredient needed for the notion of Gale duality. Suppose that $\mathbb{G} \subset \mathbb{T}^{l+n}$ is a connected subgroup of dimension n and that $L \subset \mathbb{C}^{l+n}$ is a linear subspace of dimension l not parallel to any coordinate hyperplane so that the intersection $\mathbb{G} \cap L$ has dimension 0.

DEFINITION 6.2. Suppose that we are given

(i) A primitive set $\mathcal{A} = \{0, a_1, \ldots, a_{l+n}\} \subset \mathbb{Z}^n$ and equations (6.7) defining $\mathbb{G} = \varphi_{\mathcal{A}}(\mathbb{T}^n)$ as a subgroup of $\mathbb{T}^{l+n}$. Then $\varphi_{\mathcal{A}} \colon \mathbb{T}^n \to \mathbb{G}$ is an isomorphism and $\mathcal{B} = \{\beta_1, \ldots, \beta_l\}$ is saturated.

(ii) An affine-linear isomorphism $\psi_p \colon \mathbb{C}^l \to L$ and degree one polynomials $\Lambda_1, \ldots, \Lambda_n$ on $\mathbb{C}^{l+n}$ defining L.

Let $\mathcal{H} \subset \mathbb{C}^l$ be the pullback of the coordinate hyperplanes of $\mathbb{C}^{l+n}$ along ψ_p. We say that the system of sparse polynomials on $\mathbb{T}^n$

$$\varphi_{\mathcal{A}}^*(\Lambda_1) = \varphi_{\mathcal{A}}^*(\Lambda_2) = \cdots = \varphi_{\mathcal{A}}^*(\Lambda_n) = 0 \tag{6.8}$$

with support $\mathcal{A}$ is *Gale dual* to the system of master functions

$$p(y)^{\beta_1} = p(y)^{\beta_2} = \cdots = p(y)^{\beta_l} = 1 \tag{6.9}$$

with weights $\mathcal{B}$ on the hyperplane complement $M_{\mathcal{H}}$ and vice-versa.

The following is immediate.

THEOREM 6.3. *A pair of Gale dual systems* (6.8) *and* (6.9) *define isomorphic schemes.*

This notion of Gale duality involves two different linear algebraic dualities in the sense of linear functions annihilating vector spaces. In the first duality, the degree one polynomials $p_i(y)$ defining the map ψ_p are annihilated by the degree one polynomials Λ_i which define the system of sparse polynomials (6.8). The second duality is integral, as the weights $\mathcal{B}$ form a basis for the free abelian group of integer linear relations among the nonzero elements of $\mathcal{A}$. Writing the elements of $\mathcal{B}$ as the rows of a matrix, the $l{+}n$ columns form the *Gale transform* [**63**, §5.4] of the vector configuration $\mathcal{A}$—this is the source of our terminology.

REMARK 6.4. If we restrict the domain of $\varphi_{\mathcal{A}}$ to the real numbers or to the positive real numbers, then we obtain the two forms of Gale duality which are relevant to us. Set $\mathbb{T}_{\mathbb{R}} := \mathbb{R} \setminus \{0\}$, the real torus and $\mathbb{R}_{>}$ to be the positive real numbers. Suppose that $\mathcal{A}$ is not necessarily primitive, but that the lattice index $[\mathbb{Z}^n : \mathbb{Z}\mathcal{A}]$ is odd. Then $\varphi_{\mathcal{A}} \colon \mathbb{T}^n_{\mathbb{R}} \to \mathbb{T}^{l+n}_{\mathbb{R}}$ is injective. Similarly, if $\mathbb{Z}\mathcal{B}$ has odd index in its saturation $\mathbb{Q}\mathcal{B} \cap \mathbb{Z}^{l+n}$, which is the group of integer linear relations holding on $\mathcal{A}$, then the equations (6.7) define a not necessarily connected subgroup $\mathbb{G} \subset \mathbb{T}^{l+n}$ whose real points $\mathbb{G}_{\mathbb{R}}$ lie in its connected component containing the identity. When the linear polynomials Λ_i of (6.8) and p_i of (6.9) are real and annihilate each other, then these two systems—which do not necessarily define isomorphic schemes in the complex varieties $\mathbb{T}^n$ and $M_{\mathcal{H}}$—define isomorphic real analytic sets in $\mathbb{T}^n_{\mathbb{R}}$ for (6.8) and in the complement $M^{\mathbb{R}}_{\mathcal{H}} := \mathbb{R}^l \setminus \mathcal{H}$ of the real hyperplanes defined by the p_i for (6.9).

In the version valid for the positive real numbers, we may suppose that the exponents $\mathcal{A}$ are real vectors, for if $r \in \mathbb{R}_{>}$ and $a \in \mathbb{R}$, then $r^a := \exp(a \log(x))$ is well-defined. In this way, we obtain systems of polynomials with *real exponents*. In this case, the weights $\mathcal{B}$ should be a basis for the vector space of linear relations holding on $\mathcal{A}$, and the degree one polynomials Λ_i and p_i are again real and dual to each other. The equations (6.7) for $z \in \mathbb{R}^{l+n}_{>}$ define a connected analytic subgroup of $\mathbb{R}^{l+n}_{>}$ which equals $\varphi_{\mathcal{A}}(\mathbb{R}^n_{>})$. In this generality, the polynomial system (6.8) makes sense only for $x \in \mathbb{R}^n_{>}$ and the system of master functions (6.9) only makes sense for y in the *positive chamber* Δ_p of the hyperplane complement $M^{\mathbb{R}}_{\mathcal{H}}$,

$$\Delta_p \;:=\; \{y \in \mathbb{R}^l \mid p_i(y) > 0 \quad i = 1, \ldots, l{+}n\}\,, \tag{6.10}$$

and the two systems define isomorphic real analytic sets in $\mathbb{R}^n_{>}$ for (6.8) and in Δ_p for (6.9).

The description of Gale duality in Definition 6.2 lends itself immediately to an algorithm for converting a system of sparse polynomials into an equivalent system of master functions. We describe this over $\mathbb{C}$, but it works equally well over $\mathbb{R}$ or $\mathbb{R}_{>}$. Suppose that $\mathcal{A} \subset \mathbb{Z}^n$ is a primitive collection of integer vectors and that

$$f_1(x_1, \ldots, x_n) \;=\; f_2(x_1, \ldots, x_n) \;=\; \cdots \;=\; f_n(x_1, \ldots, x_n) \;=\; 0 \tag{6.11}$$

defines a zero-dimensional subscheme of $\mathbb{T}^n$. In particular, the polynomials f_i are linearly independent. We may solve these equations for some of the monomials to obtain

$$\begin{array}{ccccc} x^{a_1} & = & g_1(x) & =: & p_1(x^{a_{1+n}}, \ldots, x^{a_{l+n}}) \\ \vdots & & & & \vdots \\ x^{a_n} & = & g_n(x) & =: & p_n(x^{a_{1+n}}, \ldots, x^{a_{l+n}}) \end{array} \tag{6.12}$$

Here, $\mathcal{A} = \{0, a_1, \ldots, a_n, a_{1+n}, \ldots, a_{l+n}\}$, and for each $i = 1, \ldots, n$, $g_i(x)$ is a polynomial with support $\{0, a_{1+n}, \ldots, a_{l+n}\}$ which is a degree one polynomial function $p_i(x^{a_{1+n}}, \ldots, x^{a_{l+n}})$ in the given l arguments. (Compare this to (6.2).) For $i = 1{+}n, \ldots, l{+}n$, set $p_i(x^{a_{1+n}}, \ldots, x^{a_{l+n}}) := x^{a_i}$.

An integer linear relation among the exponent vectors in $\mathcal{A}$,

$$b_1a_1 + b_2a_2 + \cdots + b_{l+n}a_{l+n} = 0,$$

is equivalent to the monomial identity

$$(x^{a_1})^{b_1} \cdot (x^{a_2})^{b_2} \cdots (x^{a_{l+n}})^{b_{l+n}} = 1,$$

which gives the consequence of the system (6.12)

$$p(y)^\beta := \big(p_1(x^{a_{n+1}}, \ldots, x^{a_{l+n}})\big)^{b_1} \cdots \big(p_{l+n}(x^{a_{n+1}}, \ldots, x^{a_{l+n}})\big)^{b_{l+n}} = 1,$$

where $\beta := (b_1, \ldots, b_{l+n}) \in \mathbb{Z}^{l+n}$.

Define $y_1, \ldots, y_l$ to be new variables which are coordinates for $\mathbb{C}^l$. The degree one polynomials $p_i(y_1, \ldots, y_l)$ define a hyperplane arrangement $\mathcal{H}$ in $\mathbb{C}^l$. Let $\mathcal{B} := \{\beta_1, \ldots, \beta_l\} \subset \mathbb{Z}^{l+n}$ be a basis for the $\mathbb{Z}$-module of integer linear relations among the nonzero vectors in $\mathcal{A}$. These weights $\mathcal{B}$ define a system of master functions

$$p(y)^{\beta_1} = p(y)^{\beta_2} = \cdots = p(y)^{\beta_l} = 1 \tag{6.13}$$

in the complement $M_{\mathcal{H}}$ of the arrangement.

Theorem 6.5. *The system of polynomials* (6.11) *in* $\mathbb{T}^n$ *and the system of master functions* (6.13) *in* $M_{\mathcal{H}}$ *define isomorphic schemes.*

Proof. Condition (i) in Definition 6.2 holds as $\mathcal{A}$ and $\mathcal{B}$ are both primitive and annihilate each other. The linear forms Λ_i that pullback along $\varphi_{\mathcal{A}}$ to define the system (6.12) are

$$\Lambda_i(z) = z_i - p_i(z_{n+1}, \ldots, z_{l+n}),$$

which shows that condition (ii) holds, and so the statement follows from Theorem 6.3. ∎

The example at the beginning of this chapter illustrated Gale duality, but the equations (6.3) were not of the form $p^\beta = 1$. They are, however, easily transformed into such equations, and we obtain

$$\frac{y^3(1-x-y)}{x(\frac{1}{2}-x+y)^3\left(\frac{10}{11}(1+x-3y)\right)} = \frac{x^2(1-x-y)^3}{y^2(\frac{1}{2}-x+y)\left(\frac{10}{11}(1+x-3y)\right)^2} = 1. \tag{6.14}$$

Systems of the form (6.3) may be obtained from systems of master functions by multiplying $p^\beta = 1$ by the terms of p^β with negative exponents to clear the denominators, to obtain $p^{\beta^+} = p^{\beta^-}$ and thus $p^{\beta^+} - p^{\beta^-} = 0$, where $\beta_\pm$ is the componentwise maximum of the vectors $(0, \pm\beta)$.

6.2. New fewnomial bounds

The transformation of Gale duality is the key step in establishing the new fewnomial bounds.

Theorem 6.6. *A system* (6.5) *of* n *polynomials in* n *variables having a total of* $1+l+n$ *monomials with exponents* $\mathcal{A} \subset \mathbb{R}^n$ *has at most*

$$\frac{e^2+3}{4}\, 2^{\binom{l}{2}} n^l$$

positive nondegenerate solutions.

If $\mathcal{A} \subset \mathbb{Z}^n$ *and* $\mathbb{Z}\mathcal{A}$ *has odd index in* $\mathbb{Z}^n$, *then the system has at most*

$$\frac{e^4+3}{4}\, 2^{\binom{l}{2}} n^l$$

nondegenerate real solutions.

The first bound is proven in [**17**] and the second in [**5**]. By Gale duality, Theorem 6.6 is equivalent to the next theorem.

THEOREM 6.7. *Let $p_1(y), \ldots, p_{l+n}(y)$ be degree 1 polynomials on $\mathbb{R}^l$ that, together with the constant 1, span the space of degree 1 polynomials. For any linearly independent vectors $\mathcal{B} = \{\beta_1, \ldots, \beta_l\} \subset \mathbb{R}^{l+n}$, the number of solutions to*

$$p(y)^{\beta_j} \;=\; 1 \qquad \textit{for} \quad j = 1, \ldots, l,$$

in the positive chamber Δ_p (6.10) is less than

$$\frac{e^2+3}{4}\, 2^{\binom{l}{2}} n^l\,.$$

If $\mathcal{B} \subset \mathbb{Z}^{l+n}$ and has odd index in its saturation, then the number of solutions in $M_{\mathcal{H}}^{\mathbb{R}}$ is less than

$$\frac{e^4+3}{4}\, 2^{\binom{l}{2}} n^l\,.$$

The basic idea behind the proof of Theorem 6.7 is to use the Khovanskii-Rolle Theorem, but in a slightly different form than given in Theorem 5.3. Using it in this way to establish bounds for real solutions to equations was first done in [**57**]. Given functions $g_1, \ldots, g_m$ defined on a domain D, let $V_D(g_1, \ldots, g_m)$ be their set of common zeroes. If C is a curve in D, let $\mathrm{ubc}_D(C)$ be its number of unbounded (noncompact) components in D.

THEOREM 6.8 (Khovanskii-Rolle). *Let $g_1, \ldots, g_l$ be smooth functions defined on a domain $D \subset \mathbb{R}^l$ with finitely many common zeroes and suppose that $C := V_D(g_1, \ldots, g_{l-1})$ is a smooth curve of finite type. Set J to be the Jacobian determinant, $\det(\partial g_i/\partial y_j)$, of $g_1, \ldots, g_l$. Then we have*

$$|V_D(g_1, \ldots, g_l)| \;\leq\; \mathrm{ubc}_D(C) + |V_D(g_1, \ldots, g_{l-1},\, J)|\,. \tag{6.15}$$

PROOF. This form of the Khovanskii-Rolle Theorem follows from the from the usual Rolle Theorem. Suppose that $g_l(a) = g_l(b) = 0$, for points a, b on the same component of C. Let $s(t)$ be the arclength along this component of C, measured from a point $t_0 \in C$, and consider the map,

$$\begin{array}{ccc} C & \longrightarrow & \mathbb{R}^2 \\ t & \longmapsto & (s(t), g_l(t))\,. \end{array}$$

This is the graph of a differentiable function $g(s)$ which vanishes when $s = s(a)$ and $s = s(b)$, so there is a point $s(b)$ between $s(a)$ and $s(b)$ where its derivative also vanishes, by the usual Rolle Theorem. But then c lies between a and b on that component of C, and the vanishing of $g'(s(c))$ is equivalent to the Jacobian determinant J vanishing at c.

Thus along any arc of C connecting two zeroes of g_l, the Jacobian vanishes at least once. We illustrate this in Figure 6.2

As in the proof of Theorem 5.3, the estimate (6.15) follows from the observation concerning consecutive zeroes of g_l along components of C. ∎

We first make an adjustment to the system of master functions in Theorem 6.7, replacing each master function $p(y)^\beta = p_1(y)^{b_1} \cdots p_{l+n}(y)^{b_{l+n}}$ by

$$|p(y)|^\beta \;:=\; |p_1(y)|^{b_1} \cdots |p_{l+n}(y)|^{b_{l+n}}\,.$$

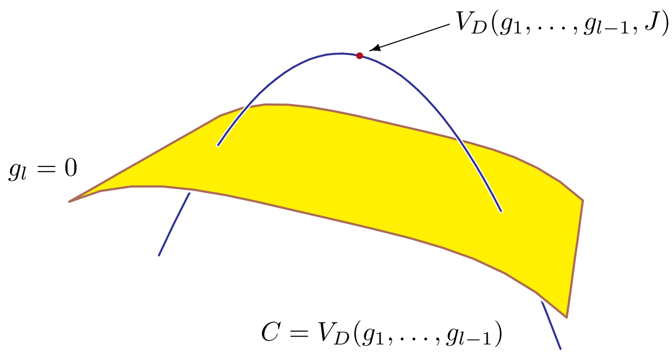


FIGURE 6.2. The Jacobian vanishes between two zeroes of g_l.

For example, if we take absolute values in the system of master functions (6.14), we obtain

$$\frac{|y|^3|1-x-y|}{|x||\frac{1}{2}-x+y|^3|\frac{10}{11}(1+x-3y)|} \;=\; \frac{|x|^2|1-x-y|^3}{|y|^2|\frac{1}{2}-x+y||\frac{10}{11}(1+x-3y)|^2} \;=\; 1\,. \tag{6.16}$$

This new system with absolute values will still have the same number of solutions in the positive chamber Δ_p as the original system, since when $y \in \Delta_p$, we have $|p_i(y)| = p_i(y)$ for $i = 1, \ldots, l+n$. Its solutions in the hyperplane complement $M_{\mathcal{H}}^{\mathbb{R}}$ will include the solutions to the system of master functions from Theorem 6.7, but there may be more solutions.

We illustrate this for the system (6.16) in Figure 6.3, which we may compare to Figure 6.1 as the system of master functions (6.14) is equivalent to the system (6.3) in the complement of the lines. In particular, among the solutions to

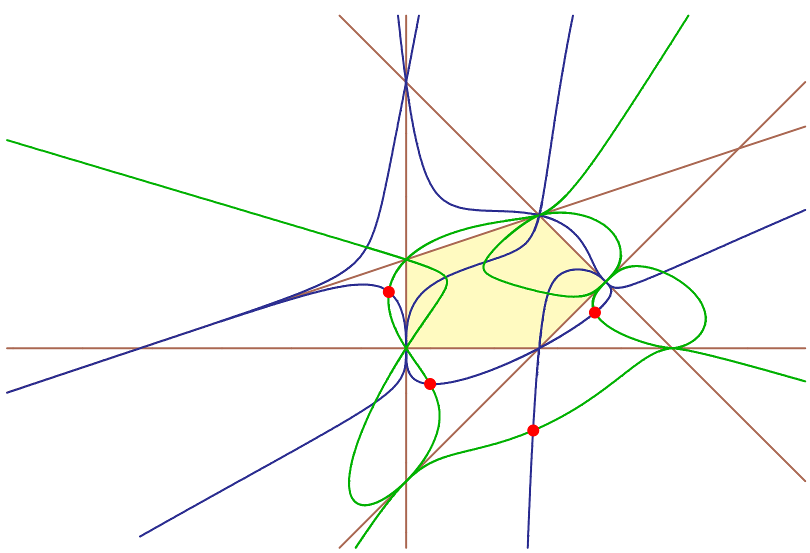


FIGURE 6.3. Curves from absolute values and additional solutions.

the system (6.16) are the three solutions to (6.3) in the positive chamber (which is shaded) as well as the three solutions to (6.3) outside the positive chamber. The system (6.16) has four additional solutions outside the positive chamber, which are marked in Figure 6.3.

We give a proof of Theorem 6.7 for systems of the form $|p(y)|^{\beta_j} = 1$ for $j = 1, \ldots, l$. This will imply the bound for systems of master functions. Taking absolute values allows nonintegral (real number) exponents in $|p(y)|^{\beta_j} = 1$, and so we shall no longer require that exponents be integral.

We make two reductions.

(1) The degree one polynomials $p_i(y)$ are in general position in that the hyperplanes in the arrangement $\mathcal{H}$ are in linear general position. That is, any j of them meet in an affine linear subspace of codimension j, if $j \leq l$, and their intersection is empty if $j > l$. We may do this, as we are bounding nondegenerate solutions, which cannot be destroyed if the $p_i(y)$ are perturbed to put the hyperplanes into this general position.

(2) Let $\mathcal{B}$ be the matrix whose rows are $\beta_1, \ldots, \beta_l$. We may assume that every minor of $\mathcal{B}$ is nonzero. This may be done by perturbing the real-number exponents in the functions $|p(y)|^{\beta_j}$. This will not reduce the number of nondegenerate solutions.

Perturbing exponents is not as drastic of a measure as it first seems. Note that in the hyperplane complement, $|p(y)|^{\beta} = 1$ defines the same set as $\log(|p(y)|^{\beta}) = 0$. If $\beta = (b_1, \ldots, b_{l+n})$, then this is simply

$$b_1 \log|p_1(y)| \;+\; b_2 \log|p_2(y)| \;+\; \cdots \;+\; b_{l+n} \log|p_{l+n}(y)| \;=\; 0\,. \tag{6.17}$$

Expressing the equations in this form shows that we may perturb the exponents without altering the structure of the equations. In fact, it is possible to arrange for this assumption on $\mathcal{B}$ to hold with the β_j rational or even integral.

EXAMPLE 6.9. We first look at these reductions in the context of the system of master functions (6.14). The hyperplane arrangement $\mathcal{H}$ is an arrangement of lines in which no three meet and no two are parallel, and thus they are in general position. The matrix of exponents is

$$\mathcal{B} \;=\; \begin{pmatrix} 2 & -2 & 3 & -1 & -2 \\ -1 & 3 & 1 & -3 & -1 \end{pmatrix}.$$

No entry and no minor of $\mathcal{B}$ vanishes.

Let us now see how the Khovanskii-Rolle Theorem applies to the system (6.16) of Figure 6.3, restricted to the positive chamber. First, take logarithms and rearrange to obtain

$$\begin{aligned} 2\log|x| - 2\log|y| + 3\log|1{-}x{-}y| - \log|\tfrac{1}{2}{-}x{+}y| - 2\log|\tfrac{10}{11}(1{+}x{-}3y)| &\;=\; 0 \\ -\log|x| + 3\log|y| + 1\log|1{-}x{-}y| - 3\log|\tfrac{1}{2}{-}x{+}y| - \log|\tfrac{10}{11}(1{+}x{-}3y)| &\;=\; 0 \end{aligned}$$

Call these functions g_1 and g_2, respectively. Their Jacobian is the rational function

$$\frac{2x^3 - 16x^2y + 12xy^2 + 6y^3 - \frac{31}{2}x^2 + 26xy - \frac{53}{2}y^2 + \frac{9}{2}x + \frac{15}{2}y - 2}{xy(1-x-y)(\frac{1}{2}{-}x{+}y)(1+x-3y)}$$

whose denominator is the product of the linear factors defining the lines in Figure 6.3. Clearing the denominator and multiplying by 2, gives a cubic polynomial

$$J_2 \;:=\; 4x^3 - 32x^2y + 24y^2x + 12y^3 - 31x^2 + 52xy - 53y^2 + 9x + 15y - 4\,.$$

Its zero set meets the curve C_1 (which is defined by $g_1 = 0$) in six points, five of which we display in Figure 6.4—the sixth is at $(3.69, -0.77)$. By the Khovanskii-

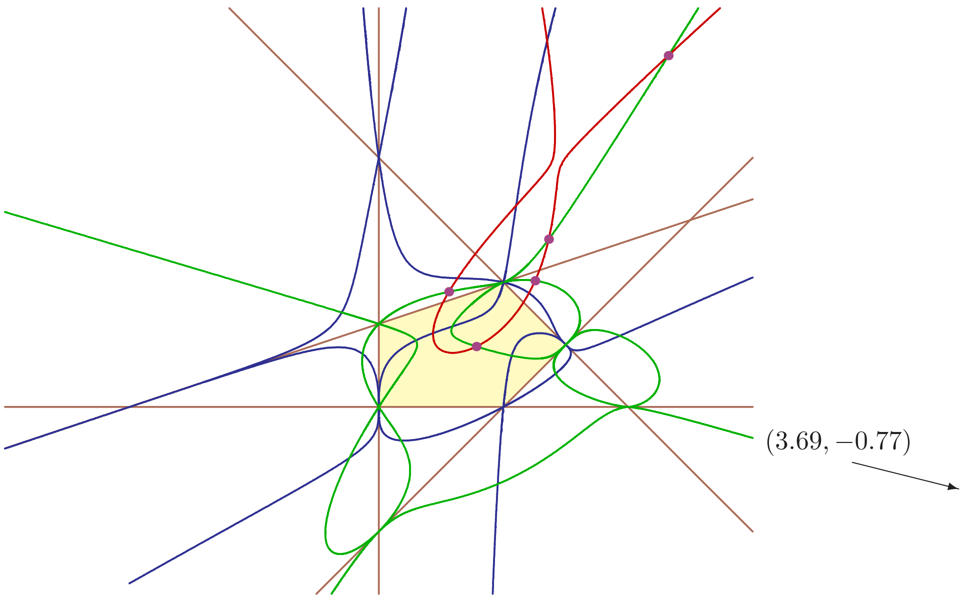


FIGURE 6.4. Gale system and Jacobian J_2.

Rolle Theorem, the number of solutions to $g_1 = g_2 = 0$ is at most this intersection number, $|V(g_1, J_2)|$, plus the number of unbounded components of C_1.

We see that C_1 consists of 14 unbounded components in the complement D of the lines, which gives the inequality

$$10 \;=\; |V_D(g_1, g_2)| \;\leq\; \mathrm{ubc}_D(C_1) + |V_D(g_1, J_2)| \;=\; 14 + 6 \;=\; 20\,.$$

We follow the suggestion in the second reduction (6.17) and replace the master functions by the logarithms of their absolute values. For each $j = 1, \ldots, l$, define

$$g_j(y) \;:=\; \log|p(y)|^{\beta_j} \;=\; \sum_{i=1}^{l+n} \beta_{i,j} \log|p_i(y)|\;,$$

where $\beta_{i,j}$ is the ith component of β_j. Observe that both $g_j = 0$ and $|p(y)|^{\beta_j} = 1$ have the same solutions in the hyperplane complement $M_{\mathcal{H}}^{\mathbb{R}}$. For each $j = 1, \ldots, l$, let $Z_j \subset M_{\mathcal{H}}^{\mathbb{R}}$ be the common zeroes of the functions $g_1, \ldots, g_j$, that is, $Z_j := V(g_1, \ldots, g_j)$.

The connected components of the complement $M_{\mathcal{H}}^{\mathbb{R}}$ are called chambers. A flat of the arrangement $\mathcal{H}$ is an affine subspace which is an intersection of some hyperplanes in $\mathcal{H}$. By our assumption that the hyperplane arrangement $\mathcal{H}$ is in general position, a flat of $\mathcal{H}$ has codimension j exactly when it is the intersection of j hyperplanes in $\mathcal{H}$.

LEMMA 6.10. *For each $j = 1, \ldots, l{-}1$, Z_j is a smooth submanifold of $M_{\mathcal{H}}^{\mathbb{R}}$ of codimension j. The closure $\overline{Z_j}$ of Z_j in $\mathbb{R}^l$ meets the arrangement $\mathcal{H}$ in a union of codimension $j{+}1$ flats. In the neighborhood of point on a codimension $j{+}1$ flat meeting $\overline{Z_j}$, Z_j has one branch in each chamber incident on that point.*

Proof. First consider the hypersurface $Z_1 \subset M^{\mathbb{R}}_{\mathcal{H}}$ defined by

$$0 \;=\; g_1(y) \;=\; \sum_{i=1}^{l+n} \beta_{i,1} \log |p_i(y)| \;. \tag{6.18}$$

This was obtained by taking the logarithm of the following equation,

$$1 \;=\; \prod_{i=1}^{l+n} |p_i(y)|^{\beta_{i,1}} \;=\; |p(y)|^{\beta_1} \;.$$

If we write $\beta^{\pm}_{i,j} := \max(0, \pm\beta_{i,j})$, we may clear the denominators to obtain

$$\prod_{i=1}^{l+n} |p_i(y)|^{\beta^+_{i,1}} \;-\; \prod_{i=1}^{l+n} |p_i(y)|^{\beta^-_{i,1}} \;=\; 0\,, \tag{6.19}$$

which agrees with (6.18) in the hyperplane complement $M^{\mathbb{R}}_{\mathcal{H}}$, but which makes sense in all of $\mathbb{R}^l$ and defined $\overline{Z}_j$.

Let $y \in \overline{Z}_1 \cap \mathcal{H}$. Then some degree one polynomial $p_i(y)$ vanishes at y, $p_i(y) = 0$. By our hypotheses on the exponents $\mathcal{B}$, $\beta_{i,1} \neq 0$. Thus one of the two terms in (6.19) vanishes, which implies the other does, as well. It follows that that for some $k \neq i$, we have $p_k(y) = 0$ where $\beta_{i,1} \cdot \beta_{k,1} < 0$. This shows that $\overline{Z}_1 \cap \mathcal{H}$ is a subset of the codimension two skeleton of $\mathcal{H}$. But this implies that $\overline{Z}_1 \cap \mathcal{H}$ is a union of codimension two flats, as its dimension is at least $l-2$.

We use induction to complete the proof of the lemma. Suppose $j > 1$. Since $\overline{Z}_{j-1} \supset \overline{Z}_j$, the intersection $\overline{Z}_j \cap \mathcal{H}$ will be contained in $\overline{Z_{j-1}} \cap \mathcal{H}$, which is a union of codimension j flats of $\mathcal{H}$. Let $y_0 \in \overline{Z}_j \cap \mathcal{H}$ lie in a codimension j flat, which, after reordering the p_i, we may assume is defined by the vanishing of the polynomials $p_1, \ldots, p_j$. Our assumption on the minors of the matrix $(\beta_{i,j})$ of exponents implies that Z_j is defined by functions of the form

$$0 \;=\; \widetilde{g}_k(y) \;:=\; \log |p_k(y)| \;+\; \sum_{i=j+1}^{l+n} \widetilde{\beta}_{i,k} \log |p_i(y)| \qquad k = 1, \ldots, j\,.$$

We obtain these equations by applying Gaussian elimination to the first j columns of the matrix of exponents. Converting these equations into binomial form, we have

$$|p_k(y)| \cdot \prod_{i=j+1}^{l+n} |p_i(y)|^{\widetilde{\beta}^+_{i,k}} \;-\; \prod_{i=j+1}^{l+n} |p_i(y)|^{\widetilde{\beta}^-_{i,k}} \;=\; 0 \qquad k = 1, \ldots, j\,. \tag{6.20}$$

Since $p_1(y_0) = p_2(y_0) = \cdots = p_j(y_0) = 0$, there is some $m > j$ with $p_m(y_0) = 0$ and $\widetilde{\beta}^-_{m,k} \neq 0$ for $k = 1, \ldots, j$. But this implies that $\overline{Z}_j \cap \mathcal{H}$ lies in a union of codimension $j{+}1$ flats. As the intersection $\overline{Z_j} \cap \mathcal{H}$ has dimension at least $l-(j{+}1)$, it must be a union of codimension $j{+}1$ flats.

Let $y_0 \in \overline{Z_j} \cap \mathcal{H}$ be in the relative interior of a codimension $j{+}1$ flat of $\mathcal{H}$. We may assume that $p_1, \ldots, p_{j+1}$ vanish at y_0 but no other polynomial p_i vanishes. Then we may solve the binomials (6.20) to obtain

$$|p_k| \;=\; |p_{j+1}|^{\widetilde{\beta}^-_{j+1,k}} \cdot \prod_{i=j+2}^{l+n} |p_i(y)|^{\widetilde{\beta}_{i,k}} \qquad \text{for} \quad k = 1, \ldots, j\,. \tag{6.21}$$

As the hyperplanes are general, we may assume that $p_1, \ldots, p_l$ are our coordinates. After restricting to a single chamber Δ of $M^{\mathbb{R}}_{\mathcal{H}}$ incident on y_0 (which amounts to choosing signs for $p_1, \ldots, p_{j+1}$), and applying an analytic change of coordinates in a neighborhood of y_0, the equations (6.21) become $p_k = f_k(p_{j+1}, \ldots, p_l)$ for $k = 1, \ldots, j$. Thus, in a neighborhood of y_0 in each chamber of $M^{\mathbb{R}}_{\mathcal{H}}$ incident on y_0, Z_j is the graph of a function, and therefore consists of a single branch. ◆

Define functions $J_l, J_{l-1}, \ldots, J_1$ by recursion,

$$J_j \ := \ \text{Jacobian of } g_1, \ldots, g_j,\ J_{j+1}, \ldots, J_l\,.$$

We would like to iteratively apply the Khovanskii-Rolle Theorem with these Jacobians. That is, if D is the hyperplane complement $M^{\mathbb{R}}_{\mathcal{H}}$ or one of its connected components, then setting $C_j := Z_{j-1} \cap V_D(J_{j+1}, \ldots, J_l)$, we would like to obtain the estimates

$$|V_D(g_1, \ldots, g_j,\ J_{j+1}, \ldots, J_l)| \ \leq\ |V_D(g_1, \ldots, g_{j-1},\ J_j, \ldots, J_l)| \ +\ \text{ubc}_D(C_j)\,,$$

but we do not know if C_j is smooth as a subset of D, or if these numbers are finite. The easiest way around this conundrum is to perturb the Jacobians.

The following is proven in [**124**] using the Cauchy-Binet Theorem.

Lemma 6.11. *There exist polynomials $F_l, \ldots, F_2, F_1$ with F_j a polynomial of degree $2^{l-j}\cdot n$ such that for each $j = l, \ldots, 2, 1$, we have that $Z_{j-1} \cap V_D(F_{j+1}, \ldots, F_l)$ defines a smooth curve C_j in D, and we have the estimate*

$$|V_D(g_1, \ldots, g_j,\ F_{j+1}, \ldots, F_l)| \ \leq\ |V_D(g_1, \ldots, g_{j-1},\ F_j, \ldots, F_l)| \ +\ \text{ubc}_D(C_j)\,,$$

and furthermore, the polynomials $F_1, \ldots, F_l$ are general given their degrees.

The main point of this lemma is that the Jacobian J of $g_1, \ldots, g_j,\ F_{j+1}, \ldots, F_l$ becomes a polynomial $\widetilde{F}_j$ of degree $2^{l-j}\cdot n$ if we multiply it by $\prod_{i=1}^{l+n} p_i(y)$. By the Khovanskii-Rolle Theorem, we have the inequality of the lemma with $\widetilde{F}_j$ in place of F_j. We would be done if the curve C_{j-1} were smooth. To guarantee this, we perturb the polynomial $\widetilde{F}_j$ to a general polynomial F_j of degree $2^{l-j}\cdot n$ so that C_{j-1} is smooth, but the estimate still holds. In fact, the estimate will hold if both $\widetilde{F}_j$ and F_j have the same sign at each point of $V(g_1, \ldots, g_j,\ F_{j+1}, \ldots, F_l)$, and this property is preserved under perturbation (as $\widetilde{F}_j$ is nonzero at these points).

We now iterate the Lemma 6.11 to estimate the number of solutions to a system of master functions as in Theorem 6.7.

$$\begin{aligned}
|V(g_1, g_2, \ldots, g_l)| \ &\leq\ |V(g_1, g_2, \ldots, g_{l-1},\ F_l)| + \text{ubc}(C_l) \\
&\leq\ |V(g_1, \ldots, g_{l-2},\ F_{l-1}, F_l)| + \text{ubc}(C_l) + \text{ubc}(C_{l-1}) \qquad (6.22)\\
&\leq\ |V(F_1, F_2, \ldots, F_l)| + \text{ubc}(C_l) + \cdots + \text{ubc}(C_1)\,.
\end{aligned}$$

Here, $V(\cdots) = V_{M^{\mathbb{R}}_{\mathcal{H}}}(\cdots)$, the common zeroes in $M^{\mathbb{R}}_{\mathcal{H}}$. Let $\text{ubc}_\Delta(C)$ count the number of unbounded components of the curve C in the positive chamber Δ_p and $V_\Delta(\cdots)$ be the common zeroes in Δ_p. Then the analog of (6.22) holds in Δ_p.

Lemma 6.12. *With these definitions, we have the estimates*

(1) $|V_\Delta(F_1, \ldots, F_l)| \ \leq\ |V(F_1, \ldots, F_l)| \ \leq\ 2^{\binom{l}{2}} n^l$.
(2) $\text{ubc}_\Delta(C_j) \ \leq\ \frac{1}{2}\binom{1+l+n}{j}\cdot 2^{\binom{l-j}{2}} n^{l-j}$.
(3) $\text{ubc}(C_j) \ \leq\ \frac{1}{2}\binom{1+l+n}{j}\cdot 2^{\binom{l-j}{2}} n^{l-j}\cdot 2^j$.

The first statement follows from Lemma 6.11 and Bézout's Theorem. For the second, recall that $C_j = Z_{j-1} \cap V(F_{j+1}, \dots, F_l)$. Since $F_1, \dots, F_l$ are general, this is transverse at each point of $\overline{Z_{j-1}} \cap \mathcal{H}$, and so C_j has one branch in Δ incident on such a point, as Z_{j-1} has a unique branch in Δ incident on such a point. Since each unbounded component of C_j has two ends, the number of unbounded components is at most half the number of points in the boundary of Δ lying in $\overline{Z_{j-1}} \cap \mathcal{H}$. By Lemma 6.10, these are points in the closure of Δ in some codimension j flat where $F_{j+1}, \dots, F_l$ vanish. The bound in the second statement of 6.12 is simply $\frac{1}{2}$ multiplied by the product of $\binom{1+l+n}{j}$ and $2^{\binom{l-j}{2}} n^{l-j}$. That is, by the number of codimension j flats in $\mathcal{H}$ (some of which meet the boundary of Δ) multiplied by the Bézout number of the system $F_{j+1} = \cdots = F_l = 0$.

Note that the bound in (2) holds for any chamber of $M_{\mathcal{H}}^{\mathbb{R}}$. We get the bound in (3) by noting that in the neighborhood of any point in the interior of a codimension j flat of $\mathcal{H}$, the complement has at most 2^j chambers, and so each point of C_j at the boundary of $M_{\mathcal{H}}^{\mathbb{R}}$ can contribute at most 2^j such ends. We get this estimate because the hyperplanes of $\mathcal{H}$ intersect transversally.

The complement $M_{\mathcal{H}}^{\mathbb{R}}$ of the hyperplane arrangement consists of many chambers. The first bound of Theorem 6.7 is in fact a bound for the number of solutions in any chamber, while the second bound is for the number of solutions in all chambers. This is smaller than what one may naively expect. The number of chambers in a generic arrangement of $1+n+l$ hypersurfaces in $\mathbb{R}^l$ is

$$\binom{l+n+2}{l} + \binom{l+n+2}{l-2} + \cdots + \begin{cases} \binom{l+n+2}{0} & \text{if } l \text{ is even} \\ \binom{l+n+2}{1} & \text{if } l \text{ is odd} \end{cases} .$$

This exceeds 2^l, which is the number of chambers cut out by l hyperplanes. Thus, we would naively expect that the ratio between $\mathrm{ubc}_\Delta(C_j)$ and $\mathrm{ubc}(C_j)$ to be this number, rather than the far smaller number 2^j. This is the source for the mild difference between the two estimates in Theorem 6.6.

We may combine the estimates of Lemma 6.12 with (6.22) to estimate the numbers $|V_\Delta(g_1, \dots, g_l)|$ and $|V(g_1, \dots, g_l)|$,

$$\begin{aligned} |V_\Delta(g_1, \dots, g_l)| &\leq \frac{1}{2} \sum_{j=1}^{l} \binom{1+l+n}{j} \cdot 2^{\binom{l-j}{2}} n^{l-j} + 2^{\binom{l}{2}} n^l , \quad \text{and} \\ |V(g_1, \dots, g_l)| &\leq \frac{1}{2} \sum_{j=1}^{l} \binom{1+l+n}{j} \cdot 2^{\binom{l-j}{2}} n^{l-j} \cdot 2^j + 2^{\binom{l}{2}} n^l . \end{aligned}$$

It is not hard to show the estimate [**17**, Eq.(3.4)]

$$\binom{1+l+n}{j} \cdot 2^{\binom{l-j}{2}} n^{l-j} \leq \frac{2^{j-1}}{j!} 2^{\binom{l}{2}} n^l ,$$

so that these estimates become

$$\begin{aligned} |V_\Delta(g_1, \dots, g_l)| &\leq \left(\tfrac{1}{2} \textstyle\sum_{j=1}^{l} \frac{2^{j-1}}{j!} + 1 \right) 2^{\binom{l}{2}} n^l \leq \tfrac{e^2+3}{4} 2^{\binom{l}{2}} n^l , \quad \text{and} \\ |V(g_1, \dots, g_l)| &\leq \left(\tfrac{1}{2} \textstyle\sum_{j=1}^{l} \frac{2^{2j-1}}{j!} + 1 \right) 2^{\binom{l}{2}} n^l \leq \tfrac{e^4+3}{4} 2^{\binom{l}{2}} n^l . \end{aligned}$$

This implies Theorem 6.7 and thus the fewnomial bounds of Theorem 6.6.

6.3. Dense fewnomials

The bounds of Theorem 6.6 can be lowered if we know more about the structure of the set of exponents. We illustrate this in a case which generalizes the near circuits of [**13**], and close with some open questions involving fewnomial bounds.

In our reduction to the a system of master functions, we had polynomials $p_i(x^{a_{1+n}}, \ldots, x^{a_{l+n}})$ which, after substituting $y_i = x^{a_{i+n}}$, became degree one polynomials. Dense fewnomials are when the exponent vectors $a_{1+n}, \ldots, a_{l+n}$ have the additional structure of being the d-fold sum of a collection of k vectors. We sketch the results of [**124**], which develops a version of Gale duality and gives bounds for dense fewnomials.

A collection $\mathcal{A} \subset \mathbb{Z}^n$ of exponent vectors is *(d, k)-dense* if there are integers d, k such that $\mathcal{A}$ admits a decomposition of the form

$$\mathcal{A} \;=\; \psi(d\triangle^k \cap \mathbb{Z}^k) \bigcup \mathcal{W}\,, \tag{6.23}$$

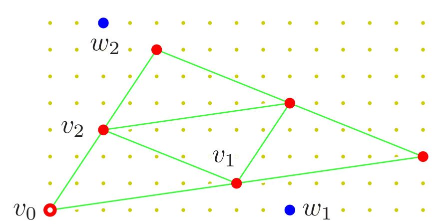
where $\mathcal{W}$ consists of n affinely independent vectors, $\psi\colon \mathbb{Z}^k \to \mathbb{Z}^n$ is an affine-linear map, and $\triangle^k$ is the unit simplex in $\mathbb{R}^k$. A *(d, k)-dense fewnomial* is a Laurent polynomial whose support $\mathcal{A} \subset \mathbb{Z}^n$ is (d, k)-dense.

A general (d, k)-dense set $\mathcal{A}$ has the form,

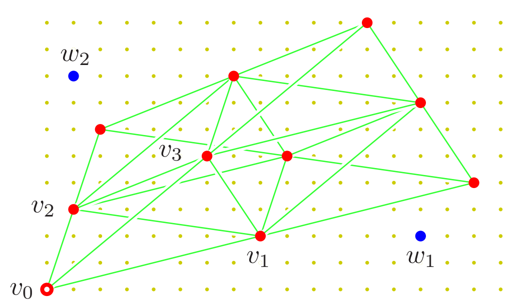
$$\mathcal{A} \;:=\; \Bigl\{v_0 \;+\; \sum_{m=1}^{k} d_m v_m \mid 0 \le d_m\,,\ \sum_i d_i \le d\Bigr\} \bigcup \mathcal{W}\,, \tag{6.24}$$

where $\mathcal{W} = \{w_1, \ldots, w_n\} \subset \mathbb{Z}^n$ is affinely independent and $v_0, v_1, \ldots, v_k$ are integer vectors. Here is an example of such a set $\mathcal{A}$ in $\mathbb{Z}^2$ ($v_0 = (0, 0)$ is the open circle).

(6.25)



For this, $n = l = d = 2$, $\mathcal{W} = \{(9, 0), (2, 7)\}$ and $v_1 = (7, 1)$ and $v_2 = (2, 3)$. Here is a $(2, 3)$-dense set in $\mathbb{Z}^2$.



The numbers of nondegenerate positive and nondegenerate real solutions to a system of dense fewnomials have bounds similar to those for ordinary fewnomials.

Theorem 6.13. *Suppose that $\mathcal{A} \subset \mathbb{Z}^n$ is (d, k)-dense. Then a system*

$$f_1(x_1, \ldots, x_n) \;=\; f_2(x_1, \ldots, x_n) \;=\; \cdots \;=\; f_n(x_1, \ldots, x_n) \;=\; 0$$

of real polynomials with support $\mathcal{A}$ has fewer than

$$\frac{e^2+3}{4}\,2^{\binom{k}{2}}\,n^k\cdot d^k$$

nondegenerate positive solutions. If the affine span of $\mathcal{A}$ is a sublattice of $\mathbb{Z}^n$ with odd index, then the number of nondegenerate real solutions is less than

$$\frac{e^4+3}{4}\,2^{\binom{k}{2}}\,n^k\cdot d^k\,.$$

This is proven in nearly the same way as Theorem 6.6, except that a modified version of Gale duality for (d,k)-dense fewnomials is used in which the degree one polynomials p_i in l variables are replaced by degree d polynomials in k variables.

When $d=1$, a dense fewnomial is an ordinary fewnomial with $k=l$, and these bounds reduce to the fewnomial bound of Theorem 6.6. When $d>1$, these bounds are a significant improvement over Theorem 6.6 as the number of monomials is $\binom{d+k}{k}+n$, so that the parameter l is $\binom{d+k}{k}-1$. When $k=1$, a (d,k)-dense fewnomial is a *near circuit* of [**13**], and the bound of Theorem 6.13 extends the bound for near circuits.

Despite this work on fewnomial bounds which refine Khovanskii's breakthrough, many important questions remain open. For example, while the bound for $X(2,2)$ has dropped from 5184 to $\frac{e^2+3}{4}2^{\binom{2}{2}}2^2\approx 20.78$, and in fact the proof gives the bound

$$2^{\binom{2}{2}}2^2\;+\;\tfrac{1}{2}2^{\binom{1}{2}}2^1\cdot 5\;+\;\tfrac{1}{2}2^{\binom{0}{2}}2^0\cdot 5\;=\;15.5\,,$$

(a polygon with five $(2+2+1)$ edges has five vertices). However, the best construction of a system of two polynomials in two variables with five monomials remains essentially that of Haas (5.10) which has five solutions. Thus all we currently know is that

$$5\;\le\;X(2,2)\;\le\;15\,.$$

New ideas are needed to improve these bounds for $X(l,n)$, as the examples in this section are nearly the limit of the ideas that led to Theorem 6.6. As important as proven bounds are constructions of fewnomial systems with many positive solutions. In fact, a lack of examples or constructions, particularly when n is fixed or when it has moderate size with respect to l, prevents us from understanding how good the known upper bound for $X(l,n)$ is.

CHAPTER 7

Lower Bounds for Sparse Polynomial Systems

In Chapters 5 and 6 we studied upper bounds on the number of real solutions to systems of polynomial equations. We turn now to the other side of the inequality (1.2). That is, results which guarantee the existence of real solutions by establishing lower bounds on the number of real solutions to certain geometric problems or systems of equations. While some work on lower bounds is quite sophisticated, we use only elementary topology and the structure of real toric varieties to develop a theory of lower bounds for sparse polynomial systems.

In Chapter 1, we discussed how work of Welschinger [**162**], Mikhalkin [**99**], and of Kharlamov, Itenberg, and Shustin [**77, 78**] combined to show that there is a nontrivial lower bound W_d for the number of real rational curves of degree d interpolating $3d-1$ points in $\mathbb{RP}^2$. Similar results were found by Pandharipande, Solomon, and Walcher [**111**] for rational curves on the quintic three-fold. For example, there are at least 15 real lines (out of 2875 complex lines) on a real smooth quintic hypersurface in $\mathbb{RP}^4$.

We also discussed earlier work of Eremenko and Gabrielov [**45**], who found a similar result for the number of real solutions to the inverse Wronski problem. They gave numbers $\sigma_{m,p} > 0$ for $m+p$ odd, and proved that if Φ is a real polynomial of degree mp then there are at least $\sigma_{m,p}$ different m-dimensional spaces of real polynomials of degree $m+p-1$ with Wronskian Φ. We will prove this in Section 8.2.

These spectacular results are but the beginning of what could be an important story for the applications of mathematics. That is, a useful theory of natural geometric problems or systems of structured polynomial equations possessing a *lower bound* on their number of real solutions. The point is that nontrivial lower bounds imply the existence of real solutions to systems of equations. A beginning of the interaction between applications and this theory of lower bounds is found in the work of Fiedler-Le Touzé [**49**] discussed in Section 1.5.

The first steps toward a theory of lower bounds for sparse polynomial systems were developed jointly with Soprunova in [**138**]. That theory concerned unmixed systems, such as those covered by Kushnirenko's Theorem. It remains an important open problem to develop a theory for mixed systems such as those which appear in Bernstein's Theorem. Example 7.16 is a step in this direction. Also, the computations reported in Chapters 13 and 14 suggest there are many systems possessing lower bounds for which there is currently no theoretical or even conjectural explanation.

This chapter develops the basic theory of lower bounds on the number of real solutions to systems of sparse polynomial equations, while the next chapter will be devoted to instances and applications of this theory.

Like the fewnomial upper bounds, which used the topological argument of the Khovanskii-Rolle Theorem, this theory of lower bounds is based on topology, specifically the notion of mapping degree. There are three steps in this theory.

(1) Realize a polynomial system as the fiber $g^{-1}(p)$ of a map

$$g : X_{\mathcal{A}} \longrightarrow \mathbb{P}^n .$$

If we restrict this to the real points of $X_{\mathcal{A}}$ and $\mathbb{P}^n$,

$$g : Y_{\mathcal{A}} \longrightarrow \mathbb{RP}^n , \tag{7.1}$$

then the number of real solutions, $\#g^{-1}(p)$, is bounded below by the mapping degree of g, if the spaces in (7.1) are orientable.

(2) Characterize when the spaces in (7.1) are orientable. This is from [**139**], which extends work of Nakayama and Nishimura [**109**].

(3) Use toric degenerations to compute the mapping degree in some cases.

These steps are presented in the following three sections. These systems are unmixed in that the polynomials all have the same Newton polytope. We close with a discussion of open problems and an example of a mixed system with a provable lower bound.

7.1. Polynomial systems as fibers of maps

Let $\mathcal{A} \subset \mathbb{Z}^n$ be a finite set of exponent vectors and consider a system of real polynomial equations with support $\mathcal{A}$,

$$f_1(x_1, \ldots, x_n) \;=\; f_2(x_1, \ldots, x_n) \;=\; \cdots \;=\; f_n(x_1, \ldots, x_n) \;=\; 0 . \tag{7.2}$$

Lemma 3.5 explains how the solution set to a system (7.2) of polynomials with support $\mathcal{A} \subset \mathbb{Z}^n$ is the pullback along $\varphi_{\mathcal{A}}$ of a linear section,

$$\varphi_{\mathcal{A}}^{-1}(L \cap X_{\mathcal{A}}) , \tag{7.3}$$

where $\varphi_{\mathcal{A}} \colon \mathbb{T}^n \ni x \mapsto [x^a \mid a \in \mathcal{A}] \in \mathbb{P}^{\mathcal{A}}$ is the parameterization map and $X_{\mathcal{A}}$ is the closure of its image, a toric variety. This correspondence is a bijection when $\mathcal{A}$ is primitive.

In this case, we can assume that $\mathcal{A}$ consists of all the integer points in its convex hull. Indeed, if we set $\mathcal{A}' = \mathrm{conv}(\mathcal{A}) \cap \mathbb{Z}^n$, then $\mathcal{A}$ and $\mathcal{A}'$ have the same convex hull and are both primitive, so by Kushnirenko's Theorem, the toric varieties $X_{\mathcal{A}}$ and $X_{\mathcal{A}'}$ have the same degrees in their natural projective embeddings into $\mathbb{P}^{\mathcal{A}}$ and $\mathbb{P}^{\mathcal{A}'}$. Furthermore $X_{\mathcal{A}}$ is the image of $X_{\mathcal{A}'}$ under the natural coordinate projection $\pi \colon \mathbb{P}^{\mathcal{A}'} - \to \mathbb{P}^{\mathcal{A}}$, and so there is a bijection between the linear sections

$$L \cap X_{\mathcal{A}} \qquad \text{and} \qquad \pi^{-1}(L) \cap X_{\mathcal{A}'} .$$

Since π commutes with $\varphi_{\mathcal{A}'}$ and $\varphi_{\mathcal{A}}$, we may replace the first linear section (7.2) by the second $\pi^{-1}(L) \cap X_{\mathcal{A}'}$. Thus, in (7.2), we may assume that $\mathcal{A} = \mathrm{conv}(\mathcal{A}) \cap \mathbb{Z}^n$. This means that the toric variety $X_{\mathcal{A}}$ is normal, which implies that it is smooth in codimension one and therefore its singular locus has codimension at least two.

Remark 7.1. The correspondence of Lemma 3.5 is a bijection of complex solutions only when $\mathcal{A}$ is primitive. It is a bijection between real solutions to the system of polynomials and real points in the linear section (7.3) only when $\varphi_{\mathcal{A}}$ is injective on $\mathbb{T}^n_{\mathbb{R}}$. This is equivalent to the lattice index $[\mathbb{Z}\mathcal{A} \colon \mathbb{Z}^n]$ being odd, so that the kernel of the map $\varphi_{\mathcal{A}}$ on $\mathbb{T}^n_{\mathbb{R}}$ is $\{1\}$. While there is not a bijection when the index is even, it is a combinatorial problem to determine the real solutions to the polynomial system given the real points in the linear section. If the lattice index $[\mathbb{Z}\mathcal{A}\colon \mathbb{Z}^n]$ is odd so that this correspondence is a bijection on real solutions, we may replace $\mathbb{Z}^n$ by $\mathbb{Z}\mathcal{A}$ so that $\mathcal{A}$ is primitive, without changing the number of real solutions. Therefore, we will assume that $\mathcal{A}$ is primitive for the remainder of this chapter.

To use mapping degree, we need to realize $L \cap X_{\mathcal{A}}$ as the fiber of a map. To that end, let $E \subset L$ be a hyperplane in L that does not meet $X_{\mathcal{A}}$ and $M \simeq \mathbb{P}^n$ a linear space that is disjoint from E. Then E has codimension $n{+}1$ in $\mathbb{P}^{\mathcal{A}}$ and the set of codimension n planes containing E is naturally identified with M, with each plane associated to its unique intersection with M. The *linear projection*

$$\pi_E \;:\; \mathbb{P}^{\mathcal{A}} \setminus E \;\longrightarrow\; M \simeq \mathbb{P}^n \tag{7.4}$$

sends a point $y \in \mathbb{P}^{\mathcal{A}} \setminus E$ to the intersection of M with span of E and y. Figure 7.1 illustrates this in $\mathbb{P}^3$, where E and M are lines. Write $\pi_E \colon \mathbb{P}^{\mathcal{A}} - \to \mathbb{P}^n$, using the

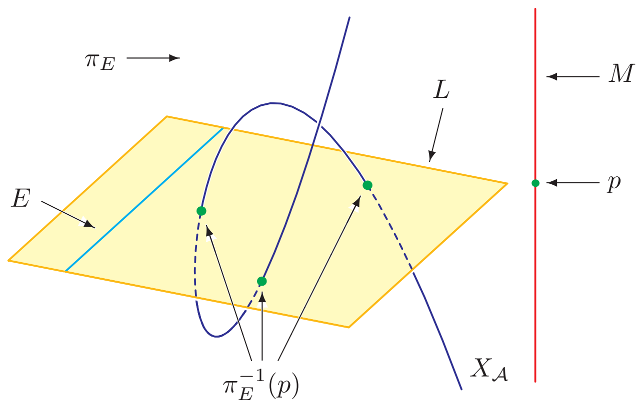


FIGURE 7.1. A linear projection π with center E.

broken arrow to indicate that π_E is a rational map that is not defined on all of $\mathbb{P}^{\mathcal{A}}$.

Write π for the restriction of the linear projection π_E to the toric variety $X_{\mathcal{A}}$. If $p := L \cap M$ is the point where L meets M, then

$$L \cap X_{\mathcal{A}} \;=\; \pi^{-1}(p)\,.$$

This is also illustrated in Figure 7.1, where $X_{\mathcal{A}}$ is a rational normal cubic curve. To study the real solutions/real points in the linear section, define the real toric variety $Y_{\mathcal{A}} := X_{\mathcal{A}} \cap \mathbb{RP}^{\mathcal{A}}$, which is also the closure in $\mathbb{RP}^{\mathcal{A}}$ of $\varphi_{\mathcal{A}}(\mathbb{T}^n_{\mathbb{R}})$. This is a consequence of our assumption that $\mathcal{A}$ is primitive. Then the real solutions to our system of polynomials correspond to points of the real linear section $L \cap Y_{\mathcal{A}}$. Equivalently, we restrict the projection π further to a map

$$g \;:\; Y_{\mathcal{A}} \;\to\; \mathbb{RP}^n\,,$$

and consider points in the fiber $g^{-1}(p)$, where $x \in \mathbb{RP}^n$. Since $Y_{\mathcal{A}}$ and $\mathbb{RP}^n$ have the same dimension, the map g may have a degree. For this, $Y_{\mathcal{A}}$ and $\mathbb{RP}^n$ must be orientable and we must fix orientations of $Y_{\mathcal{A}}$ and of $\mathbb{RP}^n$. Suppose this is the case. For a regular value $p \in \mathbb{RP}^n$ of g and a point $y \in g^{-1}(p)$, the differential map dg_y is a bijection between the tangent spaces $T_yY_{\mathcal{A}}$ and $T_p\mathbb{RP}^n$. Set $\operatorname{sign}_g(y) := 1$ if the differential dg_y preserves the orientations of the tangent spaces and $\text{sign}_g(y) := -1$ if dg_y reverses the orientations. The *mapping degree* $\text{mdeg}(g)$ of g is the sum

$$\sum_{y\in g^{-1}(p)} \text{sign}_g(y)\,.$$

The value of this sum does not depend upon the choice of a regular value x, as the target space $\mathbb{RP}^n$ is connected, but it does depend upon the choice of orientation. The value of the notion of mapping degree is the following.

THEOREM 7.2. *The number of points in a fiber $g^{-1}(p)$, for $p \in \mathbb{RP}^n$ a regular value of g, is at least the absolute value $|\,\text{mdeg}(g)|$ of the mapping degree of g.*

COROLLARY 7.3. *The absolute value of the mapping degree $|\,\text{mdeg}(g)|$ is a lower bound on the number of solutions to a system of polynomial equations modeled as a fiber of the map g.*

It is sufficient, but not necessary, that $\text{mdeg}(g) \neq 0$ for there to be a lower bound. Nonnecessity is apparently illustrated by the computer experiment on the Schubert problem $\square^9 = 42$ in the Grassmannian of 3-planes in $\mathbb{C}^6$ reported in Table 13.4. This experiment determined the numbers of real points in a fiber of the Wronski map on the real Grassmannian, which has mapping degree $\sigma_{3,3} = 0$, yet there were always at least two real points in each fiber.

7.2. Orientability of real toric varieties

To apply Corollary 7.3 to deduce a lower bound on the number of real solutions to a system of polynomials realized as the fibers of a map g, we apparently need both $\mathbb{RP}^n$ and $Y_{\mathcal{A}}$ to be orientable manifolds. There are *a priori* problems with this approach. Real projective space $\mathbb{RP}^n$ is orientable if and only if n is odd, toric varieties are typically singular, and we also need to understand the orientability of $Y_{\mathcal{A}}$. The first two objections are easily handled, and we will characterize the orientability of $Y_{\mathcal{A}}$ using a little algebraic topology, following [**139**].

For the nonorientability of of $\mathbb{RP}^n$, recall that the n-sphere $\mathbb{S}^n$ is the oriented double cover of $\mathbb{RP}^n$. Similarly, the sphere $\mathbb{S}^{\mathcal{A}}$ in $\mathbb{R}^{\mathcal{A}}$ is the oriented double cover of $\mathbb{RP}^{\mathcal{A}}$. If $Y^+_{\mathcal{A}} \subset \mathbb{S}^{\mathcal{A}}$ is the pullback of $Y_{\mathcal{A}} \subset \mathbb{RP}^{\mathcal{A}}$ along this double cover, and $g^+ \colon Y^+_{\mathcal{A}} \to \mathbb{S}^n$ is the pullback of the map g, we get a commutative diagram:

$$\begin{array}{ccccccc} g^+ & : & Y^+_{\mathcal{A}} & \subset & \mathbb{S}^{\mathcal{A}} & \overset{\pi^+_E}{\dashrightarrow} & \mathbb{S}^n \\ & & \downarrow & & \downarrow & & \downarrow \\ g & : & Y_{\mathcal{A}} & \subset & \mathbb{RP}^{\mathcal{A}} & \overset{\pi_E}{\dashrightarrow} & \mathbb{RP}^n \end{array} \tag{7.5}$$

Passing to the double cover does not affect the number of points in fibers, for if $p', p'' \in \mathbb{S}^n$ are the antipodal points covering a point $p \in \mathbb{RP}^n$, then each point in the fiber $g^{-1}(p)$ is covered by a pair of antipodal points, one in each fiber of g^+ above the points p' and p''. We will call $Y^+_{\mathcal{A}}$ a *spherical toric variety*.

Next, while the toric variety $Y_{\mathcal{A}}$ is typically singular, its singular points have codimension at least two. Indeed, as we assumed that $\mathcal{A}$ is the set of lattice points within its convex hull Δ, the complex toric variety $X_{\mathcal{A}}$ is normal and hence its singular locus has codimension at least two. Thus the singular locus of $Y_{\mathcal{A}}$ will also have codimension at least two. These facts hold for the spherical toric variety $Y^+_{\mathcal{A}}$.

The image under g^+ of the singular locus of $Y^+_{\mathcal{A}}$ in $\mathbb{S}^n$ has codimension at least two. If U is its complement, then it is connected and $g^+\colon (g^+)^{-1}(U) \to U$ is a proper map between manifolds of the same dimension and thus it will have a well-defined mapping degree, if $(g^+)^{-1}(U)$ is oriented. So it suffices to understand the orientability of $(g^+)^{-1}(U)$, which is equivalent to the orientability of the smooth locus of $Y^+_{\mathcal{A}}$, as they differ only in codimension two or greater.

If $Y_{\mathcal{A}}$ is oriented and n is odd, then Corollary 7.3 applies so there is no need to pass to the double cover of $g\colon Y_{\mathcal{A}} \to \mathbb{RP}^n$. On the other hand, it is no loss of generality to pass to this double cover, as orientations pull back. We will characterize when the smooth points of either $Y_{\mathcal{A}}$ or $Y^+_{\mathcal{A}}$ are orientable.

This characterization is given in [**139**] following the ideas of [**109**]. It uses the dual description of the convex hull Δ of $\mathcal{A}$ in terms of intersections of half-spaces, or *facet inequalities*, which have the form

$$\Delta \;=\; \bigcap_{F \text{ a facet}} \{x \in \mathbb{R}^n \;\mid\; v_F^T \cdot x \;\geq\; -b_F\}\,, \tag{7.6}$$

where, for each facet F of Δ, $v_F \in \mathbb{Z}^n$ is the unique shortest inward-pointing integer normal to F and b_F is the (signed) lattice distance of F from the origin.

For example, the symmetric hexagon of Example 3.4 is defined by the six facet inequalities, which we collect in matrix form,

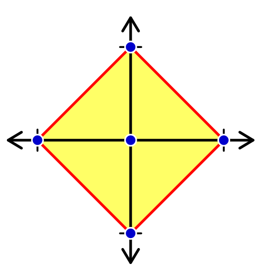
$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \;\geq\; \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}.$$

Let $\varepsilon_F \in \{\pm 1\}^n$ be the element whose ith coordinate is -1 raised to the power of the ith coordinate of v_F. That is, ε_F is the image of v_F under the natural map $\mathbb{Z}^n \twoheadrightarrow \{\pm 1\}^n$ given by reduction modulo 2.

THEOREM 7.4. *The smooth locus of $Y_{\mathcal{A}}$ is orientable if and only if there is a basis for $\{\pm 1\}^n$ such that for every facet F of Δ, the element ε_F is the product of an odd number of basis elements.*

REMARK 7.5. Lemma 2.3 of [**139**] gives the following equivalent formulations of the condition of Theorem 7.4. There is a basis of $\{\pm 1\}^n$ such that every facet element ε_F is the product of an odd number of basis elements if and only if there is an independent subset E of these facet elements such that every ε_F is a product of an odd number of elements of E, if and only if no facet element is the product of an even number of facet elements.

EXAMPLE 7.6. Consider the two-dimensional crosspolytope,



$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -1 & -1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \;\geq\; \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}.$$

For each of its facets, the element ε_F is $(-1,-1)$, so the crosspolytope satisfies Theorem 7.4; we may take the basis of $\{\pm 1\}^2$ to be $(-1,-1)$ and $(-1,1)$.

We can see this directly, as $Y_{\mathcal{A}}$ is the orientable double pillow. Half of it (a pillow) is obtained by gluing two copies of the crosspolytope along their corresponding edges. This gives four singular corners, which are glued to the four corners of a second pillow, so that the corners are locally the apices of quadratic cones. Figure 7.2 shows a double pillow projected from its ambient $\mathbb{RP}^4$ into $\mathbb{RP}^3$, displaying one pillow and part of the second near the four singular points. Each pillow is

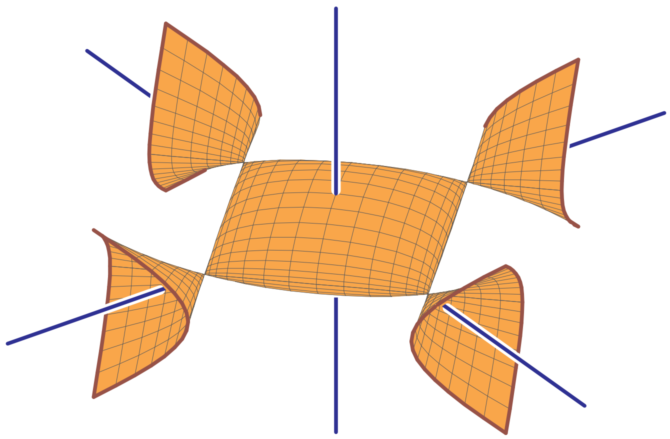


FIGURE 7.2. Double pillow in $\mathbb{R}^3$.

homeomorphic to $\mathbb{S}^2$ and is therefore orientable.

The characterization of orientability of the smooth points of $Y^+_{\mathcal{A}}$ is similar. For each facet F of Δ, set $\varepsilon^+_F := ((-1)^{b_F}, \varepsilon_F) \in \{\pm 1\}^{1+n}$.

THEOREM 7.7. *The smooth locus of $Y^+_{\mathcal{A}}$ is orientable if and only if there is a basis for $\{\pm 1\}^{1+n}$ such that for every facet F of Δ, the element ε^+_F is the product of an odd number of basis elements.*

EXAMPLE 7.8. The toric variety $Y_{\mathcal{A}}$ associated to the symmetric hexagon of Example 3.4 is not orientable as the product of the three elements ε_F, $(-1,1)$, $(1,-1)$, and $(-1,-1)$, is the identity in $\{\pm 1\}^2$. However $Y^+_{\mathcal{A}}$ is orientable, as the elements ε^+_F are $(-1,1,-1)$, $(1,-1,-1)$, and $(-1,-1,-1)$, which form a basis.

Elementary algebraic topology tells us that a connected n-dimensional manifold Y is orientable if and only if its top homology group $H_n(Y,\mathbb{Z})$ is nontrivial, if and only if Y has a fundamental cycle in homology. We prove Theorems 7.4 and 7.7 by computing the top homology groups of $Y_{\mathcal{A}}$ and $Y^+_{\mathcal{A}}$ using cellular chain complexes from explicit descriptions of $Y_{\mathcal{A}}$ and $Y^+_{\mathcal{A}}$ as cell complexes, similar to the description of the gluing of four crosspolytopes to form the double pillow in Example 7.6. Since $Y_{\mathcal{A}}$ and $Y^+_{\mathcal{A}}$ are smooth in codimension one, this is also the top homology group of their smooth points.

These cell complexes arise from the structure of real toric varieties associated to lattice polytopes, which may be found in [**52**, Ch. 4] and [**145**]. Under $\varphi_{\mathcal{A}}$, the group of units $\{\pm 1\}^n \subset \mathbb{T}^n_{\mathbb{R}}$ acts on $\mathbb{RP}^{\mathcal{A}}$ via

$$g.[x_a \mid a \in \mathcal{A}] \;=\; [g^a \cdot x_a \mid a \in \mathcal{A}]\,, \tag{7.7}$$

where $g \in \{\pm 1\}^n$. This action restricts to an action on $Y_{\mathcal{A}}$ and the orbit space $Y_{\mathcal{A}}/\{\pm 1\}^n$ is naturally identified with the polytope Δ.

The *nonnegative orthant* $\mathbb{RP}^{\mathcal{A}}_{\geq 0}$ of $\mathbb{RP}^{\mathcal{A}}$ consists of points having a representative with all coordinates nonnegative. Each orbit of $\{\pm 1\}^n$ on $Y_{\mathcal{A}}$ meets the nonnegative orthant in a unique point. Thus $Y_{\mathcal{A}} \cap \mathbb{RP}^{\mathcal{A}}_{\geq 0}$ may be identified with the orbit space. This *nonnegative part* of the toric variety $Y_{\mathcal{A}}$ is mapped homeomorphically to the polytope under the algebraic moment map $\mu_{\mathcal{A}}$, which is defined by

$$\mu_{\mathcal{A}} \; : \; [x_a \mid a \in \mathcal{A}] \;\longmapsto\; \frac{\sum_{a\in\mathcal{A}} |x_a| \cdot a}{\sum_{a\in\mathcal{A}} |x_a|} \; .$$

Birch's Theorem from algebraic statistics [**1**, p. 168] implies that this is a homeomorphism from the nonnegative part of $Y_{\mathcal{A}}$ to the polytope Δ. The fibers $\mu_{\mathcal{A}} \colon Y_{\mathcal{A}} \to \Delta$ are the orbits of $\{\pm 1\}^n$. Identifying the nonnegative part of $Y_{\mathcal{A}}$ with the polytope Δ using $\mu_{\mathcal{A}}$ yields a description of the topological space $Y_{\mathcal{A}}$ as a quotient

$$Y_{\mathcal{A}} \;=\; \bigl(\Delta \times \{\pm 1\}^n\bigr)/\sim \; , \tag{7.8}$$

where $(g,x) \sim (g,y)$ if and only if $x = y$ and $g.x = g.y$ under the action (7.7). This quotient realizes $Y_{\mathcal{A}}$ as a cell complex. Each face F of Δ corresponds to the intersection of the nonnegative part of $Y_{\mathcal{A}}$ with the coordinate subspace

$$\mathbb{P}^F \;:=\; \{x \in \mathbb{P}^{\mathcal{A}} \mid x_a = 0 \text{ if } a \notin F\} \, .$$

An element $g \in \{\pm 1\}^n$ fixes $\mathbb{P}^F$ pointwise if the element $g^a \in \{\pm 1\}$ does not depend upon the choice of $a \in F \cap \mathcal{A}$. Let $G_F \subset \{\pm 1\}^n$ be the subgroup consisting of these elements fixing $\mathbb{P}^F$ pointwise. When F is a facet, $G_F = \{1, \varepsilon_F\} = \langle \varepsilon_F \rangle$. If we let F' be the relative interior of a face F of Δ, then

$$\Delta \;=\; \coprod_F F'$$

is a realization of Δ—which is homeomorphic to a ball—as a cell complex, and

$$Y_{\mathcal{A}} \;=\; \coprod_F F' \times \bigl(\{\pm 1\}^n / G_F\bigr)$$

is a realization of $Y_{\mathcal{A}}$ as a cell complex.

Set Δ° to be the union of Δ' and F', for F a facet of Δ, which is the complement of the codimension two skeleton of Δ. Then

$$Y^\circ_{\mathcal{A}} \;:=\; \Delta^\circ \times \{\pm 1\}^n/\sim$$

is the union of all cells of dimension n and dimension $n-1$ in $Y_{\mathcal{A}}$. This consists of smooth points of $Y_{\mathcal{A}}$, and its orientability is equivalent to the orientability of the smooth points of $Y_{\mathcal{A}}$, as these two sets differ only in codimension two.

Proof of Theorem 7.4. The group $\{\pm 1\}^n$ acts transitively on the cells of $Y^\circ_{\mathcal{A}}$ of each dimension. Thus its connected components are isomorphic and $Y^\circ_{\mathcal{A}}$ is orientable if and only if each of its connected components is orientable. Thus $Y^\circ_{\mathcal{A}}$ is orientable if and only if its top integral homology group is nonzero.

We use the cellular chain complex associated to the cell decomposition of $Y^\circ_{\mathcal{A}}$,

$$Y^\circ_{\mathcal{A}} \;=\; \bigl(\Delta' \times \{\pm 1\}^n\bigr) \coprod \coprod_F F' \times \bigl(\{\pm 1\}^n / \langle \varepsilon_F \rangle\bigr) \, .$$

This is the chain complex

$$C_n \;\xrightarrow{\;\partial\;}\; C_{n-1} \, ,$$

where C_n is the free abelian group with generators $[g,\Delta]$ for $g \in \{\pm 1\}^n$, and C_{n-1} is the free abelian group with generators $[g\langle\varepsilon_F\rangle, F]$ for F a facet of Δ and $g\langle\varepsilon_F\rangle \in \{\pm 1\}^n/\langle\varepsilon_F\rangle$.

The top homology group $H_n(Y_{\mathcal{A}},\mathbb{Z})$ is the kernel of ∂. To compute it, suppose that Δ and its facets F are oriented so that

$$\partial\Delta \;=\; \sum_F F\,.$$

Consider an element Z in C_n,

$$Z \;=\; \sum_{g\in\{\pm 1\}^n} c_g \cdot [g,\Delta]\,, \tag{7.9}$$

where $c_g \in \mathbb{Z}$. Then

$$\partial(Z) \;=\; \sum_g c_g \sum_F [g\langle\varepsilon_F\rangle, F] \;=\; \sum_F \sum_{g\langle\varepsilon_F\rangle\in\{\pm 1\}^n/\langle\varepsilon_F\rangle} (c_g + c_{g\varepsilon_F})\cdot[g\langle\varepsilon_F\rangle, F]\,.$$

Thus $\partial(Z) = 0$ if and only if $c_g = -c_{g\varepsilon_F}$ for all $g \in \{\pm 1\}^n$ and facets F. Equivalently, $c_g = (-1)^k c_{g\varepsilon_{F_1}\cdots\varepsilon_{F_k}}$, when $F_1,\ldots,F_k$ are facets of Δ.

Suppose that there is a basis $e_1,\ldots,e_n$ of $\{\pm 1\}^n$ such that each ε_F is a product of an odd number of the basis elements e_i. For $g \in \{\pm 1\}^n$, set $c_g = 1$ if g is the product of an even number of the e_i and -1 if it is the product of an odd number of the e_i. Then $c_g = -c_{g\varepsilon_F}$ for all $g \in \{\pm 1\}^n$, and therefore the chain Z (7.9) is a nonzero element in the kernel of ∂, and $Y^\circ_{\mathcal{A}}$ is orientable.

Suppose there is no such basis of $\{\pm 1\}^n$. Let Z be a chain (7.9) that lies in the kernel of ∂. Then there is some ε_E for a facet E which is a product of an even number of elements ε_F for facets F. If not, then we can reduce the set $\{\varepsilon_F \mid F \text{ a facet of } \Delta\}$ to a linearly independent set and then extend it to a basis of $\{\pm 1\}^n$ in which every ε_F is a product of an odd number of basis elements. We get $\varepsilon_E = \varepsilon_{F_1}\cdots\varepsilon_{F_{2k}}$ and hence $1 = \varepsilon_E\varepsilon_{F_1}\cdots\varepsilon_{F_{2k}}$, so for every $g \in \{\pm 1\}^n$ we get

$$c_g \;=\; (-1)^{2k+1} c_{g\varepsilon_E\varepsilon_{F_1}\cdots\varepsilon_{F_{2k}}} \;=\; -c_g\,,$$

which implies that $c_g = 0$ and hence $\ker\partial = 0$ and so $Y^\circ_{\mathcal{A}}$ is nonorientable.

The same arguments using the group $\{\pm 1\}^{1+n}$ acting on $Y^+_{\mathcal{A}}$ prove Theorem 7.7.

Suppose that the real toric variety $Y_{\mathcal{A}}$ has an orientable lift $Y^+_{\mathcal{A}}$. Given any projection map π_E (7.4) whose center E is disjoint from the toric variety $X_{\mathcal{A}}$, write g for its restriction to $Y_{\mathcal{A}}$. Then the lift $g^+\colon Y^+_{\mathcal{A}} \to \mathbb{S}^n$ of g to $Y^+_{\mathcal{A}}$ has a well-defined mapping degree, whose absolute value we call the *real degree* of the map g. Its value is the following restatement of Corollary 7.3.

Corollary 7.9. *The real degree of g is a lower bound for the number of real solutions to polynomial systems arising as fibers of the map g.*

7.3. Degree from foldable triangulations

Theorems 7.4 and 7.7 provide a challenge: compute the real degree of a (or any) map g arising as a linear projection of a toric variety $Y_{\mathcal{A}}$ whose lift $Y^+_{\mathcal{A}}$ is orientable. We give a method based on toric degenerations to provide an answer to this question. It is by no means the only answer. We first describe this in terms of polynomial systems, and then give a proof which uses toric degenerations. The key notion is that of a foldable triangulation.

EXAMPLE 7.10. Consider the triangulation of the hexagon (the HSBC Bank symbol rotated 45° anti-clockwise) of Figure 7.3 with the points of $\mathcal{A}$ labeled 0, 1, and 2. Each triangle is colored by the orientation given by the cyclic order 0–1–2 of its vertices. Mapping the points of $\mathcal{A}$ to the corresponding vertices of the simplex defines a piecewise linear folding map whose mapping degree is $4 - 2 = 2$.

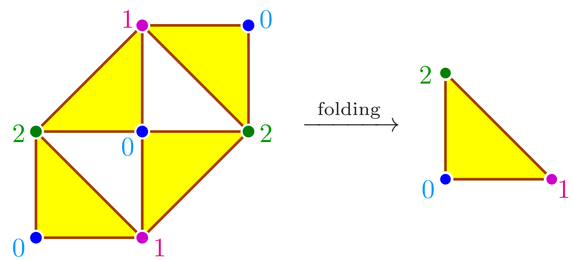
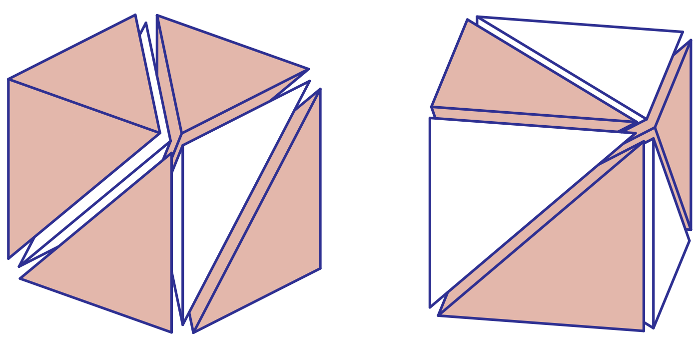


FIGURE 7.3. Foldable triangulation and folding map on the hexagon.

Suppose now that Δ is a lattice polytope and $\mathcal{A} = \Delta \cap \mathbb{Z}^n$. Let $\omega\colon \mathcal{A} \to \mathbb{N}$ be a function inducing a regular unimodular triangulation Δ_ω of Δ, as in Section 4.2. (Note that not every polytope Δ admits such a regular unimodular triangulation.) This triangulation Δ_ω is *foldable* if its facet simplices may be properly 2-colored, which is equivalent to there being a labeling of the vertices $\mathcal{A}$ of the triangulation with $n+1$ labels, where each simplex receives all $n+1$ labels [**80**]. (This is also called a balanced triangulation in the literature.) Let $\ell\colon \mathcal{A} \to \{0, 1, \ldots, n\}$ be the vertex labeling. Both the 2-coloring and the vertex-labeling are unique up to permuting the colors and labels. The difference in the number of simplices of different colors is the *signature* $\sigma(\omega)$ of the foldable triangulation. Up to a sign, it is the mapping degree of the piecewise linear combinatorial folding map from Δ to an n-simplex given by the labeling of $\mathcal{A}$, as in Figure 7.3.

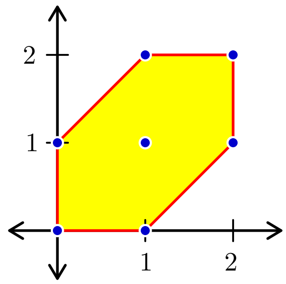
EXAMPLE 7.11. Here are two foldable triangulations of the unit cube in $\mathbb{R}^3$. The first has signature 2, while the second has signature 0.



A foldable triangulation with labeling $\ell\colon \mathcal{A} \to \{0, 1, \ldots, n\}$ and a choice of positive constants $\kappa = (\kappa_a \mid a \in \mathcal{A}) \in \mathbb{R}^{\mathcal{A}}_{>}$ defines a family of *Wronski polynomials*, $W_{\kappa,c}(x)$, which depend upon real parameters $c = (c_0, c_1, \ldots, c_n) \in \mathbb{R}^{n+1}$,

$$W_{\kappa,c}(x) \;=\; \sum_{a \in \mathcal{A}} c_{\ell(a)} \cdot \kappa_a x^a\,. \tag{7.10}$$

EXAMPLE 7.12. Suppose that $\mathcal{A}$ consists of the integer points in the hexagon,



$\kappa \in \mathbb{R}^{\mathcal{A}}_{>}$, and Δ_ω is the triangulation of Figure 7.3. A Wronski polynomial for this triangulation has the form

$$c_0\big(\kappa_{00} + \kappa_{11}xy + \kappa_{22}x^2y^2\big) \ + \ c_1\big(\kappa_{10}x + \kappa_{12}xy^2\big) \ + \ c_2\big(\kappa_{01}y + \kappa_{21}x^2y\big)\,.$$

A *system of Wronski polynomials* consists of n Wronski polynomials (7.10) where the coefficients c_i vary, but not the constants $\kappa \in \mathbb{R}^{\mathcal{A}}_{>}$. We show that a system of Wronski polynomials for a foldable regular triangulation Δ_ω will always have at least $\sigma(\omega)$ real solutions, when the corresponding spherical toric variety $Y^+_{\mathcal{A}}$ is orientable and a technical condition holds that we now explain.

As in Section 4.2, the function $\omega\colon \mathcal{A} \to \mathbb{N}$ defines an action of $\mathbb{T}$ on $\mathbb{P}^{\mathcal{A}}$,

$$t.[y_a \mid a \in \mathcal{A}] \ := \ [t^{\omega(a)}y_a \mid a \in \mathcal{A}]\,.$$

Let $E_{\omega,\kappa} \subset \mathbb{P}^{\mathcal{A}}$ be the linear space of codimension $n+1$ defined by the vanishing of the terms multiplying the coefficients c_i of a Wronski polynomial,

$$\Lambda_i(z) \ := \ \sum_{\ell(a)=i} \kappa_a z_a \qquad \text{for } i = 0, 1, \dots, n\,. \tag{7.11}$$

Then the technical condition is that the real points $t^{-1}.Y_{\mathcal{A}}$ of the toric degeneration do not meet the subspace $E_{\omega,\kappa}$ for $0 < t \leq 1$.

THEOREM 7.13. *Suppose that the set $\mathcal{A} = \Delta \cap \mathbb{Z}^n$ of integer points in a polytope Δ is primitive and $Y^+_{\mathcal{A}}$ is orientable. Let $\omega\colon \mathcal{A} \to \mathbb{N}$ be a function inducing a regular unimodular foldable triangulation Δ_ω of Δ with signature $\sigma(\omega)$, and let $\kappa \in \mathbb{R}^{\mathcal{A}}_{>}$. If $t^{-1}.Y_{\mathcal{A}} \cap E_{\omega,\kappa}$ is empty for all $0 < t \leq 1$, then a general system of Wronski polynomials* (7.10) *for ω and κ has at least $n!\,\mathrm{volume}(\Delta)$ complex solutions, at least $\sigma(\omega)$ of which are real*

EXAMPLE 7.14. Let $\mathcal{A}$ be the integer points in the hexagon of Example 7.10. The foldable triangulation is induced by a function ω that takes the value 3 at the center point $(1,1)$, the value 0 at $(0,0)$ and $(2,2)$, and otherwise takes the value 2. We check the condition on $t^{-1}.Y_{\mathcal{A}} \cap E_{\omega,\kappa}$. The center $E_{\omega,\kappa}$ is defined by

$$\kappa_{00}z_{00} + \kappa_{11}z_{11} + \kappa_{22}z_{22} \ = \ \underline{\kappa_{10}z_{10} + \kappa_{12}z_{12}} \ = \ \kappa_{01}z_{01} + \kappa_{21}z_{21} \ = \ 0\,.$$

As in Corollary 4.8, the ideal of $t^{-1}.Y_{\mathcal{A}}$ is defined by the binomials

$$z_{00}z_{11} - tz_{10}z_{01}\,, \ \underline{z_{10}z_{12} - t^2z_{11}^2}\,, \ z_{01}z_{21} - t^2z_{11}^2\,, \ \dots$$

The underlined form defining $E_{\omega,\kappa}$ implies that $\kappa_{10}z_{10} = -\kappa_{12}z_{12}$. Substituting this into underlined term defining $t^{-1}.Y_{\mathcal{A}}$ (after multiplying it by κ_{10}) gives

$$-\kappa_{12}z_{12}^2 \ - \ \kappa_{10}t^2z_{11}^2 \ = \ 0\,,$$

which has no nonzero real solutions, and so $z_{10} = z_{12} = z_{11} = 0$. Similar consequences of other equations show that $t^{-1}.Y_{\mathcal{A}}$ does not meet $E_{\omega,\kappa}$, for any real t, and thus any Wronski polynomial system for the foldable triangulation Δ_ω will have at least $\sigma(\omega) = 2$ real solutions. Figure 7.4 shows Wronski polynomials (with constants $\kappa_a = 1$), and the curves they define in the plane. This system has exactly two real solutions, which is the lower bound predicted by Theorem 7.13.

$$3(1 + xy + x^2y^2) + 5(x + xy^2) + (y + x^2y) = 0$$

$$(1 + xy + x^2y^2) - 2(x + xy^2) - 3(y + x^2y) = 0$$

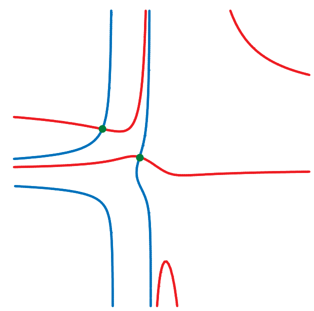


FIGURE 7.4. System of Wronski polynomials.

We prove Theorem 7.13 by showing that the systems of Wronski polynomials (7.10) are the fibers of a map

$$g \,:\, Y_{\mathcal{A}} \longrightarrow \mathbb{RP}^n$$

whose real degree is the signature $\sigma(\omega)$ of the foldable triangulation Δ_ω. This proceeds in two steps. First, the labeling function $\ell\colon \mathcal{A} \to \{0, \ldots, n\}$ coming from the foldable triangulation and the constants κ induce a linear projection

$$\pi_{\omega,\kappa} \,:\, \mathbb{P}^{\mathcal{A}} \dashrightarrow \mathbb{P}^n$$

whose center is $E_{\omega,\kappa}$. We will show that if $t > 0$ is sufficiently small, then the restriction g_t of $\pi_{\omega,\kappa}$ to $t^{-1}.Y_{\mathcal{A}}$ is a map $t^{-1}.Y_{\mathcal{A}} \to \mathbb{P}^n$ of real degree $\sigma(\omega)$. The condition that $t^{-1}.Y_{\mathcal{A}} \cap E_{\omega,\kappa} = \emptyset$ for $0 < t \leq 1$, and the definition of real degree as the degree of the lift to double covers, implies that the real degree of g_t is constant for $0 < t \leq 1$, and therefore the map $g_1 = g$ has real degree $\sigma(\omega)$.

Define the map $\pi_{\omega,\kappa}$ by the linear forms $\Lambda_i(z)$ (7.11),

$$\pi_{\omega,\kappa}(z) \;:=\; [\Lambda_0(z) : \Lambda_1(z) : \cdots : \Lambda_n(z)] \in \mathbb{P}^n\,.$$

This has center $E_{\omega,\kappa}$, and the pullback of a linear form $\sum_i c_i y_i$ is

$$\sum_{a\in\mathcal{A}} c_{\ell(a)} \kappa_a z_a\,,$$

whose pullback under $\varphi_{\mathcal{A}}$ is a Wronski polynomial (7.10). Thus a system of Wronski polynomials (7.10) is a fiber of the map $g\colon Y_{\mathcal{A}} \to \mathbb{RP}^n$ obtained by restricting $\pi_{\omega,\kappa}$ to $Y_{\mathcal{A}}$. By Corollary 7.9, the real degree of g is a lower bound for the number of real solutions to systems of Wronski polynomials. For $0 < t$, let g_t be the restriction of $\pi_{\omega,\kappa}$ to $t^{-1}.Y_{\mathcal{A}}$. By our assumption on $E_{\omega,\kappa}$, the real degree of g_t is constant for $0 < t \leq 1$. Thus Theorem 7.13 is a consequence of the following lemma.

LEMMA 7.15. *For t sufficiently small, the real degree of g_t is $\sigma(\omega)$.*

Proof. The *positive part* $Y_{\mathcal{A},>}$ of a projective toric variety $X_{\mathcal{A}}$ consists of its points with positive coordinates. It is the image of the positive orthant $\mathbb{R}^n_>$ under the parameterization map $\varphi_{\mathcal{A}}$ and is homeomorphic to the interior of the convex hull of $\mathcal{A}$ under the algebraic moment map $\mu_{\mathcal{A}}$. In particular, $Y_{\mathcal{A},>}$ is orientable.

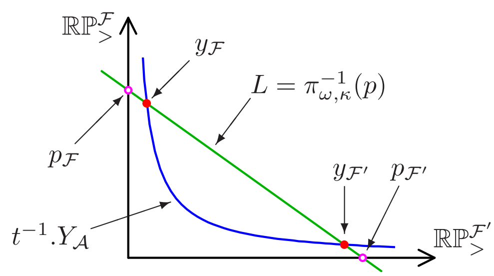
For each facet simplex $\mathcal{F}$ of Δ_ω, the map $\pi_{\omega,\kappa}$ sends a point $[y_a \mid a \in \mathcal{F}] \in \mathbb{RP}^{\mathcal{F}}$ to the point $[\kappa_a y_a \mid a \in \mathcal{F}] \in \mathbb{RP}^n$, where $\kappa_a y_a$ is the $\ell(a)$-th coordinate. For $p := [1 : 1 : \cdots : 1] \in \mathbb{RP}^n_>$ the linear space $L := \pi^{-1}_{\omega,\kappa}(p)$ meets $\mathbb{RP}^{\mathcal{F}}$ in the point $p_{\mathcal{F}} := [1/\kappa_a \mid a \in \mathcal{F}] \in \mathbb{RP}^{\mathcal{F}}_>$.

Since Δ_ω is unimodular, if we restrict the toric degeneration to the real points, Corollary 4.14 implies that

$$\lim_{t\to 0} t^{-1}.Y_{\mathcal{A}} \;=\; \bigcup_{\mathcal{F}} \mathbb{RP}^{\mathcal{F}}\,. \tag{7.12}$$

If we restrict this to $t > 0$, then the positive part $t^{-1}.Y_{\mathcal{A},>}$ degenerates to the union of positive parts $\bigcup_{\mathcal{F}} \mathbb{RP}^{\mathcal{F}}_>$.

As in Section 4.4, if $t > 0$ is sufficiently small, then the points of $L \cap t^{-1}.Y_{\mathcal{A}} = g_t^{-1}(p)$ are in bijection with the facets $\mathcal{F}$ of Δ_ω with each point having a unique closest point $p_{\mathcal{F}}$. Fix such a $t > 0$ and let $y_{\mathcal{F}}$ be the point of $g_t^{-1}(p)$ closest to $p_{\mathcal{F}}$.



Choosing orientations of $t^{-1}.Y_{\mathcal{A},>}$ and $\mathbb{RP}^n_>$, the sum

$$\sum_{y\in g_t^{-1}(p)} \operatorname{sign}_{g_t}(y) \;=\; \sum_{\mathcal{F}} \operatorname{sign}_{g_t}(y_{\mathcal{F}}) \tag{7.13}$$

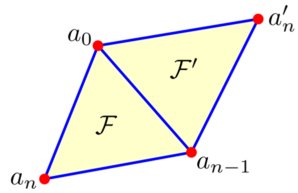
is the degree of the map g_t restricted to $t^{-1}.Y_{\mathcal{A},>}$. As the positive parts are connected and this computation is local, this sum (7.13) is identical to the sum computing the degree of g_t^+ using the fiber above the lifted point $p^+ = (1 : \cdots : 1) \in \mathbb{S}^n$.

Since $\kappa \in \mathbb{R}^{\mathcal{A}}_>$, the map is $\pi_{\omega,\kappa}$ is a homeomorphism between the positive parts $\mathbb{RP}^{\mathcal{F}}_>$ and $\mathbb{RP}^n_>$, and therefore induces orientations on $\mathbb{RP}^{\mathcal{F}}_>$ so that the map $\pi_{\omega,\kappa} \colon \mathbb{RP}^{\mathcal{F}}_> \to \mathbb{RP}^n_>$ has degree 1. As in the proof of Kushnirenko's Theorem in Section 4.3, in a neighborhood of the point $y_{\mathcal{F}}$, the map g_t is isotopic to the coordinate projection to $\mathbb{RP}^{\mathcal{F}}_>$, and therefore $\operatorname{sign}_{g_t}(y_{\mathcal{F}})$ is equal to the sign of this coordinate projection to $\mathbb{RP}^{\mathcal{F}}_>$ (by our choice of orientation of $\mathbb{RP}^{\mathcal{F}}_>$). But this coordinate projection to $\mathbb{RP}^{\mathcal{F}}_>$ induces an orientation $\mathfrak{o}_{\mathcal{F}}$ on $t^{-1}.Y_{\mathcal{A}}$, and so $\operatorname{sign}_{g_t}(y_{\mathcal{F}})$ measures the agreement of the induced orientation $\mathfrak{o}_{\mathcal{F}}$ with the fixed orientation.

We complete the proof by showing that if $\mathcal{F}$ and $\mathcal{F}'$ are adjacent facets of the triangulation Δ_ω, then the induced orientations $\mathfrak{o}_{\mathcal{F}}$ and $\mathfrak{o}_{\mathcal{F}'}$ differ, for then the sum (7.13) is equal to the signature $\sigma(\omega)$ of the triangulation.

The adjacent facets $\mathcal{F}$ and $\mathcal{F}'$ share a common face $\{a_0, \ldots, a_{n-1}\} := \mathcal{F} \cap \mathcal{F}'$ and we have $\mathcal{F} \setminus \mathcal{F}' = \{a_n\}$ and $\mathcal{F}' \setminus \mathcal{F} = \{a'_n\}$. We assume that the labeling associated to the triangulation gives $\ell(a_i) = i$ and $\ell(a'_n) = n$. Setting $z_{a_0} = 1$, $\mathbb{RP}^{\mathcal{F}}_>$ has coordinates $(z_{a_1}, \ldots, z_{a_n})$ and $\mathbb{RP}^{\mathcal{F}'}_{>}$ has coordinates $(z_{a_1}, \ldots, z'_{a_n})$, and the implied orientations agree. Pulling these back to $t^{-1}.Y_{\mathcal{A},>}$ along the two projections gives two local coordinate charts inducing the orientations $\mathfrak{o}_{\mathcal{F}}$ and $\mathfrak{o}_{\mathcal{F}'}$.

To compare these orientations, we determine the change of coordinates between these two charts. Consider the union of the facet simplices $\mathcal{F}$ and $\mathcal{F}'$ in the triangulation.



The line segment $\overline{a_n, a'_n}$ crosses the affine hull of $\{a_0, \ldots, a_{n-1}\}$ and so there is an integer linear relation

$$Na_n + Ma'_n \;=\; \alpha_0 a_0 + \alpha_1 a_1 + \cdots + \alpha_{n-1} a_{n-1}\,,$$

where N, M are positive. As in Section 6.1, this gives a valid equation for $Y_{\mathcal{A}}$,

$$z_{a_n}^M z_{a'_n}^N \;-\; z_{a_0}^{\alpha_0} \cdots z_{a_{n-1}}^{\alpha_{n-1}} \;=\; 0\,.$$

As in Section 4.3 this gives the equation on $t^{-1}.Y_{\mathcal{A}}$,

$$t^{\langle \omega, \alpha \rangle} z_{a_n}^M z_{a'_n}^N \;-\; t^{\omega(a_n)N + \omega(a'_n)M} z_{a_0}^{\alpha_0} \cdots z_{a_{n-1}}^{\alpha_{n-1}} \;=\; 0\,.$$

We can solve this in the affine chart where $z_{a_0} = 1$ to obtain,

$$z_{a_n} \;=\; \sqrt[N]{t^{\omega(a_n)N + \omega(a'_n)M - \langle \omega, \alpha \rangle} z_{a'_n}^{-M} \cdot z_{a_1}^{\alpha_1} \cdots z_{a_n}^{\alpha_n}} \;=\; z_{a'_n}^{-M/N} R\,,$$

where $R := \sqrt[N]{t^{\omega(a_n)N + \omega(a'_n)M - \langle \omega, \alpha \rangle} \cdot z_{a_1}^{\alpha_1} \cdots z_{a_n}^{\alpha_n}}$. The Jacobian matrix for the change of coordinates between these two charts has the form

$$\begin{pmatrix} 1 & \cdots & 0 & * \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & * \\ 0 & \cdots & 0 & -\frac{M}{N} z_{a'_n}^{M/N-1} \cdot R \end{pmatrix}\,,$$

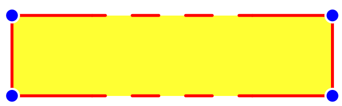
whose determinant is negative. This implies that $\mathfrak{o}_{\mathcal{F}}$ and $\mathfrak{o}_{\mathcal{F}'}$ are opposite orientations, and completes the proof.

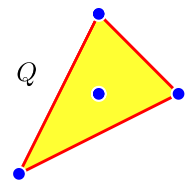
7.4. Open problems

There is much more to be done in this area. Here are some suggestions.

(1) Find other methods to give polynomial systems whose degree may be computed or estimated. For example, we give a family of systems in Section 8.3 having a lower bound on their numbers of real solutions for which $Y^+_{\mathcal{A}}$ is not orientible.
(2) Find more unbalanced triangulations (see [**81**]).
(3) Apply these ideas to specific problems from the applied sciences.
(4) Extend any of this from unmixed systems (all polynomials have the same Newton polytope) to more general mixed systems (those whose polynomials have different Newton polytopes). We end with an example in this direction which is due to Chris Hillar.

EXAMPLE 7.16. Let P and Q be the two lattice polytopes given below





A polynomial with support P has the form

$$g \ := \ A(x) \ + \ yB(X)\,,$$

where A and B are univariate polynomials in x with degree m. Let their coefficients be $a_0, \ldots, a_m$ and $b_0, \ldots, b_m$, and let h be a polynomial with support Q,

$$h \ := \ c \ + \ dxy \ + \ ex^2y \ + \ fxy^2\,.$$

By Bernstein's Theorem 1.2, a general mixed system $g(x, y) = h(x, y) = 0$ will have $2m+2$ solutions in $\mathbb{T}^2$, as $2m+2$ is the mixed volume of P and Q. (For polygons P, Q, the mixed volume is volume$(P+Q)$ − volume(P) − volume(Q).) We can compute an eliminant for this system by substituting $-A(x)$ for $yB(x)$ in $h \cdot B(x)^2$, to obtain

$$cB(x)^2 \ - \ dxA(x)B(x) \ - \ ex^2A(x)B(x) + fxA(x)^2\,.$$

This has constant term cb_0^2 and leading term $-ea_mb_m$. If $ce > 0$ and $a_mb_m > 0$, then these have different signs, which implies that the mixed system has at least one positive root (and hence at least two real roots). This may be ensured by the condition that none of the coefficients vanish and they satisfy the two linear equations, $c + e = a_m + b_m = 0$.

CHAPTER 8

Some Lower Bounds for Systems of Polynomials

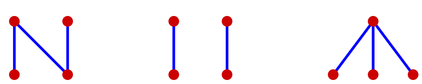
While the theory of lower bounds developed in Chapter 7 does not apply to all systems of sparse polynomials, it does have several significant applications. We discuss three. One is a system of polynomials constructed from partially ordered sets which gives a satisfying application for this theory. In fact, the theory of lower bounds was developed to explain observed phenomena for this family of systems. Another is the Wronski map in the Schubert Calculus (1.4)—the theory of lower bounds, together with the sagbi degeneration of the Grassmannian, implies Eremenko and Gabrielov's result about lower bounds for fibers of the Wronski map. The last are polynomial systems from posets that are union of incomparable chains. For these, we give a different derivation of the lower bounds and establish a new phenomenon of gaps in the possible numbers of real solutions.

The phenomena discussed here—lower bounds and gaps—will reappear in relation to the Shapiro Conjecture, particularly in Chapters 13 and 14. We are far from understanding them, and this area is ripe for progress.

8.1. Polynomial systems from posets

The motivating application of the theory of lower bounds is a class of polynomial systems associated to partially ordered sets. These are systems of Wronski polynomials (7.10) coming from regular unimodular triangulations of order polytopes of partially ordered sets. We give necessary definitions before applying the theory of Chapter 7 to give a lower bound for these systems of polynomials.

Let P be a finite partially ordered set (*poset*). Posets are often represented by their *Hasse diagrams*, which are acyclic graphs with vertices the elements of P and an edge for every minimal order relation (*cover*, $\lessdot$) in P. The order relation is induced by moving vertically along the edges. Here are posets on four elements.



An *order ideal* $I \subset P$ is a set that is closed upwards,

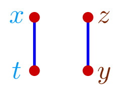
$$x \leq y \quad \text{with} \quad x \in I \;\Rightarrow\; y \in I\,.$$

We take elements of P as our variables. To an order ideal $I \subset P$ we associate a monomial whose exponent is the characteristic function of I,

$$x^I \;:=\; \prod_{x \in I} x\,.$$

Its degree is $|I|$, the cardinality of the order ideal I.

EXAMPLE 8.1. If P is the incomparable union of two chains each of length two,

(8.1) $P \;:=$ 

,

then the monomials corresponding to the order ideals of P are

$$\{\emptyset,\ x,\ z,\ tx,\ xz,\ yz,\ xyz,\ txz,\ txyz\}\,.$$

A *Wronski polynomial* for P is a polynomial of the form

$$\sum_I c_{|I|}\kappa_I x^I \;=\; \sum_{i=0}^{|P|} c_i\Bigl(\sum_{|I|=i}\kappa_I x^I\Bigr)\,, \tag{8.2}$$

the sum over all order ideals I of P, where $c_0,\ldots,c_{|P|}\in\mathbb{R}$ and $\kappa_I\in\mathbb{R}_>$. A *system of Wronski polynomials* is a system of polynomials (8.2), where the coefficients c_i vary, but the constants κ_I are the same for all polynomials in the system.

EXAMPLE 8.2. A Wronski polynomial for the poset (8.1) has the form

$$\begin{aligned} &c_4\, txyz\\ &+ c_3(txz\ +\ xyz)\\ &+ c_2(tx\ +\ xz\ +\ yz)\\ &+ c_1(x\ +\ z)\\ &+ c_0\,, \end{aligned} \tag{8.3}$$

where the coefficients $c_0,\ldots,c_4$ are real numbers. Here, all $\kappa_I = 1$.

By Kushnirenko's Theorem 3.2, the number of solutions to a system of Wronski polynomials is expected to be the normalized volume of the convex hull of their exponent vectors. This convex hull is the *order polytope $\mathcal{O}_P$* of the poset P, which is the set of all order-preserving functions,

$$f\ :\ P\ \longrightarrow\ [0,1]\,,$$

where $x\le y$ in P implies that $f(x)\le f(y)$. The integer points of $\mathcal{O}_P$ are the order-preserving maps $P\to\{0,1\}$, and thus are the characteristic functions of order ideals of P (the order ideal is $f^{-1}(1)$). Each is a vertex of $\mathcal{O}_P$, as $\mathcal{O}_P\subset[0,1]^P$. Thus $\mathcal{O}_P$ is the Newton polytope of a Wronski polynomial.

Stanley [**148**] gave a regular unimodular triangulation of $\mathcal{O}_P$. Assume that P has n elements. A *linear extension* of P is a listing $x_1,x_2,\ldots,x_n$ of the elements of P that respects the order: $x_i<x_j$ in P implies that $i<j$. Given a linear extension $x_1,x_2,\ldots,x_n$ of P, the set of all $f\in\mathcal{O}_P$ with $f(x_1)\le f(x_2)\le\cdots\le f(x_n)$ forms a unimodular simplex, and these simplices form a triangulation of $\mathcal{O}_P$. Call this the *canonical triangulation* of $\mathcal{O}_P$. The number $\lambda(P)$ of linear extensions of P is the number of simplices and thus is the normalized volume of $\mathcal{O}_P$, which is the expected number of complex solutions to a system of Wronski polynomials.

There are six linear extensions of the poset (8.1), as each is a permutation of the word $txyz$ where t precedes x and y precedes z.

The lower bound for the number of real solutions to a system of Wronski polynomials is also combinatorial. The *sign-imbalance $\sigma(P)$* of P is

$$\sigma(P)\ :=\ \Bigl|\sum \mathrm{sign}(w)\Bigr|\ , \tag{8.4}$$

the sum over all linear extensions w of P. For this, fix one linear extension of P, and then any other linear extension w is a permutation of the fixed linear extension whose sign is sign(w). The absolute value in (8.4) removes the dependence on this choice of initial linear extension.

A *chain* of length k in a poset P is sequence of covers

$$x_1 \lessdot x_2 \lessdot \cdots \lessdot x_k\,. \tag{8.5}$$

It is *maximal* if x_1 is minimal and x_k is maximal in P. A poset P is *$\mathbb{Z}_2$-graded* if the lengths all of maximal chains have the same parity. Given an element x in P, choose a chain C from x to a maximal element. Appending C to any chain (8.5) ending in x ($x = x_k$) with x_1 minimal gives a maximal chain in P. As the lengths of all maximal chains have the same parity, the parity of k is determined and so any two chains from a minimal element to x have the same parity.

We state a theorem about lower bounds for systems of polynomials from posets.

THEOREM 8.3. *A generic system of Wronski polynomials for a finite $\mathbb{Z}_2$-graded poset P has $\lambda(P)$ complex solutions, at least $\sigma(P)$ of which are real.*

We will show that the Wronski polynomials are induced by a projection $\pi_{\omega,\kappa}$ whose restriction to the toric variety of $\mathcal{O}_P$ has real degree $\sigma(P)$.

EXAMPLE 8.4. We saw in Example 1.16 that the poset P (8.1) has sign-imbalance two. Every maximal chain has length two, so P satisfies the hypotheses of Theorem 8.3, and so we conclude that a system of four equations involving polynomials of the form (8.3) has six solutions, at least two of which are real. We saw this in Table (1.1) which reported on a computer experiment that determined the number of real solutions in 10,000,000 Wronski systems for P.

PROOF OF THEOREM 8.3. We use Theorem 7.13. We first show that $\mathcal{O}_P \cap \mathbb{Z}^n$ is primitive and the canonical triangulation of $\mathcal{O}_P$ is unimodular. Writing Y_P for the real toric variety associated to $\mathcal{O}_P$, we show that Y_P^+ is orientable if and only if P is $\mathbb{Z}_2$-graded. Then we give a function ω which induces the canonical triangulation and show that it is foldable with signature $\sigma(P)$. The labeling function of the canonical triangulation shows that Wronski polynomials of P, as defined in (8.2), are Wronski polynomials (7.10) for this triangulation. We complete the proof by showing that the center of projection does not meet the toric degeneration (Lemma 8.6).

Let P be a poset with n elements and order polytope $\mathcal{O}_P$. A linear extension $x_1, x_2, \ldots, x_n$ of P gives a chain of order ideals

$$\emptyset \;\subset\; \{x_1\} \;\subset\; \{x_1, x_2\} \;\subset\; \cdots \;\subset\; \{x_1, x_2, \ldots, x_n\}\,,$$

whose corresponding simplex in the canonical triangulation has vertices

$$(0,\ldots,0)\,,\ (1,0,\ldots,0)\,,\ (1,1,0,\ldots,0)\,,\ \ldots\,,\ (1,\ldots,1)\,. \tag{8.6}$$

This generates $\mathbb{Z}^n$, so $\mathcal{O}_P \cap \mathbb{Z}^n$ is primitive and the simplex is unimodular which implies that the canonical triangulation of $\mathcal{O}_P$ is unimodular.

LEMMA 8.5. *The smooth locus of Y_P^+ is orientable if and only if P is $\mathbb{Z}_2$-graded.*

PROOF. Let $\widehat{P}$ be the poset obtained by adjoining a new maximum element $\widehat{1}$ to P. For $x \in \widehat{P}$, let $\mathbf{e}_x \in \mathbb{Z}^{\widehat{P}}$ be the standard basis element with a 1 in position x

and 0 elsewhere. The order polytope $\mathcal{O}_P$ has three types of facet inequalities

$$\begin{array}{rcll} x & \geq & 0 & \text{if } x \in P \text{ is minimal} \\ x-y & \geq & 0 & \text{if } y \lessdot x \text{ is a cover in } P \\ -x & \geq & -1 & \text{if } x \in P \text{ is maximal} \end{array} .$$

Letting the coordinate $\widehat{1}$ hold the constants in these inequalities, there are two types of facet vectors which are indexed by the minimal elements and covers of $\widehat{P}$,

$$\begin{array}{rcll} v_{0<x} & = & \mathbf{e}_x & \text{if } x \in P \text{ is minimal} \\ v_{y\lessdot x} & = & \mathbf{e}_x - \mathbf{e}_y & \text{if } y \lessdot x \text{ is a cover in } \widehat{P} \end{array} .$$

We define a basis $\{f_q \mid q \in \widehat{P}\}$ of $\mathbb{Z}^{\widehat{P}}$ consisting of facet vectors for $\mathcal{O}_P$. When $q \in \widehat{P}$ is minimal, set $f_q := v_{0<q} = \mathbf{e}_q$, and otherwise choose any cover $p \lessdot q$ in $\widehat{P}$ and set $f_q := v_{p\lessdot q} = \mathbf{e}_q - \mathbf{e}_p$. This is a basis for $\mathbb{Z}^{\widehat{P}}$ as the matrix expressing it in terms of the standard basis is upper triangular with 1s on the diagonal with respect to any linear extension of P.

Let $y_1 \lessdot y_2 \lessdot \cdots \lessdot y_k$ be a chain in P with y_1 minimal. Then

$$\mathbf{e}_{y_k} = v_{0<y_1} + v_{y_1\lessdot y_2} + \cdots + v_{y_{k-1}\lessdot y_k}\,, \tag{8.7}$$

as this is the telescoping sum $\mathbf{e}_{y_1} + (\mathbf{e}_{y_2} - \mathbf{e}_{y_1}) + \cdots + (\mathbf{e}_{y_k} - \mathbf{e}_{y_{k-1}})$.

We will show that every facet vector is the sum of an odd number of the $\pm f_q$ if and only P is $\mathbb{Z}_2$-graded. Under the map $\mathbb{Z}^{\widehat{P}} \twoheadrightarrow \{\pm 1\}^{\widehat{P}}$ given by reduction modulo 2, a sum of an odd number of the $\pm f_q$ becomes a product of an odd number of basis elements in $\{\pm 1\}^{\widehat{P}}$, and so the lemma follows by Theorem 7.7 and Remark 7.5.

The *grade* $\mathrm{gr}(x)$ of an element $x \in \widehat{P}$ is the length k of the longest chain

$$x_1 \lessdot x_2 \lessdot \cdots \lessdot x_k = x \tag{8.8}$$

of elements of P that ends in x. Necessarily, x_1 is minimal.

Suppose that P is $\mathbb{Z}_2$-graded. We show that every facet vector is the sum of an odd number of the $\pm f_q$. If x is minimal, the facet vector $v_{0<x} = f_x$, so there is nothing to show. Suppose that $x \in \widehat{P}$ is not minimal. We assume by way of induction that every facet vector $v_{z\lessdot y}$ with $\mathrm{gr}(y) < \mathrm{gr}(x)$ is a sum of an odd number of the $\pm f_q$. Let $z \lessdot x$ be a cover in $\widehat{P}$. If $f_x = v_{z\lessdot x}$, then it is already an odd sum of the $\pm f_q$. Otherwise, $f_x = v_{y\lessdot x} = \mathbf{e}_x - \mathbf{e}_y$ with $y \neq z$. Let

$$y_1 \lessdot \cdots \lessdot y_k \lessdot y \qquad \text{and} \qquad z_1 \lessdot \cdots \lessdot z_l \lessdot z \tag{8.9}$$

be chains in P ending in y and z with y_1 and z_1 minimal. Then

$$\begin{aligned} v_{z\lessdot x} &= \mathbf{e}_x - \mathbf{e}_z = \mathbf{e}_x - \mathbf{e}_y + \mathbf{e}_y - \mathbf{e}_z = f_x + \mathbf{e}_y - \mathbf{e}_z \\ &= f_x + (v_{0<y_1} + \cdots + v_{y_k\lessdot y}) - (v_{0<z_1} + \cdots + v_{z_l\lessdot z})\,. \end{aligned} \tag{8.10}$$

The last equality uses (8.7) and the chains (8.9). Since $\mathrm{gr}(x)$ exceeds the grade of any element in the chains (8.9), each facet vector in (8.10) is an odd sum of the $\pm f_q$. Since $\widehat{P}$ is $\mathbb{Z}_2$-graded, $y \lessdot x$ and $z \lessdot x$ imply that the lengths of the chains (8.9) have the same parity, so (8.10) contains an even number of facet vectors. Replacing each facet vector by an odd sum of the $\pm f_q$ results in an odd sum of the $\pm f_q$.

Thus if P is $\mathbb{Z}_2$-graded, then every facet vector is an odd sum of the $\pm f_q$. If P is not $\mathbb{Z}_2$-graded, then there is an element $x \in \widehat{P}$ with $\mathrm{gr}(x)$ minimal having two covers $y \lessdot x$ and $z \lessdot x$ with chains (8.9) having lengths of different parities. In this case, each cover in the expression (8.9) is an odd sum of the $\pm f_q$, and substituting these sums into (8.9) gives an expression of a facet vector as an even sum of the $\pm f_q$, which implies that Y_P^+ is not orientable.

We now show that the canonical triangulation of $\mathcal{O}_P$ is regular and compute its signature. For an order ideal I of P, set $\omega(I) := |I|^2$. This gives a function (also written ω) on characteristic functions of order ideals, $\omega(a_I) = \omega(I) = |I|^2$. Let $\mathcal{O}_{P,\omega} \subset \mathbb{R}^{1+n}$ be the lift of the order polytope given by ω,

$$\mathcal{O}_{P,\omega} \;:=\; \operatorname{conv}\{(\omega(a_I), a_I) \mid a_I \in \mathcal{O}_P \cap \mathbb{Z}^n\}\,.$$

The induced regular triangulation is the canonical triangulation. To see this, let $x_1, x_2, \ldots, x_n$ be a linear extension of P. Let $z_1, \ldots, z_n$ be the corresponding coordinate functions on $\mathbb{R}^n$ and consider the linear form Λ,

$$\Lambda(z_1, \ldots, z_n) \;:=\; \sum_{i=1}^{n} (2i-1) z_i\,.$$

If a_I is the characteristic function of an order ideal I, then $\Lambda(a_I) \geq |I|^2 = \omega(a_I)$ with equality only if a_I is a vertex of the simplex (8.6) corresponding to the linear extension $x_1, x_2, \ldots, x_n$. Thus $\mathcal{O}_P$ lies under the graph of Λ and meets it in a facet lying over this simplex. We conclude that the canonical triangulation of $\mathcal{O}_P$ is induced by ω.

The canonical triangulation is foldable as the map $\ell(a_I) = |I|$ takes a different value on each vertex of a simplex (8.6) in the canonical triangulation. To compute its signature, note that two linear extensions $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ of P define adjacent simplices when they differ in only one vertex. If this is the jth vertex, then $x_j = y_{j+1}$ and $y_j = x_{j+1}$, but for $i \neq j, j{+}1$, $x_i = y_i$. Thus these two linear extensions differ by an adjacent transposition in the symmetric group, which implies that the signature of the canonical triangulation is equal to the sign-imbalance of the poset P.

Given a positive constant κ_I for each order ideal I, the center $E_{\omega,\kappa}$ of the projection $\pi_{\omega,\kappa}$ giving the Wronski polynomials is defined by the linear equations

$$\sum_{|I|=i} \kappa_I z_I \;=\; 0\,, \tag{8.11}$$

where $\{z_I \mid I \text{ an order ideal of } P\}$ are the coordinate functions for the projective space containing the toric variety X_P of the order polytope $\mathcal{O}_P$.

Lemma 8.6 below completes the proof of Theorem 8.3.

Lemma 8.6. *For every $t \neq 0$, the translated toric variety $t^{-1}.X_P$ is disjoint from the center $E_{\omega,\kappa}$ of projection.*

Proof. Hibi [**68**] showed that the ideal of X_P is generated by the binomials

$$z_I z_J \;-\; z_{I\cap J} z_{I\cup J}\,, \tag{8.12}$$

for any two order ideals I, J (the intersection and union of order ideals are again order ideals, and the set of order ideals forms a distributive lattice). Substituting $z_I = x^I$ shows that these binomials are valid on X_P.

Thus the toric degeneration $t^{-1}.X_{\mathcal{A}}$ is cut out by the binomial equations

$$t^{|I|^2+|J|^2} z_I z_J \;-\; t^{|I\cap J|^2+|I\cup J|^2} z_{I\cap J} z_{I\cup J} \;=\; 0\,. \tag{8.13}$$

We show that if $t \neq 0$, then these equations imply that $z_I = 0$ for every I.

The unique order ideal I with $|I| = 0$ is $\emptyset$, so (8.11) with $i = 0$ gives $z_\emptyset = 0$, as $\kappa_\emptyset \neq 0$. Suppose that for $|K| < i$, the equations (8.11) and (8.13) imply that $z_K = 0$. If $I \neq J$ are order ideals with $|I| = |J| = i$, then I and J are incomparable with $|I \cap J| < i$ and so $z_{I\cap J} = 0$. Then the binomial equation (8.13) implies that $z_I z_J = 0$, as $t \neq 0$. Since all pairwise products of variables z_I with $|I| = i$ vanish, the equations (8.11) imply that all z_I with $|I| = i$ vanish. Lastly, if there is a unique order ideal I with $|I| = i$, then (8.11) implies that $z_I = 0$.

8.2. Sagbi degenerations

Eremenko and Gabrielov's result (Theorem 1.14) on the degree of the Wronski map on the Grassmannian [**45**] can be deduced from Theorem 8.3. The Grassmannian $\mathrm{Gr}(p, m+p)$ of p-planes in $(m+p)$-space admits a degeneration to the toric variety associated to the poset $C_{m,p}$ that is the product of a chain of length m with a chain of length p. The degree of a linear projection inducing the Wronski map (1.4) is preserved under this degeneration and the map on the toric variety $Y_{C_{m,p}}$ induces the Wronski polynomials (8.2) of Section 8.1. Thus the real degree of the Wronski map in Schubert Calculus equals the sign-imbalance of the poset $C_{m,p}$, which was shown by White [**163**] to be $\sigma_{m,p}$ (1.10). We explain this here.

The Grassmannian $\mathrm{Gr}(p, m+p)$ is the set of p-dimensional linear subspaces (p-planes) in $\mathbb{C}^{m+p}$. It admits two complementary systems of coordinates. For the first, define the *Stiefel manifold* $\mathrm{St}(p, m+p)$ to be the set of p by $(m+p)$-matrices of full rank. The row space of a matrix in $\mathrm{St}(p, m+p)$ is a p-plane H in $\mathbb{C}^{m+p}$, and this defines a surjective map $\mathrm{St}(p, m+p) \twoheadrightarrow \mathrm{Gr}(p, m+p)$ whose fibers consist of all the matrices with a given row space. The group $GL(p)$ of invertible row operations acts transitively on each fiber. General principles then imply that $\mathrm{Gr}(p, m+p)$ is an algebraic manifold of dimension

$$\dim(\mathrm{St}(p, m+p)) \;-\; \dim(GL(p)) \;=\; p(m+p) \;-\; p^2 \;=\; mp\,.$$

Under the map $\mathrm{St}(p, m+p) \twoheadrightarrow \mathrm{Gr}(p, m+p)$, entries of matrices in $\mathrm{St}(p, m+p)$ give global *Stiefel coordinates* for $\mathrm{Gr}(p, m+p)$.

A point $H \in \mathrm{Gr}(p, m+p)$ in the Grassmannian gives a vector space inclusion $H \hookrightarrow \mathbb{C}^{m+p}$ whose top (pth) exterior power,

$$\mathbb{C} \;\simeq\; \wedge^p H \;\longhookrightarrow\; \wedge^p \mathbb{C}^{m+p}\,,$$

is the inclusion of a one-dimensional linear space, and thus a point in the projective space $\mathbb{P}(\wedge^p \mathbb{C}^{m+p})$. This defines the *Plücker embedding*

$$\mathrm{Gr}(p, m+p) \;\longrightarrow\; \mathbb{P}(\wedge^p \mathbb{C}^{m+p})\,,$$

which realizes the Grassmannian $\mathrm{Gr}(p, m+p)$ as a projective variety.

The basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_{m+p}\}$ for $\mathbb{C}^{m+p}$ induces a basis $\mathbf{e}_\alpha := \mathbf{e}_{\alpha_1} \wedge \mathbf{e}_{\alpha_2} \wedge \ldots \wedge \mathbf{e}_{\alpha_p}$ indexed by $\alpha\colon 1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_p \leq m+p$ for $\wedge^p \mathbb{C}^{m+p}$. Write $\binom{[m+p]}{p}$ for this set of indexing sequences. Plücker coordinates $(p_\alpha \mid \alpha \in \binom{[m+p]}{p})$ for $\wedge^p \mathbb{C}^{m+p}$ and $\mathbb{P}(\wedge^p \mathbb{C}^{m+p})$ form the basis dual to $\{\mathbf{e}_\alpha\}$.

The Plücker coordinates of points $H \in \mathrm{Gr}(p, m+p)$ are minors of matrices $M = (m_{i,j}) \in \mathrm{St}(p, m+p)$ with row space H. The row space H of M has a basis

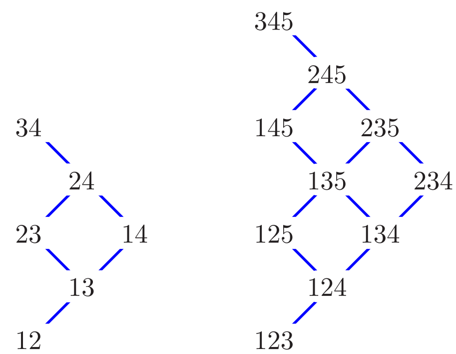
$$h_i \;:=\; \sum_j m_{i,j} \mathbf{e}_j \qquad i = 1, \ldots, p\,.$$

Then $h_1 \wedge h_2 \wedge \ldots \wedge h_p$ spans $\wedge^p H$. Expanding this in the basis $\{\mathbf{e}_\alpha\}$ gives

$$\sum_\alpha p_\alpha(M) \cdot \mathbf{e}_\alpha\,,$$

where $p_\alpha(M)$ is the determinant of the p by p submatrix of M given by its columns indexed by α. These $p_\alpha(M)$ depend on H, up to a global multiplicative constant coming from the determinant of a change of basis for H. Call $[p_\alpha(M) \mid \alpha \in \binom{[m+p]}{p}]$ *Plücker coordinates* for $H \in \mathbb{P}(\wedge^p\mathbb{C}^{m+p})$.

The set of indices $\binom{[m+p]}{p}$ form the *Bruhat order* under componentwise comparison, $\alpha \leq \beta \Leftrightarrow \alpha_i \leq \beta_i$ for every i. Here are the Bruhat posets $\binom{[4]}{2}$ and $\binom{[5]}{3}$.



This is in fact a lattice with meet and join,

$$(\alpha \wedge \beta)_i \;=\; \min\{\alpha_i, \beta_i\} \qquad \text{and} \qquad (\alpha \vee \beta)_i \;=\; \max\{\alpha_i, \beta_i\}\,.$$

Thus $14 \wedge 23 = 13$ and $14 \vee 23 = 24$.

For $\alpha \in \binom{[m+p]}{p}$, set $|\alpha| := \sum_i(\alpha_i - i)$. For the minimal and maximal elements, we have $|12\ldots p| = 0$ and $|m{+}1, \ldots, m{+}p| = mp$. Let $\kappa_\alpha \in \mathbb{R}_>$ for $\alpha \in \binom{[m+p]}{p}$ be any choice of positive constants. We define the *generalized Wronski map* Wr_κ: $\mathrm{Gr}(p, m{+}p) \to \mathbb{P}^{mp}$ by the restriction of the projection

$$\pi_\kappa \;:\; [p_\alpha \mid \alpha \in \tbinom{[m+p]}{p}] \;\longmapsto\; \Big[\sum_{|\alpha|=i} \kappa_\alpha p_\alpha \mid i = 0, \ldots, mp\Big]\,. \tag{8.14}$$

REMARK 8.7. Let $\alpha \in \binom{[m+p]}{p}$ and set $\beta\colon \beta_1 < \cdots < \beta_m$ to be the complement to α in the set $\{1, \ldots, m{+}p\}$. If we set

$$\kappa_\alpha \;:=\; (-1)^{|\beta|} \prod_{i<j} (\beta_j - \beta_i)\,,$$

then the projection π_κ (8.14) gives the Wronski map in Schubert calculus (1.4) of Section 1.4, by the computation leading to (10.8). We will see that the projection π_κ gives a Wronski map for the canonical triangulation of the order polytope $\mathcal{O}_{m,p}$ of the poset $C_{m,p}$. This connection is the source of the terminology Wronski in Chapter 7.

Recall that the sign-imbalance of the poset $C_{m,p}$ is the number $\sigma_{m,p}$ (1.10).

THEOREM 8.8 (Eremenko and Gabrielov [**45**]). *For any regular value $x \in \mathbb{RP}^{mp}$ of the real generalized Wronski map Wr_κ there are at least $\sigma_{m,p}$ points of the real Grassmannian in $\mathrm{Wr}_\kappa^{-1}(x)$.*

The *Plücker ideal* $I_{m,p}$ is the ideal of the Grassmannian $\mathrm{Gr}(p, m+p)$ in its Plücker embedding. This has a classical description due to Hodge [**71**, §XIV.9] that was reinterpreted by Sturmfels [**152**, §3.1].

LEMMA 8.9. *The Plücker ideal $I_{m,p}$ is minimally generated by quadratic polynomials with one generator $g_{\alpha,\beta}$ for each pair $\alpha, \beta \in \binom{[m+p]}{p}$ of incomparable elements of the Bruhat order. The generator $g_{\alpha,\beta}$ has the form*

$$g_{\alpha,\beta} \;=\; \underline{p_\alpha p_\beta \;-\; p_{\alpha\wedge\beta}p_{\alpha\vee\beta}} \;+\sum c_{\gamma,\delta}p_\gamma p_\delta\,, \tag{8.15}$$

where the indices γ, δ of every nonzero term in the sum satisfy $\gamma < \alpha\wedge\beta$, $\alpha\vee\beta < \delta$, and the equality $\gamma \cup \delta = \alpha \cup \beta$ of multisets.

For example, the single quadratic generator of $I_{2,2}$ is

$$\underline{p_{14}p_{23} \;-\; p_{13}p_{24}} \;+\; p_{12}p_{34}\,. \tag{8.16}$$

The structure of the quadratic polynomials (8.15) implies that the Grassmannian admits a toric degeneration to a toric variety defined by the underlined binomial of the generators $g_{\alpha,\beta}$. This degeneration of the Grassmannian is called the *sagbi degeneration*. This terminology is explained in [**154**, Ch. 11]. Let $\omega\colon \binom{[m+p]}{p} \to \mathbb{Z}$ be defined by $\omega(\alpha) := \sum_i (p-i)\alpha_i$. This induces a map on exponents of Plücker coordinates where the weight of p_α is $\omega(\alpha)$. We give the technical proof of the following lemma at the end of this section.

LEMMA 8.10. *The initial ideal* $\mathrm{in}_\omega(I_{m,p})$ *is generated by the quadratic binomials*

$$\mathrm{in}_\omega(g_{\alpha,\beta}) \;=\; p_\alpha p_\beta \;-\; p_{\alpha\wedge\beta}p_{\alpha\vee\beta} \qquad \text{for } \alpha,\beta \in \tbinom{[m+p]}{p} \text{ incomparable}.$$

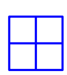
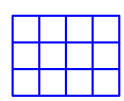
We can see this in (8.16) by computing the ω-weights of its terms.

$$\begin{aligned}
\omega(p_{14}p_{23}) &= 1+0\cdot 4 \;+\; 2+0\cdot 3 \;=\; 3\,,\\
\omega(p_{13}p_{24}) &= 1+0\cdot 3 \;+\; 2+0\cdot 4 \;=\; 3\,,\\
\omega(p_{12}p_{34}) &= 1+0\cdot 2 \;+\; 3+0\cdot 4 \;=\; 4\,.
\end{aligned}$$

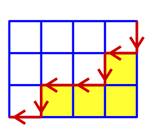
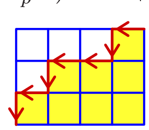
Under the action of $\mathbb{T}$ given by ω, the limit of $t^{-1}.\mathrm{Gr}(p, m+P)$ as $t \to 0$ is the variety of the initial ideal. We identify this initial scheme.

LEMMA 8.11. *The initial scheme* $\mathrm{in}_\omega \mathrm{Gr}(p, m+p)$ *is the toric variety* $X_{C_{m,p}}$ *associated to the poset $C_{m,p}$ which is a product of chains of lengths m and p.*

PROOF. This follows from a standard isomorphism between the Bruhat poset $\binom{[m+p]}{p}$ and the lattice of order ideals in $C_{m,p}$. Write the poset $C_{m,p}$ as a p by m array of boxes where boxes to the right or below are are greater in $C_{m,p}$.

$C_{2,2} \iff$  $C_{3,4} \iff$ 

An order ideal I of $C_{m,p}$ is a collection of boxes in the south east corner of this array. It is enclosed by a path from the north east corner to the south west corner in the edges of the boxes which has length $m+p$ with p vertical steps. The positions of the vertical steps give a sequence $\alpha(I) \in \binom{[m+p]}{p}$ with $|\alpha(I)| = |I|$.

 $\iff$ 136  $\iff$ 257

Under this isomorphism, the quadratics $\text{in}_\omega(g_{\alpha,\beta})$ of Lemma 8.10 and of (8.12) for $C_{m,p}$ coincide.

Proof of Theorem 8.8. As in Section 7.2, neither the real Grassmannian nor the projective space $\mathbb{RP}^{mp}$ are necessarily orientable. However, both have orientable double covers obtained by pulling back to the spheres over projective spaces. The oriented Grassmannian $\text{Gr}^+(p, m+p)$ parameterizes oriented p-planes in $\mathbb{R}^{m+p}$. We show that the degree of the map

$$\text{Wr}^+_\kappa \;:\; \text{Gr}^+(p, m+p) \;\longrightarrow\; \mathbb{S}^{mp}$$

is the sign-imbalance $\sigma_{m,p}$ of the poset $C_{m,p}$.

As every term but the first $(p_\alpha p_\beta)$ of the generators $g_{\alpha,\beta}$ and $\text{in}_\omega(g_{\alpha,\beta})$ has a factor p_γ with $\gamma < \alpha, \beta$, the same argument as in Lemma 8.6 implies that neither any translate of the Grassmannian $t^{-1}.\,\text{Gr}(p, m+p)$ nor the initial scheme $\text{in}_\omega(\text{Gr}(p, m+p))$ meets the center π_κ of the projection. The same is true for their double covers, and so the degree of the map Wr^+_κ will equal the degree of the restriction of π^+_κ to $Y^+_{C_{m,p}}$.

Maximal chains of $C_{m,p}$ have length $m+p-1$, so $C_{m,p}$ is $\mathbb{Z}_2$-graded, and therefore $Y^+_{C_{m,p}}$ is orientable by Theorem 7.7. The labeling function of the canonical triangulation of $\mathcal{O}_{C_{m,p}}$ sends an order ideal I to its cardinality $|I|$. Comparing (8.11) to (8.14) shows that the projection $\pi_{\omega,\kappa}$ associated to the canonical triangulation is equal to the projection π_κ, and thus the degree of the restriction of π^+_κ to $Y^+_{C_{m,p}}$ is the sign-imbalance $\sigma_{m,p}$ of $C_{m,p}$.

Proof of Lemma 8.10. Since $\{\alpha_i, \beta_i\} = \{(\alpha \wedge \beta)_i, (\alpha \vee \beta)_i\}$,

$$\omega(\alpha) \;+\; \omega(\beta) \;=\; \omega(\alpha \wedge \beta) \;+\; \omega(\alpha \vee \beta)\,.$$

Thus the terms of the underlined initial binomial of $g_{\alpha,\beta}$ have the same weight under ω. We show that the remaining terms have strictly larger weight.

Suppose that $c_{\gamma\delta} p_\gamma p_\delta$ is a nonzero term of $g_{\alpha,\beta}$ in the sum of (8.15). Then $\gamma \cup \delta = \alpha \cup \beta$ as multisets, and also $\gamma < \alpha \wedge \beta$ and $\alpha \vee \beta < \delta$. It follows that the pair (γ, δ) is obtained from the pair $(\alpha \wedge \beta, \alpha \vee \beta)$ by interchanging k elements γ' of $\alpha \wedge \beta$ with k elements δ' of $\alpha \vee \beta$,

$$\gamma \;=\; (\alpha \wedge \beta \setminus \{\gamma'\}) \cup \{\delta'\} \qquad \text{and} \qquad \delta \;=\; (\alpha \vee \beta \setminus \{\delta'\}) \cup \{\gamma'\}\,.$$

The condition that $\gamma < \alpha \wedge \beta$ (and also that $\alpha \vee \beta < \delta$) is equivalent to $\delta' < \gamma'$ under coordinatewise comparison in $\binom{[m+p]}{k}$.

For example, if $\alpha = 12569$ and $\beta = 23478$, then $\alpha \wedge \beta = 12\underline{468}$ and $\alpha \vee \beta = 2\underline{357}9$. If we let $\gamma' = 468 \subset \alpha \wedge \beta$ and $\delta' = 357 \subset \alpha \vee \beta$, then $\gamma = 12357 < 24689 = \delta$ and we have $\delta' < \gamma'$.

Rather than interchange subsets γ' and δ' of size k in one step, we could instead do this in k steps, first interchanging their smallest elements, then their next smallest, and so on. Thus it suffices to prove the claim: If $\zeta \leq \eta$ in $\binom{[m+p]}{p}$ and we have indices indices a, b with $\eta_b < \zeta_a$ where $\gamma := (\zeta \setminus \{\zeta_a\}) \cup \{\eta_b\}$ and $\delta := (\zeta \setminus \{\eta_a\}) \cup \{\zeta_b\}$ are in $\binom{[m+p]}{p}$, then

$$\text{(8.17)} \qquad \omega(p_\gamma p_\delta) - \omega(p_\zeta p_\eta) \;=\; \omega(\gamma) \;+\; \omega(\delta) \;-\; (\omega(\zeta) \;+\; \omega(\eta)) \;>\; 0\,.$$

To see that this is sufficient, set $\zeta = \alpha \wedge \beta$ and $\eta = \alpha \vee \beta$.

To prove the claim, let $\zeta, \eta, a, b, \delta$, and γ be as above. Let i be minimal such that $\eta_b < \zeta_i$ and j be maximal such that $\eta_j < \zeta_a$. Necessarily, $b \leq i, j \leq a$. Then γ

is obtained from ζ by replacing the elements $\zeta_i,\dots,\zeta_a$ by $\eta_b,\zeta_i,\dots,\zeta_{a-1}$ and δ is obtained from η by replacing the elements $\eta_b,\dots,\eta_j$ by $\eta_{b+1},\dots,\eta_j,\zeta_a$, specifically

$$\begin{array}{rcl}\gamma & : & \zeta_1 < \cdots\cdots < \zeta_{i-1} < \eta_b < \zeta_i < \cdots < \zeta_{a-1} < \zeta_{a+1} < \cdots < \zeta_p \\ \delta & : & \eta_1 < \cdots < \eta_{b-1} < \eta_{b+1} < \cdots < \eta_j < \zeta_a < \eta_{j+1} < \cdots\cdots < \eta_p\end{array}$$

(This representation is where we use that $\gamma,\delta \in \binom{[m+p]}{p}$.) We compute the terms in the inequality (8.17), taking advantage of cancellations

$$\begin{aligned}\omega(\gamma)-\omega(\zeta) &= (p-i)\eta_b + (p-(i+1))\zeta_i + \cdots + (p-a)\zeta_{a-1} \\ &\quad - \big((p-i)\zeta_i + (p-(i+1))\zeta_{i+1} + \cdots + (p-a)\zeta_a\big) \\ &= (p-i)\eta_b \;-\; (\zeta_i + \cdots + \zeta_{a-1}) \;-\; (p-a)\zeta_a\,.\end{aligned}$$

and

$$\begin{aligned}\omega(\delta)-\omega(\eta) &= (p-b)\eta_{b+1} + \cdots + (p-(j-1))\eta_j + (p-j)\zeta_a \\ &\quad - \big((p-b)\eta_b + \cdots + (p-(j-1))\eta_{j-1} + (p-j)\eta_j\big) \\ &= (p-j)\zeta_a \;+\; \eta_{b+1} + \cdots + \eta_j \;-\; (p-b)\eta_b\,.\end{aligned}$$

Adding these intermediate computations gives

$$\eta_{b+1} + \cdots + \eta_j + (a-j)\zeta_a \;-\; \big((i-b)\eta_b + \zeta_i + \cdots + \zeta_{a-1}\big)\,.$$

To see that this difference is positive, observe that

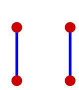
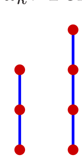
$$(\eta_{b+1},\dots,\eta_{j-1},\eta_j,\underbrace{\zeta_a,\dots,\zeta_a}_{a-j}) \;>\; (\underbrace{\eta_b,\dots,\eta_b}_{i-b},\zeta_i,\zeta_{i+1},\dots,\zeta_{a-1})\,,$$

under componentwise comparison of sequences of length $a+b$. Indeed, η_b is the minimum of these numbers and ζ_a is the maximum, so the components involving either ζ_a or η_b are in strict order. We need only show that if $i-b+a-j < a-b$, so that there are other components, than these remaining components are in order. That is, if $i \le k < j$, then $\zeta_k \le \eta_k$, but this follows from our assumption that $\zeta < \eta$. This completes the proof of the lemma.

8.3. Incomparable chains, factoring polynomials, and gaps

We consider a special case of the polynomial systems of Section 8.1 where the posets are unions of incomparable chains and the constants κ_I are all equal to 1. (This gives the systems in Examples 1.16 and 8.4.) We are able to completely analyze these systems of Wronski polynomials as their solutions correspond to the groupings of linear factors in a univariate polynomial. These systems give examples of posets P for which Y_P^+ is not orientable, yet for which there are provable lower bounds on the numbers of real solutions. These systems also have the new property of gaps in their numbers of real solutions.

Let $\mathbf{a} := (a_1,\dots,a_k)$ be positive integers with $a_1+\cdots+a_k = n$. The poset $P_{\mathbf{a}}$ is the union of incomparable chains of lengths $a_1,\dots,a_k$. For example,

P_{22} =  and P_{34} = 

The number of linear extensions of the poset $P_{\mathbf{a}}$ is the multinomial coefficient $\binom{n}{a_1,\ldots,a_k}$ which is defined by the recursion

$$\binom{n}{a_1,\ldots,a_k} \;=\; \binom{n-a_k}{a_1,\ldots,a_{k-1}} \cdot \binom{n}{a_k, n-a_k}, \tag{8.18}$$

where $\binom{n}{a,b}=0$ unless $a+b=n$ in which case $\binom{n}{a,b}=\frac{n!}{a!b!}$, the binomial coefficient.

Theorem 8.12. *The number, ρ, of real solutions to a general system of Wronski polynomials for the poset $P_{\mathbf{a}}$ with all constants $\kappa_I=1$ satisfies the inequality*

$$\binom{\lfloor\frac{n}{2}\rfloor}{\lfloor\frac{a_1}{2}\rfloor,\ldots,\lfloor\frac{a_k}{2}\rfloor} \;\leq\; \rho \;\leq\; \binom{n}{a_1,\ldots,a_k}\,.$$

The maximum and minimum are both attained. Moreover, at most

$$1 \;+\; \left\lfloor\frac{a_1}{2}\right\rfloor \;+\; \left\lfloor\frac{a_2}{2}\right\rfloor \;+\;\cdots\;+\; \left\lfloor\frac{a_k}{2}\right\rfloor$$

distinct values of ρ can occur, unless all a_i are even, in which case the number of distinct values becomes $n/2$.

This establishes lower bounds for the number of real solutions to these systems of Wronski polynomials, and the new phenomenon of gaps. The polynomial systems of Example 1.16 were Wronski systems when $\mathbf{a}=(2,2)$. As we saw in Table 1.1, only the lower bound of $2=\binom{2}{1,1}$ and the upper bound of $6=\binom{4}{2,2}$ were achieved, but not four, as only $4/2=2$ distinct values can occur. For a more extreme example, suppose that $\mathbf{a}=(4,4,5)$, then the minimum is $\binom{6}{2,2,2}=90$, the maximum is $\binom{13}{4,4,5}=90090$, and the following table gives the $7=1+\lfloor\frac{4}{2}\rfloor+\lfloor\frac{4}{2}\rfloor+\lfloor\frac{5}{2}\rfloor$ different values of ρ that can occur.

ρ	90	210	666	2226	7434	25410	90090

We will prove Theorem 8.12 by reinterpreting a system of Wronski polynomials as a problem of grouping factors in a univariate polynomial. First, for each $j=1,\ldots,k$, let $x^{(j)}_1 > x^{(j)}_2 > \cdots > x^{(j)}_{a_j}$ be variables corresponding to the jth chain in $P_{\mathbf{a}}$. Order ideals are indexed by lists $I=(i_1,\ldots,i_k)$ of integers with $0\leq i_j\leq a_j$ for each j, with the corresponding order ideal

$$\{x^{(1)}_1,\ldots,x^{(1)}_{i_1}\,,\;\ldots\;,\,x^{(j)}_1,\ldots,x^{(j)}_{i_j}\,,\;\ldots\;,\,x^{(k)}_1,\ldots,x^{(k)}_{i_j}\}\,,$$

and the monomial x^I is

$$x^I \;:=\; \prod_{j=1}^{k} x^{(j)}_1\cdots x^{(j)}_{i_j}\,.$$

This has degree $|I|=i_1+\cdots+i_k$. Wronski polynomials have the form $\sum_I c_{|I|}\kappa_I x^I$.

We make an invertible monomial change of variables, from which the alternative formulation is evident. For each $j=1,\ldots,k$, let $y^{(j)}_1 > y^{(j)}_2 > \cdots > y^{(j)}_{a_j}$ be new variables corresponding to the jth chain in $P_{\mathbf{a}}$, and set $y^{(j)}_0 := 1$. If we set $y^{(j)}_i := x^{(j)}_1\cdots x^{(j)}_i$, then $x^{(j)}_i = y^{(j)}_i/y^{(j)}_{i-1}$, and so the change of variables $x^{(j)}_i \Leftrightarrow y^{(j)}_i$ is invertible. In these new variables, the monomial of an order ideal becomes

$$y^I \;:=\; y^{(1)}_{i_1}\cdot y^{(2)}_{i_2}\cdots y^{(k)}_{i_k}\,,$$

and a Wronski polynomial with $\kappa_I = 1$ has the form $\sum_I c_{|I|} y^I$, which is

$$\sum_{i_1,\dots,i_k} c_{i_1+\cdots+i_k} y^{(1)}_{i_1} y^{(2)}_{i_2} \cdots y^{(k)}_{i_k} \;=\; \sum_{d=0}^{n} c_d \Bigl(\sum_{\substack{i_1,\dots,i_k \\ i_1+\cdots+i_k=d}} y^{(1)}_{i_1} y^{(2)}_{i_2} \cdots y^{(k)}_{i_k} \Bigr) .$$

A general system of Wronski polynomials is equivalent to one of the form

$$\sum_{\substack{i_1,\dots,i_k \\ i_1+\cdots+i_k=d}} y^{(1)}_{i_1} y^{(2)}_{i_2} \cdots y^{(k)}_{i_k} \;=\; b_d \qquad \text{for} \quad d = 1, 2, \dots, n\,. \tag{8.19}$$

Suppose that we have a solution $(y^{(j)}_i)^{i=1,\dots a_j}_{j=1,\dots,k}$ to (8.19). For each $j = 1, \dots, k$ form the monic univariate polynomial

$$g_j(t) \;:=\; t^{a_j} \;+\; \sum_{i=1}^{a_j} y^{(j)}_i t^{a_j - i} \;.$$

By (8.19), we have

$$g_1(t) g_2(t) \cdots g_k(t) \;=\; t^n \;+\; \sum_{i=1}^{n} b_j t^{n-j} \;=:\; g(t)\,.$$

Similarly, any such factorization of the monic polynomial $g(t)$ into monic factors where $\deg(g_i) = a_i$ gives a solution to (8.19). The invertible change of coordinates $x^{(j)}_i \Leftrightarrow y^{(j)}_i$ and this calculation proves the following theorem.

THEOREM 8.13. *Solutions to a general system of Wronski polynomials for the poset $P_{\mathbf{a}}$ with constants $\kappa_I = 1$ are equivalent to different factorizations of a general monic univariate polynomial $g(t)$ of degree $n = a_1 + \cdots + a_k$ into monic polynomials $g_1(t), \dots, g_k(t)$, where $g_j(t)$ has degree a_j.*

Consider this for $\mathbf{a} = (2, 2)$. We have a monic quartic polynomial $g(t)$ with distinct roots that we wish to factor into two monic quadratics, $g(t) = g_1(t) g_2(t)$, where the order of the factors matters. There are $6 = \binom{4}{2}$ ways to do this, as we first distribute a pair of the roots of $g(t)$ to $g_1(t)$, and then $g_2(t) = g(t)/g_1(t)$. If all the roots of $g(t)$ are real, this gives $\binom{4}{2}$ real factorizations. If $g(t)$ has complex roots, then it has at least one irreducible (over $\mathbb{R}$) quadratic factor, $q(t)$. There are two factorizations, either $g_1(t) = q(t)$ and $g_2(t) = g(t)/q(t)$, or vice-versa. Thus there are either two or six factorizations, and therefore either two or six solutions to a Wronski polynomial system for $P_{(2,2)}$, which we have already seen.

PROOF OF THEOREM 8.12. By Theorem 8.13, we need only establish the conclusions of Theorem 8.12 for the problem of monic factorization. Factorizations

$$g_1(t) g_2(t) \cdots g_k(t) \;=\; g(t)\,, \tag{8.20}$$

where $g_j(t)$ is a monic polynomial of degree a_j for $j = 1, \dots, k$ and $g(t)$ is monic of degree $n = a_1 + \cdots + a_k$ with distinct roots correspond to distributions of the roots of $g(t)$ among the polynomials $g_1(t), \dots, g_k(t)$, with $g_j(t)$ receiving a_j roots. This number of distributions, and hence factorizations, is a standard counting problem [**149**, Ch. 1] whose solution is the multinomial coefficient $\binom{n}{a_1,\dots,a_k}$.

Suppose now that $g(t)$ is a real polynomial with r real roots and c pairs of complex conjugate roots, all of which are distinct. In every factorization of $g(t)$

into real monic polynomials, each factor must receive both roots in a complex conjugate pair. This limits the possible numbers of such real factorizations.

If every root of $g(t)$ is real, then every factorization is real. Thus the upper bound $\binom{n}{a_1,\ldots,a_k}$ is attained. Since each $g_j(t)$ of odd degree must receive a real root, there are no real factorizations when $g(t)$ has fewer than $|\{j \mid a_j \text{ is odd}\}|$ real roots. Thus the minimum number of factorizations is 0 when more than one a_j is odd. In this case $\lfloor n/2\rfloor > \lfloor a_1/2\rfloor + \cdots + \lfloor a_k/2\rfloor$, and so the multinomial coefficient

$$\binom{\lfloor\frac{n}{2}\rfloor}{\lfloor\frac{a_1}{2}\rfloor,\ldots,\lfloor\frac{a_k}{2}\rfloor} \tag{8.21}$$

vanishes, and the lower bound is also attained in this case.

When all the a_j are even and $g(t)$ has no real roots, then n is even and there are $c = n/2$ pairs of complex conjugate roots to distribute among the polynomials $g_1(t),\ldots,g_k(t)$, with the polynomial $g_j(t)$ receiving $a_j/2$ conjugate pairs. The number of these is the multinomial coefficient (8.21). Suppose that exactly one of the numbers, a_j, is odd, and $g(t)$ has the minimum number $r = 1$ real roots. This root must be distributed to $g_j(t)$. Replacing a_j by a_j-1, reduces to the case with all a_j even, and the number of factorizations is again the multinomial coefficient (8.21).

If $g(t)$ has more real roots than the minimum, there are at least as many real factorizations as when it had the minimum number of real roots—simply pairing the real roots in any way recovers (8.21) real factorizations.

The last statement, the number of different possibilities for the number of real factorizations, follows as this number depends only upon the number of real roots of $g(t)$, and $\rho = 0$ unless $g(t)$ has at least $|\{j \mid a_j \text{ is odd}\}|$ real roots. When the a_i are all even, there is no difference between the cases of $r = 0$ or $r = 2$, so there are $n/2$ different possibilities.

We close with three remarks about these systems.

Remark 8.14. The lower bound in Theorem 8.12 is in fact the signature of the poset $P_{\mathbf{a}}$, and so for these posets, the lower bound of Theorem 8.3 is attained.

To see this, observe that precomposing a linear extension with the inverse of the extension where elements of the jth chain preceed elements of the $(j+1)$st chain identifies the set of linear extensions with the set $S^{\mathbf{a}}$ of minimal coset representatives of the parabolic subgroup $S_{a_1} \times S_{a_2} \times \cdots \times S_{a_k}$ of the symmetric group S_n. The *length* $\lg(w)$ of a permutation w is its number of inversions, $\#\{i < j \mid w(i) > w(j)\}$, and its sign is $\mathrm{sign}(w) = (-1)^{\lg(w)}$. The length generating function for $S^{\mathbf{a}}$ is the q-multinomial coefficient, (the case $k = 2$ is [**149**, Prop. 1.3.7]),

$$\sum_{w\in S^{\mathbf{a}}} q^{\lg(w)} \;=\; \binom{n}{a_1,a_2,\ldots,a_k}_q\,. \tag{8.22}$$

This is defined by the same recursion (8.18) as the ordinary multinomial coefficient, but in terms of the q-binomial coefficients

$$\binom{a+b}{a,b}_q \;:=\; \frac{(1-q^{a+b})(1-q^{a+b-1})\cdots(1-q^2)(1-q)}{(1-q^a)\cdots(1-q^2)(1-q)\cdot(1-q^b)\cdots(1-q^2)(1-q)}\,. \tag{8.23}$$

We evaluate (8.22) at $q = -1$ to compute the sign-imbalance of P. If c is odd, then $1 - q^c = 2$ when $q = -1$. For even exponents, we have

$$1 - q^{2c} \;=\; (1-q^2)(1+q^2+q^4+\cdots+q^{2c-2})\,.$$

Now consider (8.23) when $q = -1$. If both a and b are odd, then (8.23) has one more factor with an even exponent in its numerator then in its denominator, and so it vanishes at $q = -1$. Otherwise (8.23) has the same number of factors with even exponents in its numerator as in its denominator, and so we may cancel all factors of $(1 - q^2)$. Substituting $q = -1$, factors with odd exponents c becomes 2, and these cancel as there is the same number of such factors in the numerator and denominator. Since $(1 + q^2 + q^4 + \cdots + q^{2c-2}) = c$ when $q = -1$, we see that

$$\binom{a+b}{a,b}_{q=-1} \;=\; \binom{\lfloor\frac{a}{2}\rfloor + \lfloor\frac{b}{2}\rfloor}{\lfloor\frac{a}{2}\rfloor, \lfloor\frac{b}{2}\rfloor} .$$

The recursion (8.18) for the q-multinomial coefficients, together with the generating function (8.22), gives

$$\sum_{w \in S^{\mathbf{a}}} (-1)^{\lg(w)} \;=\; \binom{\lfloor\frac{n}{2}\rfloor}{\lfloor\frac{a_1}{2}\rfloor, \ldots, \lfloor\frac{a_k}{2}\rfloor} .$$

Remark 8.15. The possible lengths of maximal chains in the poset $P_{\mathbf{a}}$ are the components of $\mathbf{a}$, and so $P_{\mathbf{a}}$ is $\mathbb{Z}_2$-graded if and only if the numbers a_j all have the same parity. By Lemma 8.5, the toric variety $Y^+_{P_{\mathbf{a}}}$ is orientible if and only if the numbers a_j have the same parity. Nevertheless, Theorem 8.12 gives lower bounds which are non-trivial in some cases when exactly one of the numbers a_j is odd, with the rest even. For example, when for $\mathbf{a} = (4, 4, 5)$, the lower bound is 90.

In this way, Theorem 8.12 extends Theorem 8.3, giving lower bounds for some posets P when Y^+_P is not orientible. This suggests that the theory of Chapter 7 may admit an extension to more systems of sparse polynomials.

Remark 8.16. The most distinctive feature of the polynomial systems for the posets $P_{\mathbf{a}}$ is the existence of gaps in the possible numbers of real solutions. As we saw in the proof of Theorem 8.12, the number of real solutions depends upon the number r of real roots of $g(t)$ and the number c of pairs of complex conjugate roots. There is in fact a generating function for these numbers of real solutions, as Ira Gessel once explained to the author.

Lemma 8.17. *The coefficient of* $x_1^{a_1} \cdots x_k^{a_k}$ *in* $(x_1 + \cdots + x_k)^r (x_1^2 + \cdots + x_k^2)^c$ *is the number of factorizations*

$$g_1(t) \cdot g_2(t) \cdots g_k(t) \;=\; g(t)$$

where $g(t)$ *is a monic real polynomial of degree* $r + 2c = n$ *with* r *distinct real roots and* c *distinct pairs of complex conjugate roots, and* $g_j(z)$ *is a monic real polynomial of degree* a_j *for* $j = 1, \ldots, k$.

Proof. This is a standard application of generating functions, as described in Chapter 1 of [**149**]. We have r red balls and c cyan balls to distribute among k boxes such that if r_i is the number of red balls in box i and c_i is the number of cyan balls in box i, then $r_i + 2c_i = a_i$.

This phenomenon of gaps is not isolated. Many other systems of real polynomials or geometric problems appear to exhibit gaps. Sections 13.3 and 14.2 contain some examples. Unlike lower bounds, we have no current framework with which to understand this phenomenon of gaps.

CHAPTER 9

Enumerative Real Algebraic Geometry

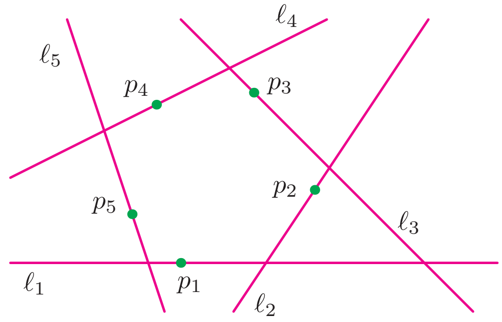
Enumerative geometry is the art of counting geometric figures satisfying conditions imposed by other, fixed, geometric figures. For example, in 1848, Steiner [**150**] asked how many plane conics are tangent to five given conics? His answer, $6^5 = 7776$, turned out to be incorrect, and in 1864 Chasles [**29**] gave the correct answer of 3264. The methods Chasles used were later systematized and used to great effect by Schubert [**128**], who codified the field of enumerative geometry.

This classical work always concerned *complex* figures. It was only in 1984 that the question of reality was posed by Fulton [**51**, p. 55]: "The question of how many solutions of real equations can be real is still very much open, particularly for enumerative problems." He goes on to ask: "For example, how many of the 3264 conics tangent to five general conics can be real?" He later determined that all can be real, but did not publish that result. Independently, Ronga, Tognoli, and Vust [**119**] gave a careful proof that all 3264 can be real.

There are now many geometric problems for which it is known that all solutions can be real. We describe some of these problems, beginning with the problem of conics, then presenting examples coming from kinematics, computational geometry, and hyperplane arrangements before ending with the Schubert Calculus.

9.1. 3264 real conics

The basic idea of the arguments of Fulton and of Ronga, Tognoli, and Vust is to deform the same special configuration. Suppose that $\ell_1, \ldots, \ell_5$ are the lines supporting the edges of a convex pentagon and $p_i \in \ell_i$, $i = 1, \ldots, 5$ are points in the interior of the corresponding edge.



The points in this example are $\{(0,0), (\frac{11}{4}, \frac{3}{2}), (\frac{3}{2}, \frac{7}{2}), (-\frac{1}{2}, \frac{13}{4}), (-1,1)\}$, and the corresponding slopes of the lines are $0, \frac{3}{2}, -1, \frac{1}{2}, -3$.

For every subset S of the lines, there are $2^{\min(|S|, 5-|S|)}$ conics that are tangent to the lines in S and that meet the $5-|S|$ points not on the lines of S. This is the number of complex conics, and it does not depend upon the configuration of (generic) points and lines. However, when the points and lines are chosen in convex position, then all such conics will be real. Altogether, this gives

$$2^0\binom{5}{0} + 2^1\binom{5}{1} + 2^2\binom{5}{2} + 2^2\binom{5}{3} + 2^1\binom{5}{4} + 2^0\binom{5}{5} = 102$$

real conics, that, for each $i = 1, \ldots, 5$ either meet p_i or are tangent to ℓ_i. We draw these in Figure 9.1. Since our pentagon was asymmetric, exactly 51 of these conics

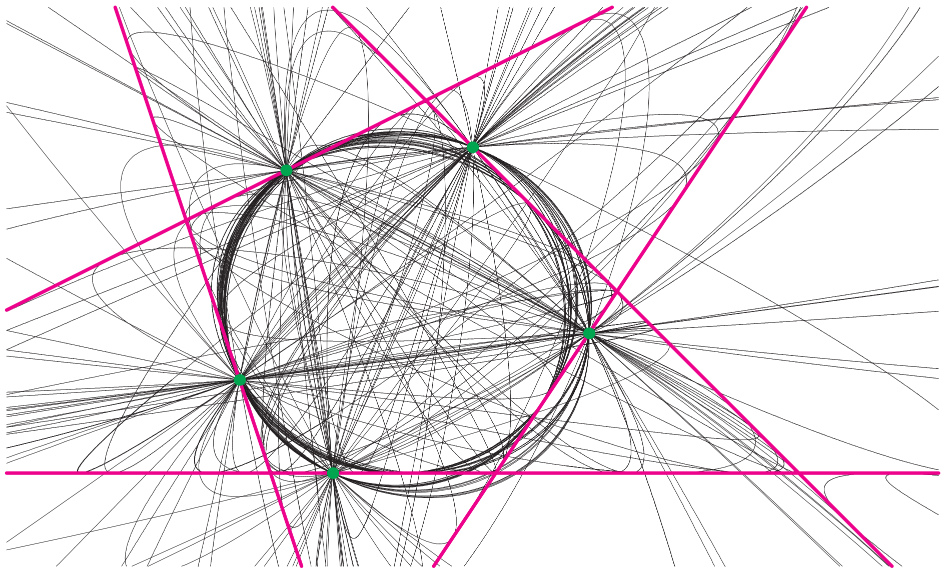
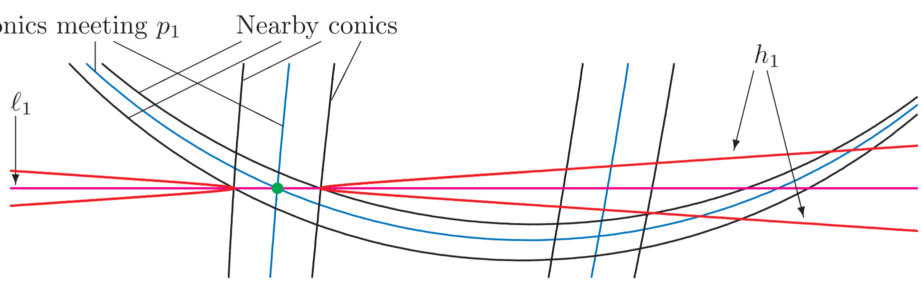


FIGURE 9.1. 102 conics.

meet each point p_i and none of the 51 conics tangent to ℓ_i are tangent at p_i.

The idea now is to replace each pair $p_i \in \ell_i$ by a hyperbola h_i that is close to the pair $p_i \in \ell_i$, in that h_i lies close to its asymptotes, which are two lines close to ℓ_i that meet at p_i. If we do this for one pair $p_i \in \ell_i$, then, for every conic in our configuration, there will be two nearby conics tangent to h_i. To see this, suppose that $i = 1$. Then the set C of conics which satisfy one of the 2^4 conditions "meet p_j" or "tangent to ℓ_j" for each $j = 2, 3, 4, 5$ will form an irreducible curve C. For each conic in C that meets p_1, there will be two nearby conics in C tangent to h_1 near p_1, and for each conic in C tangent to ℓ_1, there will be two nearby conics in C tangent to each of the two nearby branches of h_1. Figure 9.2 illustrates this when C is the curve of conics tangent to ℓ_2, ℓ_3, ℓ_4, and ℓ_5, showing the conics in the family C. We show some conics in the family near the conic that meets p_1.



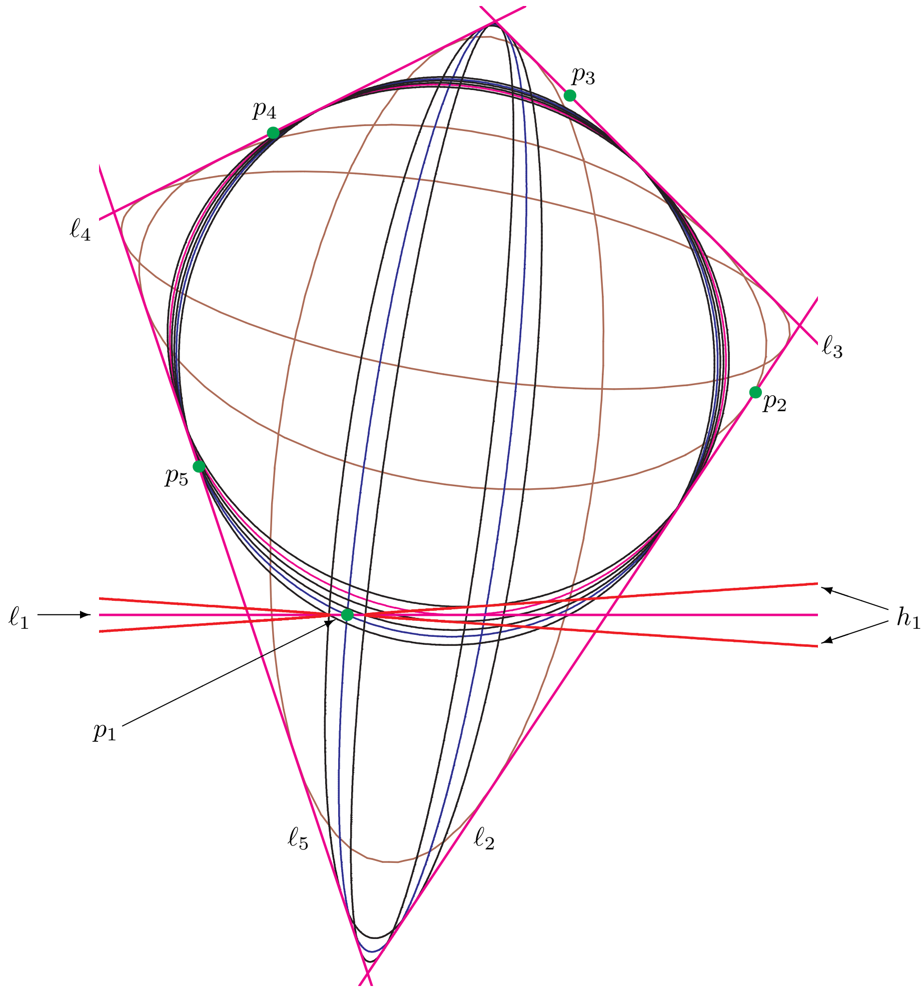
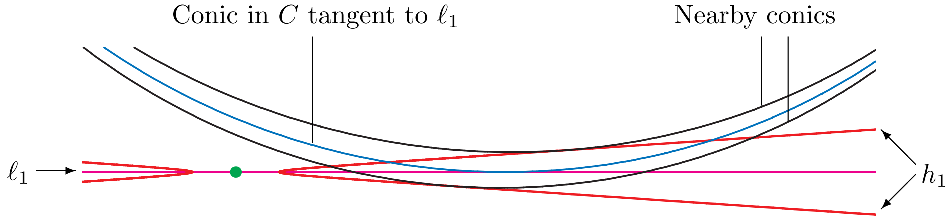


FIGURE 9.2. Family of conics.

Here are conics in the family C near the conic tangent to ℓ_1.

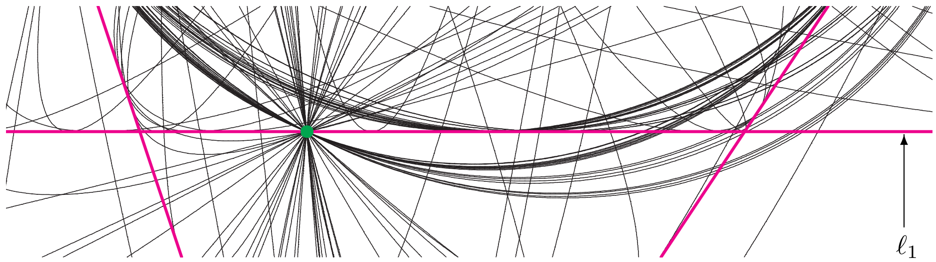


The hyperbola in these pictures is

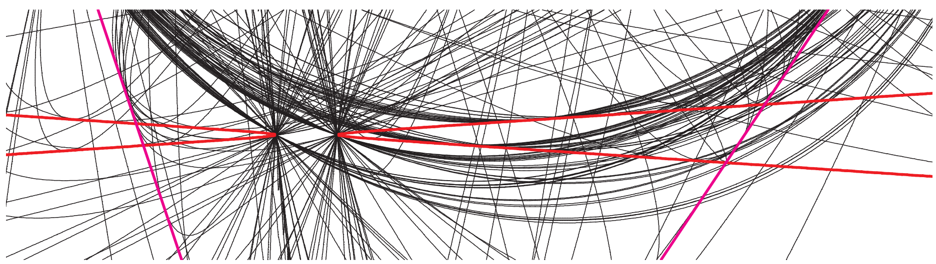
$$h_1 : \qquad (y - \tfrac{1}{15}x)(y + \tfrac{1}{15}x) + \tfrac{1}{15000} = 0,$$

which is close enough to its asymptotes, and these pass through p_1 and are close close enough to ℓ_1 so that for each of the 102 conics of Figure 9.1, there are two nearby conics in C tangent to h_1 that pass through the other points or are tangent to the other lines.

We illustrate this doubling, by first showing the configuration of 102 conics of Figure 9.1 in the neighborhood of ℓ_1,



and then the resulting 204 conics in the same region. Every conic in the first picture is replaced by two nearby conics in the second.



This construction works as explained because no tangent direction to a conic in Figure 9.1 through p_1 meets h_1. It was possible to find such a hyperbola h_1, as no conic was tangent to ℓ_1 at p_1. Figure 9.3 shows the resulting 204 conics that are tangent to h_1 and, for each $i = 2, 3, 4, 5$ either contain p_i or are tangent to ℓ_i.

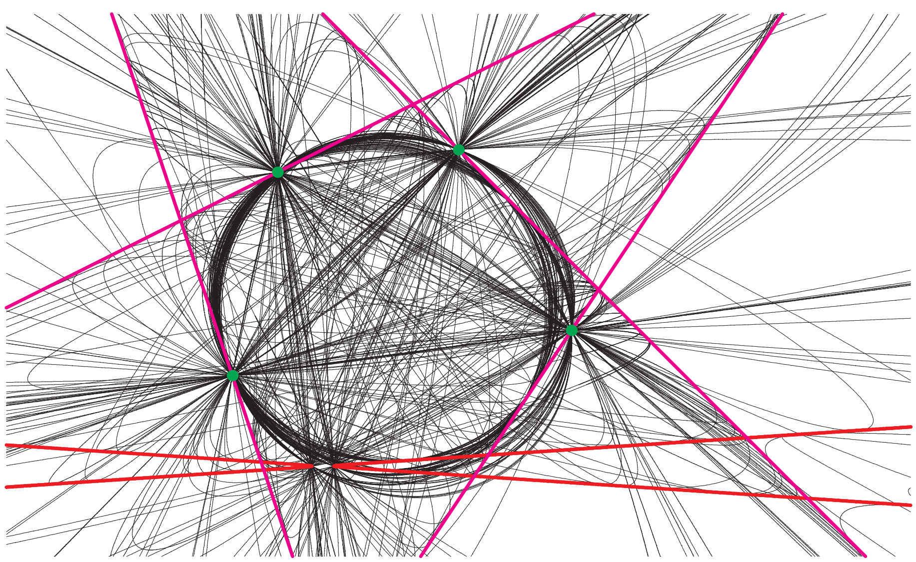


FIGURE 9.3. 204 conics.

If we now replace $p_2 \in \ell_2$ by a similar nearby hyperbola, then the 204 conics become 408. Replacing $p_3 \in \ell_3$ by a nearby hyperbola, will give 816 conics. Continuing with $p_4 \in \ell_4$ gives 1632, and finally replacing $p_5 \in \ell_5$ with a hyperbola gives five hyperbolae, $h_1, \ldots, h_5$ for which there are $2^5 \cdot 102 = 3264$ real conics tangent to each h_i. In this way, the classical problem of 3264 conics can have all of its solutions be real. Observe that this discussion also gives a derivation of the number 3264 without reference to intersection theory [**54**].

9.2. Some geometric problems

There are many geometric problems for which it known to be possible that all solutions are real. We present some from four diverse areas. The first is the Stewart platform from kinematics, and is representative of many other realizability problems in this field. The second is a classical interpolation problem involving plane rational which exhibits a lower bound on its number of solutions and which influenced Welschinger's work on invariants. Then we give a problem of lines tangent to spheres from computational geometry. The last example is the problem of the critical points of real master functions, which arose in the theory of hyperplane arrangements, but has applications in algebraic statistics and optimization. It has the property that all of its solutions are real.

Example 9.1 (The Stewart-Gough platform). The position of a rigid body in $\mathbb{R}^3$ has six degrees of freedom (three rotations and three translations). This is exploited in kinematics, giving rise to the *Stewart-Gough platform* [**62, 151**]: Suppose that we have six fixed points $A_1, \ldots, A_6$ in space and six points $B_1, \ldots, B_6$ on a rigid body B (the framework of Figure 9.4). The body is controlled by varying each distance l_i between the fixed point A_i and the point B_i on B. This may be accomplished by attaching rigid actuators between spherical joints located at

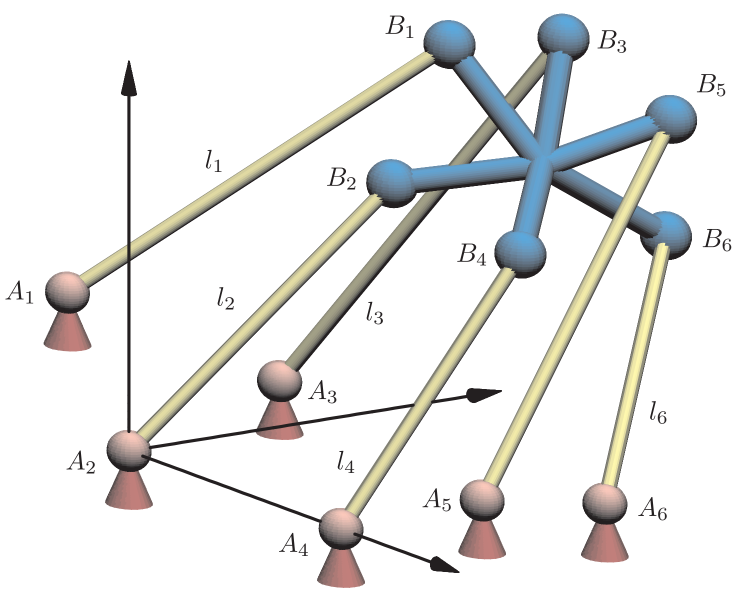


Figure 9.4. A Stewart platform.

the points A_i and B_i, or by suspending the platform from a ceiling with cables. (Apparently, this configuration is often used in factories.)

Given a position of the body B, the distances $l_1, \ldots, l_6$ are uniquely determined. A fundamental problem is the inverse question (in kinematics, this is called the forward problem): Given a platform (positions of the A_i fixed and the relative positions of the B_i specified) and a sextuple of distances $l_1, \ldots, l_6$, what is the position of the platform?

It had long been understood that several positions were possible for a given sextuple of lengths. This led to the following enumerative problem:

For a given (or general) Stewart platform, how many (complex)
positions are there for a generic choice of the distances $l_1, \ldots, l_6$?
How many of these can be real?

In the early 1990's, several approaches (numerical experimentation [**117**], intersection theory [**120**], Gröbner bases [**92**], resultants [**101**], and algebra [**102**]) each showed that there are 40 complex positions of a general Stewart platform. The obviously practical question of how many positions could be real remained open until 1998, when Dietmaier introduced a novel method involving numerical homotopy to find a platform and value of the distances $l_1, \ldots, l_6$ with all 40 positions real.

THEOREM 9.2 (Dietmaier [**37**]). *All* 40 *positions can be real.*

EXAMPLE 9.3 (Real rational cubics through 8 points in $\mathbb{P}^2_{\mathbb{R}}$). In Section 1.5 we remarked that there are 12 singular (rational) cubic curves containing eight general points in the plane. We will derive that number in the process of explaining Kharlamov's treatment of this question over the real numbers.

THEOREM 9.4 ([**33**, Proposition 4.7.3]). *Given eight general points in $\mathbb{P}^2_{\mathbb{R}}$, there are at least eight real rational cubics containing them, and there are choices of the eight points for which all* 12 *rational cubics are real.*

A homogeneous cubic has ten coefficients, so the set of plane cubics is naturally identified with nine-dimensional projective space. Let $p_1, \ldots, p_8$ be general points in $\mathbb{RP}^2$. As the condition for a cubic to contain a point p_i is linear in the coefficients of the cubic, there is a pencil (a $\mathbb{P}^1$) of cubics through these eight points. Let P, Q be two distinct cubics in this pencil, which is then parameterized by $sP + tQ$ for $[s, t] \in \mathbb{P}^1$. By Bézout's Theorem, the cubics P and Q, and hence every cubic in the pencil, vanish at a ninth point, p_9.

It is not hard to see that there is a unique cubic in the pencil that vanishes at any point $p \in \mathbb{RP}^2 - \{p_1, \ldots, p_9\}$. A little harder, but still true, is that there is a unique cubic in the pencil with any given tangent direction at some point p_i. In this way, we have maps

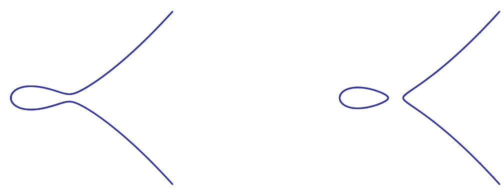
$$\begin{array}{ccc} Z & := & \mathrm{Bl}_{\{p_1,\ldots,p_9\}}\mathbb{RP}^2 \\ C \downarrow & & \downarrow \pi \\ \mathbb{RP}^1 & & \mathbb{RP}^2 \end{array} \tag{9.1}$$

where $\mathrm{Bl}_{\{p_1,\ldots,p_9\}}\mathbb{RP}^2$ is the blow-up of $\mathbb{RP}^2$ in the 9 points, which is obtained by removing each point p_i and replacing it with the tangent directions $\mathbb{RP}^1 \simeq S^1$ to $\mathbb{RP}^2$ at p_i. The map π is the blow-down, and the map C associates a point of Z to the unique curve in the pencil that contains that point.

Because Z is a blow-up, we may compute its Euler characteristic to obtain

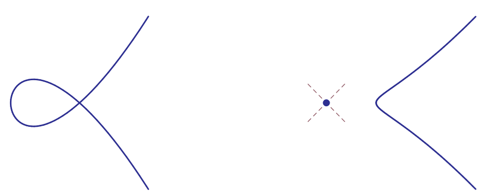
$$\begin{aligned}\chi(Z) &= \chi(\mathbb{RP}^2) - 9\cdot\chi(\text{pt}) + 9\cdot\chi(\mathbb{S}^1)\\ &= 1 - 9 + 0\\ &= -8\,.\end{aligned}$$

The key to Theorem 9.4 is to compute the Euler characteristic of Z a second way using the map $C\colon Z \to \mathbb{RP}^1 \simeq \mathbb{S}^1$. The fibers of this map are the cubic curves in the pencil. Smooth real cubics either have one or two topological components,



and hence are homeomorphic to one or two copies of $\mathbb{S}^1$. In either case, their Euler characteristic is zero, and so the Euler characteristic of Z, -8, is the sum of the Euler characteristics of singular fibers of the map C.

Because the points $p_1, \ldots, p_8$ were general, the only possible singular fibers are nodal cubics, and there are two types of real nodal cubics.



The first is the topological join of two circles and has Euler characteristic -1, while the second is the disjoint union of a circle with a point and therefore has Euler characteristic 1. Thus, if n is the number of nodal cubics in the pencil and s is the number with a solitary point, then we have $s - n = -8$. The same argument over the complex numbers implies there are 12 complex rational cubics, and so we also have $n + s \leq 12$. (To see that this number is 12, note that the diagram (9.1) leads to the computation of Euler characteristic

$$\chi(Z_{\mathbb{C}}) = \chi(\mathbb{P}^2) - 9 + 9\cdot\chi(\mathbb{P}^1) = 3 - 9 + 18 = 12\,.$$

For the computation over $\mathbb{P}^1$, observe that smooth complex cubics have genus 1 and Euler characteristic zero, so only but singular complex cubics contribute and they all have Euler characteristic 1.)

This system, $s - n = -8$ and $s + n = 12$, has three solutions with n, s non-negative integers, $(n, s) \in \{(8,0), (9,1), (10,2)\}$. Thus there are at least eight real cubics through the eight points, and this is the derivation of the Welschinger invariant $W_3 = 8$. Moreover, if there are two cubics in the pencil with solitary points, then all 12 rational cubics will be real. Figure 9.5 shows two cubics which generate such a pencil. We conclude that there will be 12 real rational cubics interpolating any subset of eight of the nine points where these two cubics meet.

The question of how many of the N_d (1.8) rational curves of degree d which interpolate $3d{-}1$ points in $\mathbb{RP}^2$ can be real remains open.

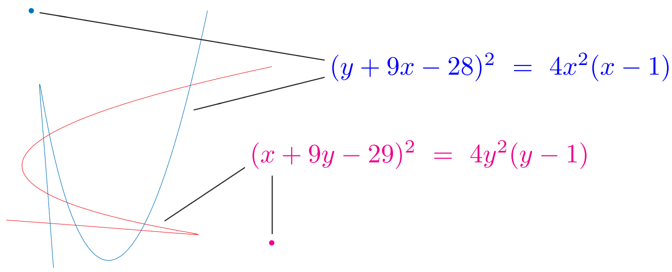


FIGURE 9.5. Two cubics generating a pencil with 12 real rational cubics.

EXAMPLE 9.5 (Common tangent lines to spheres). How many common tangent lines are there to $2n-2$ spheres in $\mathbb{R}^n$? For example, when $n = 3$, how many common tangent lines are there to four spheres in $\mathbb{R}^3$? (The number $2n-2$ is the dimension of the space of lines in $\mathbb{R}^n$ and is necessary for there to be finitely many common tangents.) Despite its simplicity, this question does not seem to have been asked classically, but rather arose in computational geometry. Macdonald, Pach, and Theobald [**97**] gave an elementary argument that four spheres with the same radius in $\mathbb{R}^3$ can have at most 12 common tangents. Then they considered the symmetric configuration where the spheres are centered at the vertices of a regular tetrahedron. If the spheres overlap pairwise, but no three have a common point, then there will be exactly 12 common real tangents, as illustrated in Figure 9.6.

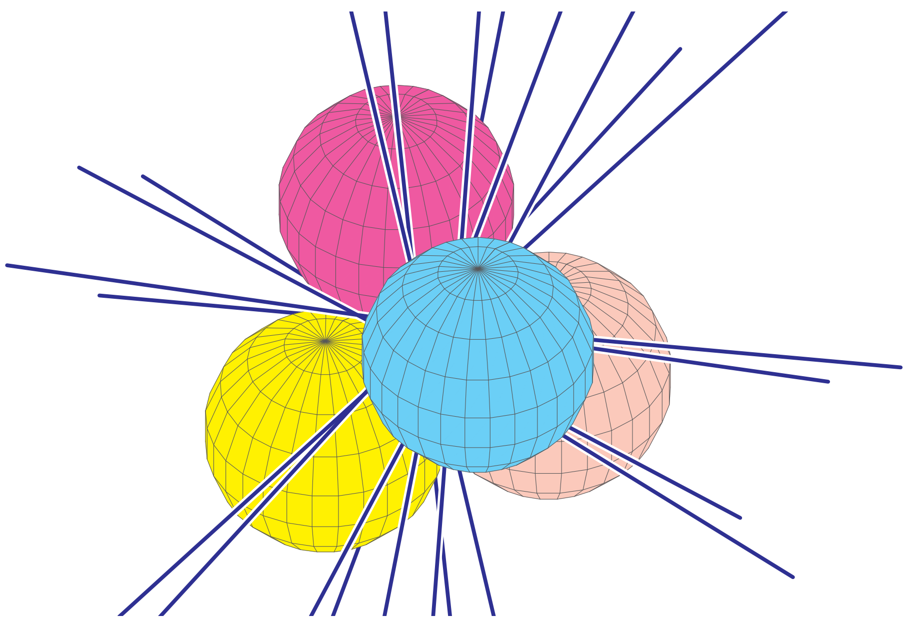


FIGURE 9.6. Four spheres with 12 common tangents.

The general case was established soon after that [**147**].

Theorem 9.6. *$2n-2$ general spheres in $\mathbb{R}^n$ ($n \geq 3$) have $3 \cdot 2^{n-1}$ complex common tangent lines, and there are $2n-2$ such spheres with all common tangent lines real.*

The same elementary arguments of Macdonald, Pach, and Theobald give a bound valid for all n and for spheres of any radius, and a generalization of the symmetric configuration of Figure 9.6 gives a configuration of $2n-2$ spheres having $3 \cdot 2^{n-1}$ common real tangents.

Megyesi [**98**] showed that this result for $n = 3$ remains true if the spheres have coplanar centers (Figure 9.7), but that there can only be eight common real

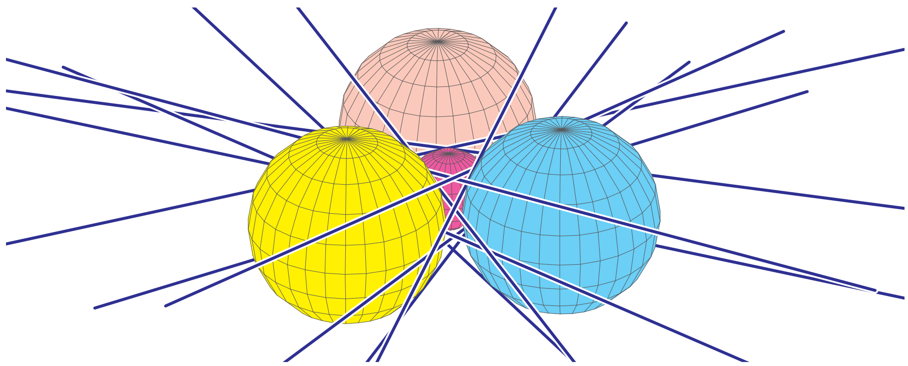


Figure 9.7. Four spheres with coplanar centers and 12 common tangents.

tangents (out of 12 complex ones) if the spheres have the same radii (Figure 9.8).

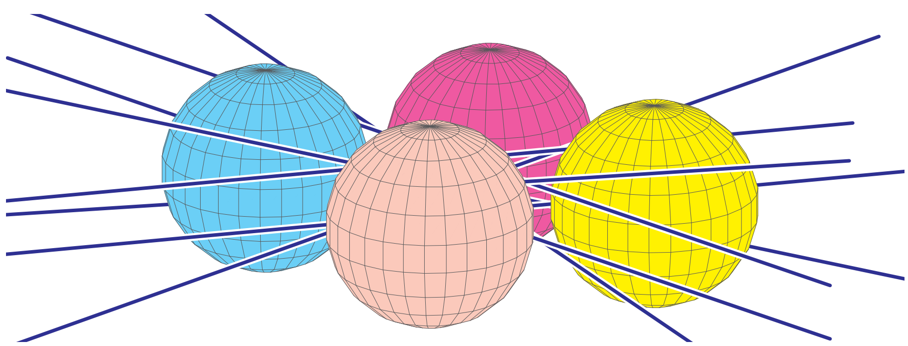


Figure 9.8. Four equal spheres with coplanar centers and eight common tangents.

The spheres in Figures 9.6 and 9.7 are not disjoint, in fact their union is connected. Fulton asked if it were possible for four disjoint spheres to have 12 common real tangents. A perturbation of the configuration of Figure 9.7 gives four pairwise disjoint spheres with 12 common tangents, as we show in Figure 9.9 The three large spheres have radius $4/5$ and are centered at the vertices of an equilateral triangle of side length $\sqrt{3}$, while the smaller sphere has radius $1/4$ and is centered on the axis of symmetry of the triangle, but at a distance of $35/100$ from the plane of the triangle. It remains an open question whether it is possible for four disjoint unit spheres to have 12 common tangents.

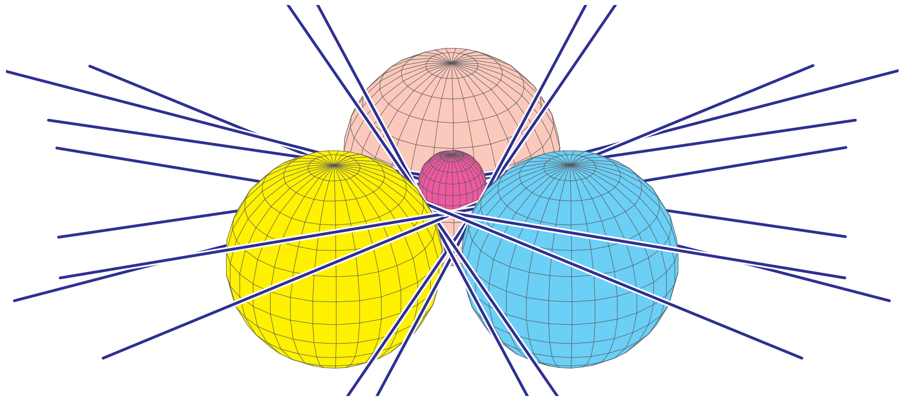


FIGURE 9.9. Four disjoint spheres with 12 common tangents.

EXAMPLE 9.7 (Critical points of master functions). Our last sampling of geometric problems that can have all their solutions be real is one that has only real solutions. This involves hyperplane arrangements and arose in Varchenko's study of quasiclassical asymptotics of the Knizhnik-Zamolodchikov equation with values in certain $\mathfrak{sl}_2$-representations [**161**]. It concerns the critical points of certain real master functions, in the sense of Section 6.1.

Let $p_1(x), \ldots, p_m(x)$ be real degree one polynomials on $\mathbb{C}^n$. Their product $\prod_i p_i(x) = 0$ defines an arrangement $\mathcal{H}$ of real hyperplanes in $\mathbb{C}^n$. Let $\beta = (b_1, \ldots, b_m)$ be positive integral weights for $\mathcal{H}$ and consider the master function,

$$p(x)^\beta \ := \ p_1(x)^{b_1} \cdot p_2(x)^{b_2} \cdots p_m(x)^{b_m} ,$$

on the complement $M_{\mathcal{H}} := \mathbb{C}^n \setminus \mathcal{H}$ of the arrangement. The critical points of $p(x)^\beta$ in $M_{\mathcal{H}}$ are defined by the system of equations,

$$\frac{\partial}{\partial x_i} p(x)^\beta \ = \ 0 , \qquad \text{for} \quad i = 1, \ldots, n .$$

The hyperplanes in $\mathcal{H}$ divide the real points $M^{\mathbb{R}}_{\mathcal{H}}$ of M into connected components, called *chambers*. Each chamber is a polyhedron, and some chambers are bounded and some are unbounded. We give Varchenko's Theorem in this context. A hyperplane arrangement is *essential* if the normal vectors of its hyperplanes span $\mathbb{R}^n$.

THEOREM 9.8. *Suppose that $\mathcal{H}$ is an essential arrangement of hyperplanes. Then all critical points of $p(x)^\beta$ are real and nondegenerate and they lie in the bounded chambers of $M^{\mathbb{R}}_{\mathcal{H}}$, with one critical point in each bounded chamber.*

Consider a nontrivial example. Suppose that $n = 2$ and we have four degree one polynomials,

$$p_1 \ = \ x\,, \quad p_2 \ = \ y\,, \quad p_3 \ = \ 2x + y - 2\,, \quad p_4 \ = \ x - 2y + 1\,.$$

For unit weights $b_i = 1$, we seek critical points of $p^\beta = xy(2x + y - 2)(x - 2y + 1)$. Taking logarithmic derivatives, these are the solutions to

$$\frac{1}{x} + \frac{2}{2x + y - 2} + \frac{1}{x - 2y + 1} \ = \ \frac{1}{y} + \frac{1}{2x + y - 2} - \frac{2}{x - 2y + 1} \ = \ 0\,.$$

Also consider the directional derivative $(\partial_x - 2\partial_y)\log(p^\beta)$. If we clear denominators of these three derivatives, we obtain

$$\begin{aligned} e &:= 2y^2 + 6xy - 5y - 6x^2 + 2 \;=\; 0\,, \\ f &:= 3y^2 + 3xy - 5y - x^2 + 1 \;=\; 0\,, \qquad \text{and} \\ g &:= 2y^2 - 10xy - y + 2x^2 + 2x \;=\; 0\,. \end{aligned}$$

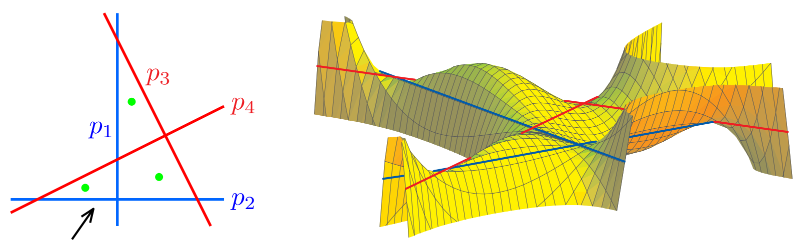
We may eliminate y^2 with the combination $h := 3e - 2f = 12xy - 5y - 16x^2 + 4$, and both y^2 and xy with $k := (-9e + 8f - 3g)/2 = 4y + 20x^2 - 3x - 5$. Finally, $(4x-1)k - 2h$ eliminates y,

$$l \;:=\; 80x^3 - 24x^2 - 15x + 3\,.$$

We may solve k and l to get the three solutions

$$(-0.40093, 0.14555),\;\; (0.17930, 1.22372), \;\; \text{and} \;\; (0.52163, 0.28073)\,.$$

These are the three critical points, with one in each of the three bounded regions in the complement of the lines. The line arrangement and the critical points are on the left below, and the graph of the master function is on the right. (The arrow in the line arrangement indicates the direction of view for the graph.)



PROOF OF THEOREM 9.8. We first show that each bounded chamber contains a unique nondegenerate critical point of $p(x)^\beta$, and then show there are no other critical points. Write each polynomial as $p_i(x) = a_i + \mathbf{v}_i \cdot x$, where $a_i \in \mathbb{R}$ and $\mathbf{v}_i \in \mathbb{R}^n$. The hyperplane arrangement is essential so the vectors $\mathbf{v}_i$ span $\mathbb{R}^n$.

The critical points of $p(x)^\beta$ in a bounded chamber Δ are equal to those of $|p(x)|^\beta$, and therefore to the critical points of its logarithm,

$$b_1 \log|p_1(x)| \;+\; b_2 \log|p_2(x)| \;+\; \cdots \;+\; b_m \log|p_m(x)|\,. \tag{9.2}$$

This is a sum of concave functions on Δ. As its limit is $-\infty$ at every boundary point and Δ is bounded, it has a unique maximum, and therefore a unique critical point in Δ. This critical point is nondegenerate, as the Hessian of (9.2) is negative definite on Δ. Indeed, each summand has a negative semidefinite Hessian whose null space is the hyperplane $\mathbf{v}_i \cdot x = 0$. As the hyperplane arrangement $\mathcal{H}$ is essential, these null spaces have intersection $\{0\}$.

Now suppose that x lies in an unbounded component of $M^{\mathbb{R}}_{\mathcal{H}}$. Then there is a ray $x + \mathbb{R}_{\geq}\mathbf{u}$ emanating from x which does not meet $\mathcal{H}$. Thus

$$p_i(x + s\mathbf{u}) \;=\; p_i(x) + s\mathbf{v}_i \cdot \mathbf{u} \;=\; 0\,,$$

has no solutions with $s \geq 0$ for any i, and so $\mathbf{v}_i \cdot \mathbf{u}/p_i(x) \geq 0$ for all i. But then the logarithmic directional derivative of p^β in the direction $\mathbf{u}$ at x,

$$D_{\mathbf{u}} \log|p(x)|^\beta \;=\; b_1\frac{\mathbf{v}_i \cdot \mathbf{u}}{p_1(x)} \;+\; b_2\frac{\mathbf{v}_i \cdot \mathbf{u}}{p_2(x)} \;+\; \cdots \;+\; b_m\frac{\mathbf{v}_i \cdot \mathbf{u}}{p_m(x)}\,, \tag{9.3}$$

is strictly positive, because it is a sum of nonnegative terms, not all of which are zero, as the hyperplane arrangement is essential. Thus $p(x)^\beta$ has no critical points in unbounded regions of $M_{\mathcal{H}}^{\mathbb{R}}$.

Lastly, let $x \in M_{\mathcal{H}} \setminus \mathbb{R}^n$ be a nonreal point in the hyperplane complement. Write x as a sum of its real and imaginary parts, $x = y + \sqrt{-1}\mathbf{u}$ with $y, \mathbf{u} \in \mathbb{R}^n$. Then $p_i(x) = p_i(y) + \sqrt{-1}\mathbf{v}_i \cdot \mathbf{u}$ is the decomposition of $p_i(x)$ into its real and imaginary parts. From (9.3), we can see that the imaginary part of the logarithmic directional derivative is

$$\Im(D_{\mathbf{u}} \log |p(x)|^\beta) \;=\; -b_1\frac{(\mathbf{v}_1\cdot\mathbf{u})^2}{|p_1(x)|^2} \;-\; b_2\frac{(\mathbf{v}_2\cdot\mathbf{u})^2}{|p_2(x)|^2} \;-\; \cdots \;-\; b_m\frac{(\mathbf{v}_m\cdot\mathbf{u})^2}{|p_m(x)|^2}\,,$$

which is strictly negative. Thus $p(x)^\beta$ has no critical points in $M_{\mathcal{H}} \setminus \mathbb{R}^n$, which completes the proof.

This argument leads to the same conclusion if the weights β are positive real numbers. In that case, $p(x)^\beta$ should be interpreted as a multivalued function on $M_{\mathcal{H}}$—this is the context in which Varchenko studied this system.

Theorem 9.8, and more generally the critical point equations

$$D_{\mathbf{u}} \log p(x)^\beta \;=\; b_1\frac{v_1\cdot u}{a_1+\mathbf{v}_1\cdot x} \;+\; b_2\frac{v_1\cdot u}{a_2+\mathbf{v}_1\cdot x} \;+\; \cdots \;+\; b_m\frac{v_1\cdot u}{a_m+\mathbf{v}_1\cdot x} \;=\; 0\,,$$

have played a recent role in some applications of algebraic geometry. This includes maximum likelihood estimation for linear statistical models [**28**] and [**110**, § 1.2.1] and the analysis of the central path of interior point methods in linear programming [**32**]. These applications, particularly the last one, used that all solutions to these equations were all real.

9.3. Schubert Calculus

The largest class of problems which have been studied from the perspective of having all solutions be real come from the classical Schubert Calculus of enumerative geometry, which involves linear spaces meeting other linear spaces. The simplest nontrivial example illustrates some of the vivid geometry behind this class of problems. Consider the following question:

How many line transversals are there to four given lines in space?

To answer this problem of four lines, first consider three lines. They lie on a unique hyperboloid. (See Figure 9.10.) This hyperboloid has two rulings by lines. The three lines are in one ruling, and the other ruling (which is drawn on the hyperboloid in Figure 9.10) consists of the lines which meet the three given lines.

The fourth line will meet the hyperboloid in two points (the hyperboloid is defined by a quadratic polynomial). Through each point of intersection there will be one line in the second family, and that line will meet our four given lines. In this way, we see that the answer to the question is 2. Note that the fourth line may be drawn so that it meets the hyperboloid in two real points, and both solution lines will be real when this happens.

Let $\mathrm{Gr}(p, m{+}p)$ be the *Grassmannian* of p-dimensional linear subspaces (*p-planes*) in an $(m{+}p)$-dimensional vector space, which is an algebraic manifold of dimension mp. The Schubert Calculus involves fairly general incidence conditions imposed on p-planes H. These general conditions are imposed by *flags*, which are

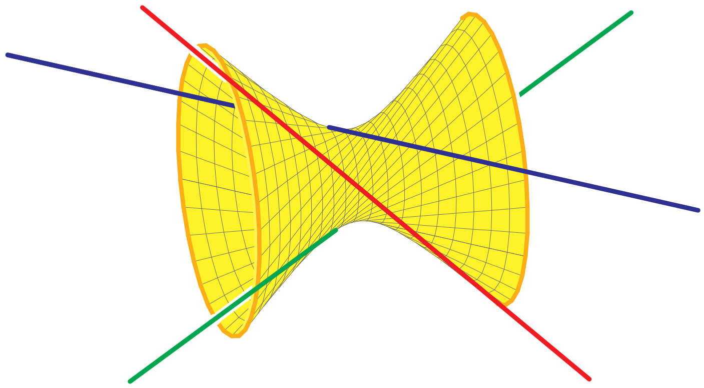


FIGURE 9.10. Hyperboloid containing three lines.

sequences of linear subspaces, one contained in the next. More specifically, a flag $F_\bullet$ is a sequence

$$F_\bullet \ : \ F_1 \ \subset \ F_2 \ \subset \ \cdots \ \subset \ F_{m+p-1} \ \subset \ F_{m+p}\,,$$

where F_i is a linear subspace having dimension i.

These general conditions are called *Schubert conditions* and are indexed by sequences α in the Bruhat poset $\binom{[m+p]}{p}$. The set of all p-planes satisfying the condition α imposed by the flag $F_\bullet$ is a *Schubert variety*, defined by

$$X_\alpha F_\bullet \ := \ \{H \in \mathrm{Gr}(p, m+p) \ \mid \ \dim(H \cap F_{\alpha_j}) \geq j \quad \text{for} \quad j = 1, \ldots, p\}\,.$$

This subvariety of the Grassmannian has dimension $|\alpha| := \alpha_1 - 1 + \cdots + \alpha_p - p$. Its codimension is $mp - |\alpha|$.

Write $\square$ for the Schubert condition $(m, m+2, \ldots, m+p)$, the unique Schubert condition α with $|\alpha| = mp - 1$. If K is a linear subspace of codimension p and $F_\bullet$ is a flag with $F_m = K$, then

$$X_\square F_\bullet \ := \ X_{(m,m+2,\ldots,m+p)} F_\bullet \ = \ \{H \ \mid \ H \cap K \neq \{0\}\}\,. \tag{9.4}$$

The condition that H meets K is a *simple Schubert condition* and $X_\square F_\bullet$ is a *simple Schubert variety*. An important class of Schubert conditions are those involving only a single subspace in the flag called *special Schubert conditions*. There are two types of special Schubert conditions. Let $0 < a \leq m$ and $0 < b \leq b$, then the corresponding *special Schubert varieties* are

$$X_{m+1-a,m+2,\ldots,m+p} F_\bullet \ := \ \{H \ \mid \ H \cap F_{m+1-a} \neq \{0\}\}\,,$$
$$X_{m,m+1,\ldots,m-1+b,m+1+b,\ldots,m+p} F_\bullet \ := \ \{H \ \mid \ \mathrm{span}\{H, F_{m-1+b}\} \neq \mathbb{C}^{m+p}\}\,.$$

A list $\alpha^1, \ldots, \alpha^n$ of Schubert conditions satisfying the numerical condition

$$\sum_{i=1}^{n} (mp - |\alpha^i|) \ = \ mp \tag{9.5}$$

will be called a *Schubert problem*. The reason for this terminology is that the numerical condition (9.5) implies that the expected dimension of an intersection of Schubert varieties given by $\alpha^1, \ldots, \alpha^n$ is zero. For a Schubert problem $\alpha^1, \ldots, \alpha^n$, the Kleiman-Bertini Theorem [**86**] implies that if flags $F^1_\bullet, \ldots, F^n_\bullet$ are in general position, then the intersection

$$X_{\alpha^1}F^1_\bullet \cap X_{\alpha^2}F^2_\bullet \cap \cdots \cap X_{\alpha^n}F^n_\bullet \tag{9.6}$$

is transverse and zero-dimensional, so there will be finitely many complex p-planes H which satisfy the Schubert condition α^i imposed by flag $F^i_\bullet$ for $i = 1, \ldots, n$. This number of p-planes may be computed using algebraic algorithms from the Schubert Calculus or combinatorial algorithms based on the Littlewood-Richardson rule [**54**].

Schubert problems are another class of geometric problems that can have all their solutions be real.

THEOREM 9.9. *For any Schubert problem $\alpha^1, \alpha^2, \ldots, \alpha^n$ for $\mathrm{Gr}(p, m+p)$, there exist real flags $F^1_\bullet, F^2_\bullet, \ldots, F^n_\bullet$ such that the intersection* (9.6) *is transverse with all points real.*

Theorem 9.9 was proved in several stages. First, when $p = 2$ [**140**], and then for any p, but only for special Schubert conditions [**141**], and then finally for general Schubert conditions by Vakil [**159**].

EXAMPLE 9.10 (Quantum Schubert Calculus). A related geometric problem that can have all of its solutions be real arises in the quantum Schubert calculus. Given points $s_1, \ldots, s_{d(m+p)+mp} \in \mathbb{P}^1$ and m-planes $K_1, \ldots, K_{d(m+p)+mp}$ in $\mathbb{C}^{m+p}$, there are finitely many rational curves $\gamma\colon \mathbb{P}^1 \to \mathrm{Gr}(p, m+p)$ of degree d so that

$$\gamma(s_i) \;\cap\; K_i \;\neq\; \{0\} \qquad i = 1, 2, \ldots, d(m+p) + mp\,. \tag{9.7}$$

These are *simple quantum Schubert conditions*. More generally, one could (but we will not) impose the condition that the p-plane $\gamma(s_i)$ lie in some predetermined Schubert variety. The number of solutions to such problems are certain Gromov-Witten invariants of the Grassmannian, and may be computed by the quantum Schubert Calculus [**12, 75, 134, 158**].

THEOREM 9.11 ([**142**]). *There exist points $s_1, s_2, \ldots, s_{d(m+p)+mp} \in \mathbb{RP}^1$ and m-planes $K_1, K_2, \ldots, K_{d(m+p)+mp} \subset \mathbb{R}^{m+p}$ so that every rational curve $\gamma\colon \mathbb{P}^1 \to \mathrm{Gr}(p, m+p)$ of degree d satisfying* (9.7) *is real.*

EXAMPLE 9.12 (Theorem of Mukhin, Tarasov, and Varchenko). In May of 1995, Boris Shapiro communicated to the author a remarkable conjecture that he and his brother Michael had made concerning reality in the Schubert Calculus. They conjectured that there would only be real points in a zero-dimensional intersection of Schubert varieties given by flags osculating the rational normal curve. Subsequent computation [**121, 143**] gave strong evidence for the conjecture and revealed that the intersection should be transverse. Partial results were obtained [**141, 46**], and the full conjecture was proven by Mukhin, Tarasov, and Varchenko [**104**]. They later gave a second proof [**106**], which is different from their original proof and gave a proof of transversality.

This *Shapiro Conjecture* has been a motivating conjecture for the study of reality in the Schubert Calculus with several interesting (and as-yet-unproven) generalizations that we will discuss in subsequent chapters. Let γ be the rational normal (or moment) curve in $\mathbb{C}^{m+p}$, which we will take to be the image of the map

$$\gamma(t) \;=\; (1,\ t,\ t^2,\ \ldots,\ t^{m+p-1}) \;\in\; \mathbb{C}^{m+p}\,,$$

defined for $t \in \mathbb{C}$. Given $t \in \mathbb{C}$, the *osculating flag $F_\bullet(t)$* is the flag of subspaces whose i-plane is the linear span of the first i derivatives of γ, evaluated at t

$$F_i(t) \ := \ \mathrm{span}\{\gamma(t),\ \gamma'(t),\ \gamma''(t),\ \ldots,\ \gamma^{(i-1)}(t)\}\,.$$

We may also define $F_\bullet(\infty)$ to be the limit as $s \to 0$ of $F_\bullet(\frac{1}{s})$, to get a family of flags $F_\bullet(t)$ for $t \in \mathbb{P}^1$. Here is the strongest form of the Shapiro Conjecture that has been proven [**106**].

THEOREM 9.13. *For any Schubert problem $\alpha^1, \ldots, \alpha^n$ for $\mathrm{Gr}(p, m+p)$ and for every choice of n distinct points $s_1, \ldots, s_n \in \mathbb{RP}^1$, the intersection*

$$X_{\alpha^1}F_\bullet(s_1) \cap X_{\alpha^2}F_\bullet(s_2) \cap \cdots \cap X_{\alpha^n}F_\bullet(s_n)$$

is transverse with all points real.

Example 1.10 discussed this in the context of the problem of four lines. For that, the Shapiro Conjecture asserts that given any four lines tangent to the real rational normal curve, there will always be two lines meeting them, and both will be real. We also gave a more elementary formulation of (a special case of) the Shapiro Conjecture in terms of the Wronski map in Schubert Calculus. The remainder of this book will explore what we know and do not know about the Shapiro Conjecture and some of its extensions and generalizations.

CHAPTER 10

The Shapiro Conjecture for Grassmannians

In Example 9.12 we discussed the Shapiro Conjecture (Theorem of Mukhin, Tarasov, and Varchenko), which asserts that all solutions to a problem in the Schubert Calculus were real, when the flags were chosen to osculate a real rational normal curve. We also presented this same conjecture and result in Section 1.4, but phrased in terms of the Wronski map. Our purpose here is reconcile these two different points of view. This not only connects these two formulations and links them to the discussion in Section 8.2, but provides a foundation for the rest of this book.

We work in the Grassmannian $\mathrm{Gr}(p, m{+}p)$ of p-planes in $\mathbb{C}^{m+p}$. Recall from Section 9.3 that Schubert varieties are indexed by elements of the Bruhat poset $\alpha \in \binom{[m+p]}{p}$. Given a flag $F_\bullet$, the corresponding Schubert variety is

$$X_\alpha F_\bullet \;=\; \{H \in \mathrm{Gr}(p, m{+}p) \;\mid\; \dim(H \cap F_{\alpha_j}) \geq j \quad \text{for} \quad j = 1, \ldots, p\}\,.$$

This has dimension $|\alpha| := \sum_j (\alpha_j - j)$.

In Example 9.12, we considered the rational normal curve γ in $\mathbb{C}^{m+p}$, which we took to be the image of the map

$$\gamma(t) \;=\; (1,\ t,\ t^2,\ \ldots,\ t^{m+p-1}) \;\in\; \mathbb{C}^{m+p}\,,$$

defined for $t \in \mathbb{C}$. For a point $t \in \mathbb{C}$, the osculating flag $F_\bullet(t)$ is the flag of subspaces whose i-plane is the linear span of $\gamma(t)$ and the the first $i{-}1$ derivatives of γ, evaluated at t,

$$F_i(t) \;:=\; \mathrm{span}\{\gamma(t),\ \gamma'(t),\ \gamma''(t),\ \ldots,\ \gamma^{(i-1)}(t)\}\,.$$

We define $F_\bullet(\infty)$ to be the limit as $s \to 0$ of $F_\bullet(\frac{1}{s})$, to get a family of flags $F_\bullet(t)$ for $t \in \mathbb{P}^1$.

We investigate the Shapiro Conjecture for Grassmannians (Theorem of Mukhin, Tarasov, and Varchenko).

Theorem 9.13. *If $\alpha^1, \ldots, \alpha^n \in \binom{[m+p]}{p}$ are Schubert conditions that satisfy $\sum_i (mp - |\alpha^i|) = mp$, then for every choice of n distinct points $s_1, \ldots, s_n \in \mathbb{RP}^1$, the intersection*

$$X_{\alpha^1} F_\bullet(s_1) \cap X_{\alpha^2} F_\bullet(s_2) \cap \cdots \cap X_{\alpha^n} F_\bullet(s_n)$$

is transverse with all points real.

We show that a special case of this theorem is equivalent to the statement (Theorem 1.9) of the Shapiro Conjecture from Chapter 1, prove this special case in an asymptotic sense, and discuss Grassmann duality which helps to understand the different formulations of these results.

10.1. The Wronski map and Schubert Calculus

In the special case when all the Schubert conditions are simple (so that $\alpha = \square = (m, m+2, \ldots, m+p)$ and $|\alpha| = mp-1$), the Shapiro Conjecture has another formulation in terms of the Wronski map. The *Wronskian* of a list $f_1(t), \ldots, f_m(t)$ of polynomials of degree $m+p-1$ is the determinant

$$\mathrm{Wr}(f_1, f_2, \ldots, f_m) \;:=\; \det\left(\left(\frac{d}{dt}\right)^{j-1} f_i(t)\right)_{i,j=1,\ldots,m}, \tag{10.1}$$

which is a polynomial of degree mp, when the polynomials $f_1(t), \ldots, f_m(t)$ are generic among all polynomials of degree $m+p-1$.

Up to a scalar factor, the Wronskian depends only upon the linear span of the polynomials $f_1(t), \ldots, f_m(t)$. Removing these ambiguities, gives the *Wronski map*

$$\mathrm{Wr} \;:\; \mathrm{Gr}(m, \mathbb{C}_{m+p-1}[t]) \;\longrightarrow\; \mathbb{P}^{mp} \;=\; \mathbb{P}(\mathbb{C}_{mp}t)\;, \tag{10.2}$$

from the Grassmannian of m-planes in the space of polynomials of degree $m+p-1$ to the space of polynomials of degree mp, modulo scalars.

Let us begin with the moment (rational normal) curve. For $t \in \mathbb{C}$, set

$$\gamma(t) \;=\; (1, t, t^2, \ldots, t^{m+p-1})^T \;\in\; \mathbb{C}^{m+p}\,.$$

Let $\Gamma = \Gamma(t)\colon \mathbb{C}^m \to \mathbb{C}^{m+p}$ be the map such that

$$\Gamma(\mathbf{e}_i) \;=\; \gamma^{(i-1)}(t)\,, \tag{10.3}$$

the $(i-1)$-th derivative of γ. (We take $\mathbf{e}_1, \mathbf{e}_2, \ldots$ to be the standard basis vectors of the vector space in which we are working.)

A polynomial f corresponds to a linear form (also written f):

$$f \;:\; \mathbb{C}^{m+p} \longrightarrow \mathbb{C} \qquad \text{so that} \qquad f\circ\gamma(t) \;=\; f(t)\,.$$

The matrix in the definition (10.1) of the Wronskian is the matrix of the composition

$$\mathbb{C}^m \xrightarrow{\;\Gamma(t)\;} \mathbb{C}^{m+p} \xrightarrow{\;\Psi\;} \mathbb{C}^m\,, \tag{10.4}$$

where the rows of Ψ are the linear forms defining the polynomials $f_1, \ldots, f_m$. Let H be the kernel of the map Ψ. Choosing a basis, we may consider H to be a map $\mathbb{C}^p \xrightarrow{\;H\;} \mathbb{C}^{m+p}$, which we sum with $\Gamma(t)$ to get a map

$$\mathbb{C}^{m+p} \;=\; \mathbb{C}^m \oplus \mathbb{C}^p \xrightarrow{\;(\Gamma(t):H)\;} \mathbb{C}^{m+p}\,.$$

This map is invertible if and only if the composition $\Psi\circ\Gamma(t)$ (10.4) is invertible. Thus, up to a constant (depending on the choice of basis for H), we have

$$\mathrm{Wr}(f_1, f_2, \ldots, f_m) \;=\; \det(\Gamma(t) \,:\, H)\,, \tag{10.5}$$

as both are polynomials of the same degree with the same roots. (Strictly speaking, this argument requires the Wronskian to have distinct roots. The general case follows via a limiting argument.)

We obtain a useful formula for the Wronskian when we expand the determinant (10.5) along the columns of $\Gamma(t)$

$$\mathrm{Wr}(f_1, f_2, \ldots, f_m) \;=\; \det(\Gamma(t) \,:\, H) \;=\; \sum_{\alpha} (-1)^{|\alpha|} p_\alpha(\Gamma(t)) \cdot p_{\alpha^c}(H)\,. \tag{10.6}$$

Here, the sum is over all $\alpha \in \binom{[m+p]}{m}$, which are choices of m distinct rows of the matrix $\Gamma(t)$. Also, $\alpha^c := [m+p] - \alpha \in \binom{[m+p]}{p}$ are the complimentary rows of H, and $p_\alpha(\Gamma(t))$ is the αth maximal minor of $\Gamma(t)$, which is the determinant of the submatrix of $\Gamma(t)$ formed by the rows in α, and similarly for $p_{\alpha^c}(H)$. (These are Plücker coordinates of the row spans of H and $\Gamma(t)$ as in Section 8.2.)

There is a similar expansion for the Wronskian using the composition (10.4). Take the top exterior power ($\wedge^m$) of this composition,

$$\mathbb{C} \;=\; \wedge^m\mathbb{C}^m \xrightarrow{\;\wedge^m\Gamma(t)\;} \wedge^m\mathbb{C}^{m+p} \xrightarrow{\;\wedge^m\Psi\;} \wedge^m\mathbb{C}^m \;=\; \mathbb{C}\,,$$

where we have used the ordered basis of $\mathbb{C}^m$ so that $\wedge^m\mathbb{C}^m = \mathbb{C}\cdot\mathbf{e}_1\wedge\dots\wedge\mathbf{e}_m$ to identify $\wedge^m\mathbb{C}^m$ with $\mathbb{C}$. Then $\wedge^m\Gamma(t)$ is a vector in $\wedge^m\mathbb{C}^{m+p}$ and $\wedge^m\Psi$ is a linear form on $\wedge^m\mathbb{C}^{m+p}$. If we use the basis $\mathbf{e}_\alpha := \mathbf{e}_{\alpha_1}\wedge\dots\wedge\mathbf{e}_{\alpha_m}$ for $\wedge^m\mathbb{C}^{m+p}$, where $\alpha \in \binom{[m+p]}{m}$, then the Wronskian has the form,

$$\mathrm{Wr}(f_1, f_2, \dots, f_m) \;=\; \sum_{\alpha\in\binom{[m+p]}{m}} p_\alpha(\Gamma(t))\cdot p_\alpha(\Psi)\,. \qquad (10.7)$$

Here, $p_\alpha(\Gamma(t))$ and $p_\alpha(\Psi)$ are the αth coordinates of the corresponding vector/linear form, which are the αth maximal minors of the corresponding matrices or Plücker coordinates. (This expansion (10.7) is the familiar Cauchy-Binet formula for the determinant of a composition.) Equating these two expressions for the Wronskian gives an interesting equality (again up to a constant)

$$\sum_{\alpha\in\binom{[m+p]}{m}} (-1)^{|\alpha|} p_\alpha(\Gamma(t))\cdot p_{\alpha^c}(H) \;=\; \sum_{\alpha\in\binom{[m+p]}{m}} p_\alpha(\Gamma(t))\cdot p_\alpha(\Psi)\,.$$

We explore some geometric consequences of the formulas (10.5) and (10.6). Let H be the column space of the matrix H, which is the kernel of the map Ψ. Then H is a point in the Grassmannian $\mathrm{Gr}(p, m+p)$. From the definition (10.3) of Γ, we see that the column space of the matrix $\Gamma(t)$ is the m-plane $F_m(t)$ osculating the rational normal curve γ at the point $\gamma(t)$. From (10.5) and (10.6), we see that s is a zero of the Wronskian $\Phi(t) := \mathrm{Wr}(f_1(t), f_2(t), \dots, f_m(t))$ if and only if

$$0 \;=\; \det(\Gamma(s) \,:\, H)\,.$$

This implies that there is a linear dependence among the columns of this matrix and thus there is a nontrivial intersection between the subspaces $F_m(s)$ and H.

Suppose that a polynomial $\Phi(t)$ has roots $s_1, \dots, s_{mp}$. Let $f_1(t), \dots, f_m(t)$ be univariate polynomials of degree $m+p-1$ whose associated linear forms cut out a p-plane H in $\mathbb{C}^{m+p}$. Then $\Phi(t)$ is the Wronskian of the polynomials $f_1(t), \dots, f_m(t)$

$\Leftrightarrow$ H meets the m-plane $F_m(s_i)$ nontrivially for each $i = 1, \dots, mp$,

$\Leftrightarrow$ H lies in the Schubert variety $X_{\square}F_\bullet(s_i)$ (9.4) for each $i = 1, \dots, mp$.

If the roots $s_1, \dots, s_{mp}$ are all real, then the Shapiro Conjecture (Theorem 9.13) asserts that all such p-planes H are real, and there are the expected number of them. Thus the second part of Theorem 1.9 is a consequence of Theorem 9.13.

Second part of Theorem 1.9. *If the polynomial $\Phi(t) \in \mathbb{P}^{mp}$ has simple real roots then there are $\#_{m,p}$ real points in* $\mathrm{Wr}^{-1}(\Phi)$.

Recall from Chapter 1 the formula for the degree of the Wronski map,

$$\#_{m,p} \;=\; \frac{1!2!\cdots(m-1)!\cdot[mp]!}{p!(p+1)!\cdots(m+p-1)!}\,, \tag{1.5}$$

which is the number of inverse images of a regular value of the Wronski map. The first part of Theorem 1.9, which asserts that all points are real in a fiber of the Wronski map over a polynomial with only real roots, follows from the second by the construction of Theorem 10.1.

10.2. Asymptotic form of the Shapiro Conjecture

It is not too hard to show that the conclusion of the Shapiro Conjecture holds when all conditions α^i are simple, $\alpha^i = \square$, for *some* $s_1,\dots,s_{mp}\in\mathbb{R}$. We establish the following asymptotic form of the Shapiro Conjecture from [**141**].

Theorem 10.1. *There exist numbers $s_1,\dots,s_{mp}\in\mathbb{R}$ such that the intersection*

$$X_{\square}F_\bullet(s_1)\cap X_{\square}F_\bullet(s_2)\cap\cdots\cap X_{\square}F_\bullet(s_{mp})$$

is transverse with all points real.

The proof uses a version of Schubert's principle of degeneration to special position and the same ideas can be used to establish similar results for related varieties, such as Theorem 9.11 on rational curves in Grassmannians.

Interchanging α with α^c, the expansion (10.6) becomes (up to a sign)

$$\det(\Gamma(t)\,:\,H) \;=\; \sum_{\alpha\in\binom{[m+p]}{p}} (-1)^{|\alpha|}p_{\alpha^c}(\Gamma(t))\cdot p_\alpha(H)\,.$$

We convert this into a very useful form by expanding the minor $p_{\alpha^c}(\Gamma(t))$. Let $\alpha\in\binom{[m+p]}{m}$. Observe that the determinant of the rows indexed by α in $\Gamma(t)$

$$\det\begin{pmatrix} t^{\alpha_1-1} & (\alpha_1-1)t^{\alpha_1-2} & \dots & \frac{(\alpha_1-1)!}{(\alpha_1-m)!}t^{\alpha_1-m}\\ t^{\alpha_2-1} & (\alpha_2-1)t^{\alpha_2-2} & \dots & \frac{(\alpha_2-1)!}{(\alpha_2-m)!}t^{\alpha_2-m}\\ \vdots & \vdots & \ddots & \vdots\\ t^{\alpha_m-1} & (\alpha_m-1)t^{\alpha_m-2} & \dots & \frac{(\alpha_m-1)!}{(\alpha_m-m)!}t^{\alpha_m-m} \end{pmatrix}$$

is equal to

$$t^{|\alpha|}\cdot\det\begin{pmatrix} 1 & (\alpha_1-1) & \dots & \frac{(\alpha_1-1)!}{(\alpha_1-m)!}\\ 1 & (\alpha_2-1) & \dots & \frac{(\alpha_2-1)!}{(\alpha_2-m)!}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & (\alpha_m-1) & \dots & \frac{(\alpha_m-1)!}{(\alpha_m-m)!} \end{pmatrix} \;\overset{!}{=}\; t^{|\alpha|}\cdot\det\begin{pmatrix} 1 & \alpha_1 & \dots & \alpha_1^{m-1}\\ 1 & \alpha_2 & \dots & \alpha_2^{m-1}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & \alpha_m & \dots & \alpha_m^{m-1} \end{pmatrix}.$$

(The second equality is via column operations.) We recognize this last determinant as the Vandermonde, $\prod_{i<j}(\alpha_j-\alpha_i)$. Write κ_α for the product

$$(-1)^{|\alpha|}\cdot\prod_{i<j}(\alpha_j-\alpha_i)\,.$$

Since $|\alpha^c| = mp - |\alpha|$, we obtain the expansion for the Wronskian (up to a sign).

$$\det(\Gamma(t)\,:\,H) \;=\; \sum_{\alpha\in\binom{[m+p]}{p}} t^{mp-|\alpha|}\,\kappa_{\alpha^c}\,p_\alpha(H)\,. \tag{10.8}$$

Collecting together the coefficients of t^i for $i = 0,\ldots,mp$, shows that the resulting map to $\mathbb{P}(\mathbb{C}_{mp}[t])$ is the restriction to the Grassmannian of a generalized Wronski map (8.14) with constants κ_{α^c}.

For the purpose of display, we transpose all matrices, replacing column vectors by row vectors. Let $H \in \mathrm{Gr}(p, m{+}p)$ be represented as the row space of a p by $(m{+}p)$-matrix (a point in the Stiefel manifold as in Section 8.2), and apply Gaussian elimination with pivoting from bottom to top and right to left to H to obtain a unique representative matrix of the form

$$H \;=\; \mathrm{span}\begin{pmatrix} * & \cdots & * & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ * & \cdots & * & 0 & * & \cdots & * & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \ddots & & \vdots & \vdots & & \vdots\\ * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & * & 1 & 0 & \cdots & 0 \end{pmatrix}. \tag{10.9}$$

The entries $*$ indicate an unspecified element of our field ($\mathbb{R}$ or $\mathbb{C}$).

The set of columns containing the leading 1s (pivots) is a discrete invariant of the linear subspace H. Let $\alpha \in \binom{[m+p]}{p}$ be the positions of the pivots, that is, α_i is the column of the leading 1 in row i. Observe that $p_\beta(H) = 0$ unless $\beta_i \leq \alpha_i$ for every i. This coordinatewise comparison defines the *Bruhat order* on the indices $\binom{[m+p]}{p}$, which was investigated in Section 8.2.

The set of linear spaces whose row reduced echelon forms (10.9) have pivots in the columns of α forms a topological cell of dimension $|\alpha|$, called the *Schubert cell* and written X°_α. The undetermined entries $*$ in (10.9) show that it is isomorphic to $\mathbb{A}^{|\alpha|}$, the affine space of dimension $|\alpha|$.

To determine which linear spaces are in the closure of a Schubert cell, let M_α be the set of matrices with full rank p where the entries in row i are undetermined up to column α_i, and are 0 thereafter. These matrices have the form

$$\begin{pmatrix} * & \cdots & * & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ * & \cdots & * & * & \cdots & * & 0 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & & \vdots & \vdots & & \vdots & & \ddots & & 0 & \cdots & 0\\ * & \cdots & * & * & \cdots & * & * & \cdots & * & 0 & \cdots & 0 \end{pmatrix},$$

where the last undetermined entry $*$ in row i is in column α_i. This is a closed subset of the set of p by $(m{+}p)$-matrices of full rank p. The pivots β of a matrix $M \in M_\alpha$ occur weakly to the left of the columns indexed by α, so that $\beta \leq \alpha$, and all such β occur.

This shows that the set of p-planes H parameterized by matrices in M_α is the union of the Schubert cells indexed by β for $\beta \leq \alpha$ in the Bruhat order. This is a closed subset of the Grassmannian, in fact it is one of the *Schubert varieties* defined in Chapter 9. To see this, let $\mathbf{e}_1,\ldots,\mathbf{e}_{m+p}$ be basis vectors corresponding to the columns of our matrices. For each $i = 1,\ldots,m{+}p$ let F_i be the linear span of the vectors $\mathbf{e}_1,\ldots,\mathbf{e}_i$. From the form of matrices in M_α, we see that if H is the row space of a matrix in M_α, then we have

$$\dim(H \cap F_{\alpha_j}) \;\geq\; j \qquad \text{for } j = 1,\ldots,p\,.$$

This defines the Schubert variety $X_\alpha F_\bullet$. Note that if $H \in X_\alpha F_\bullet$, then $p_\beta(H) = 0$ unless $\beta \leq \alpha$. Write $X^\circ_\alpha F_\bullet$ for the Schubert cell consisting of those H of the form (10.9).

The key lemma in our proof of Theorem 10.1 is due essentially to Schubert [**130**].

LEMMA 10.2. *For any* $\alpha \in \binom{[m+p]}{p}$,

$$X_\alpha F_\bullet \cap \{H \mid p_\alpha(H) = 0\} \;=\; \bigcup_{\beta \lessdot \alpha} X_\beta F_\bullet\,,$$

as schemes.

Here $\beta \lessdot \alpha$ means that α *covers* β that is, $\beta < \alpha$, but there is no index μ in the Bruhat order with $\beta < \mu < \alpha$. This is easy to see set-theoretically, as for $H \in X_\alpha F_\bullet$ we have $p_\beta(H) = 0$ unless $\beta \leq \alpha$.

It is also easy to see that this is true on the generic point of each Schubert variety $X_\beta F_\bullet$ for $\beta \lessdot \alpha$. Fix some index β with $\beta \lessdot \alpha$. Then there is a unique index k with $\beta_k = \alpha_k - 1$, and for all other indices i, $\beta_i = \alpha_i$. Consider the subset of the matrices M_α, where we require the entries in row i and column β_i to be 1, and write x_{k,α_k} for the entry in row k and column α_k

$$\begin{pmatrix}
* & \cdots & * & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
\vdots & & \vdots & 0 & & \ddots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\
* & \cdots & * & 0 & * & \cdots & * & 1 & x_{k,\alpha_k} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
* & \cdots & * & 0 & * & \cdots & * & 0 & * & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \ddots & & \vdots & \vdots & & \vdots\\
* & \cdots & * & 0 & * & \cdots & * & 0 & * & * & \cdots & * & 1 & 0 & \cdots & 0
\end{pmatrix}. \tag{10.10}$$

The row spans of these matrices form a dense subset of the Schubert variety $X_\alpha F_\bullet$, and therefore define a coordinate patch for $X_\alpha F_\bullet$. If we set $x_{k,\alpha_k} = 0$, then we get all matrices of the form (10.9), but for the index β.

If H is the row space of a matrix in this set (10.10), then $p_\alpha(H) = x_{k,\alpha_k}$. Thus on this coordinate patch for $X_\alpha F_\bullet$, the vanishing $p_\alpha = 2$ of the Plücker coordinate cuts out the Schubert variety $X_\beta F_\bullet$, scheme-theoretically. Repeating this local argument for each $\beta \lessdot \alpha$, proves (10.2), at least at the generic point of each component $X_\beta F_\bullet$ (which is sufficient for our purposes). More careful arguments show this is true even at the level of their homogeneous ideals.

We prove a statement which implies Theorem 10.1 using induction on the Bruhat order (sometimes called *Schubert induction*).

LEMMA 10.3. *There exist numbers* $s_1, \ldots, s_{mp} \in \mathbb{R}$ *such that for* $\alpha \in \binom{[m+p]}{p}$,

$$X_\alpha F_\bullet \;\cap\; \bigcap_{i=1}^{|\alpha|} X_{\square} F_\bullet(s_i) \tag{10.11}$$

is transverse with all points of intersection real.

The statement of Theorem 10.1 is the case $\alpha = m{+}1, m{+}2, \ldots, m{+}p$, when $X_\alpha F_\bullet$ is the Grassmannian.

REMARK 10.4. It is not hard to see (it is equivalent to the Plücker formula [**115**] for rational curves and was noted by Eisenbud and Harris [**40**, Theorem 2.3]) that the intersection (10.11) lies in the Schubert cell $X^\circ_\alpha F_\bullet$ for the index α. That is, every point H in the intersection (10.11) has row-reduced echelon form (10.9), for the index α. To see this, suppose that $H \in X_\beta F_\bullet$. Then the expression (10.12) derived below for H to lie in $X_{\square}F_\bullet(t)$ is a polynomial of degree $|\beta|$ in t, and thus H lies in $X_{\square}F_\bullet(t)$ for at most $|\beta|$ different values of t, which implies that H cannot lie in the intersection (10.11).

PROOF OF LEMMA 10.3. When $\alpha = 1, 2, \ldots, p$, then $|\alpha| = 0$ and the Schubert variety $X_\alpha F_\bullet$ consists of the single point $\{F_p\}$. Thus the base case of the induction to prove Lemma 10.3 is trivial, as there is no intersection to contend with.

Suppose that we have real numbers $s_1, \ldots, s_j$ such that, for each index α with $|\alpha| \leq j$ the intersection (10.11) is transverse with all points real. Let $\alpha \in \binom{[m+p]}{p}$ with $|\alpha| = j+1$. By (10.8) the intersection $X_\alpha F_\bullet \cap X_{\square}F_\bullet(t)$ is defined by the single polynomial equation

$$\sum_\beta t^{mp-|\beta|}\kappa_{\beta^c} p_\beta(H) \ = \ 0 \qquad \text{for } H \in X_\alpha F_\bullet\,.$$

Since $p_\beta(H) = 0$ unless $\beta \leq \alpha$, this becomes

$$\sum_{\beta \leq \alpha} t^{mp-|\beta|}\kappa_{\beta^c} p_\beta(H) \ = \ 0\,.$$

Dividing by the lowest power $t^{mp-|\alpha|}$ of t, this becomes

$$\sum_{\beta\leq\alpha} t^{|\alpha|-|\beta|}\kappa_{\beta^c} p_\beta(H) \ = \ \kappa_{\alpha^c} p_\alpha(H) \ + \ t \cdot \sum_{\beta<\alpha} t^{|\alpha|-|\beta|-1}\kappa_{\beta^c} p_\beta(H) \ = \ 0\,. \tag{10.12}$$

Since $\kappa_{\alpha^c} \neq 0$, we see that in the limit as $t \to 0$, this equation becomes $p_\alpha(H) = 0$. Using the Schubert's Lemma 10.2, we compute the scheme-theoretic limit

$$\lim_{t\to 0}\,(X_\alpha F_\bullet \cap X_{\square}F_\bullet(t)) \ = \ X_\alpha F_\bullet \cap \{H \mid p_\alpha(H) = 0\} \ = \ \bigcup_{\beta \lessdot \alpha} X_\beta F_\bullet\,. \tag{10.13}$$

By our induction assumption on j, each intersection

$$X_\beta F_\bullet \ \cap \ \bigcap_{i=1}^{j} X_{\square}F_\bullet(s_i)$$

is transverse with all points real, and by Remark 10.4, every point of the intersection lies in the Schubert cell $X^\circ_\beta F_\bullet$. Since the Schubert cells are disjoint, we conclude that the intersection

$$\Big(\bigcup_{\beta\lessdot\alpha} X_\beta F_\bullet\Big) \ \cap \ \bigcap_{i=1}^{j} X_{\square}F_\bullet(s_i)$$

is transverse with all points real. By the computation of the limit (10.13) and the observation that transversality is preserved by small perturbations, we see that there is a number $0 < \epsilon_\alpha$ such that if $0 < t \leq \epsilon_\alpha$ then

$$X_\alpha F_\bullet \ \cap \ X_{\square}F_\bullet(t) \ \cap \ \bigcap_{i=1}^{j} X_{\square}F_\bullet(s_i)$$

is transverse with all points real.

We complete the induction by setting s_{j+1} to be the minimum of the numbers ϵ_α where $|\alpha| = j + 1$. This completes the proof of Lemma 10.3.

Similar asymptotic arguments are behind the proof of Theorem 9.11, which proved reality in the quantum Schubert Calculus, as well as results for the classical flag manifolds and for the orthogonal Grassmannian [**144**].

Remark 10.5. The proof of Lemma 10.3 used induction to show that the intersection (10.11) is transverse with all points real. In fact, it gives an inductive method to construct all the points of intersection. The induction began with $\alpha = 1,\ldots,p$ so that $|\alpha| = 0$ and the Schubert variety $X_\alpha F_\bullet$ consists of the single point $\{F_p\}$. When $|\alpha| = j+1$, the limit

$$\lim_{t\to 0}\Bigl(X_\alpha F_\bullet \cap X_{\square}F_\bullet(t)\Bigr) \ \cap\ \bigcap_{i=1}^{j} X_{\square}F_\bullet(s_i) \ =\ \bigcup_{\beta\lessdot\alpha} X_\beta F_\bullet \ \cap\ \bigcap_{i=1}^{j} X_{\square}F_\bullet(s_i)$$

shows that each point in the intersection (10.11) is connected to a point in

$$\bigcup_{\beta\lessdot\alpha} X_\beta F_\bullet \ \cap\ \bigcap_{i=1}^{j} X_{\square}F_\bullet(s_i) \ =\ \bigcup_{\beta\lessdot\alpha}\Bigl(X_\beta F_\bullet \ \bigcap_{i=1}^{j} X_{\square}F_\bullet(s_i)\Bigr) \tag{10.14}$$

along a path as t ranges from s_{j+1} to 0, and the union on the right is disjoint.

For the inductive construction, suppose that all points in the set (10.14) have been previously constructed as $\beta \lessdot \alpha$ implies that $|\beta| = j$. Starting at one of the points in (10.14) and tracing the path from $t = 0$ back to $t = s_{j+1}$ gives a point in the intersection (10.11) for α with $|\alpha| = j+1$, and all such points in the intersection (10.11) arise in this manner. Following paths along a general curve in $\mathbb{C}$ (as opposed to the line segment $[0, s_{j+1}]$) constructs points in the intersection (10.11) where s_{j+1} is any complex number. This is the germ of the idea behind the numerical *Pieri homotopy algorithm*, which was proposed in [**72**] and implemented in [**73**]. Its power was demonstrated in [**93**], which used the Pieri homotopy algorithm to compute all solutions to a Schubert problem on $\mathrm{Gr}(3,9)$ with 17589 solutions. The Pieri homotopy algorithm does not use the flags $F_\bullet(t)$, but rather a different family degenerating to the standard flag which gives equations that are more stable numerically.

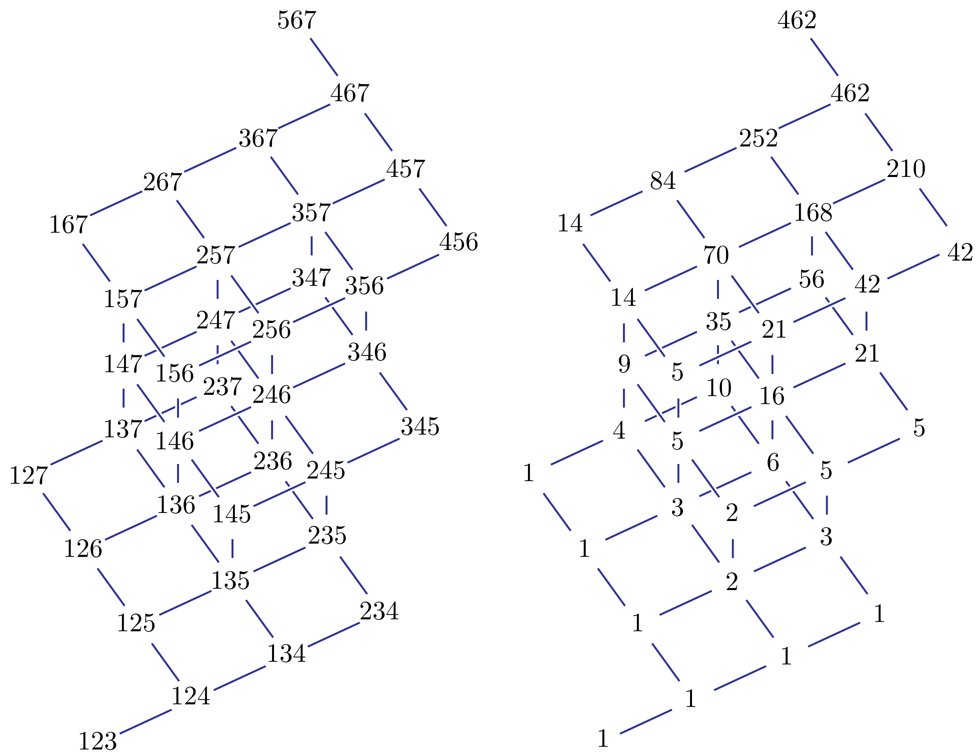
If $d(\alpha)$ is the number of points in the intersection (10.11), then this limiting process also gives the recursion for $d(\alpha)$ along the Bruhat order,

$$d(1,2,\ldots,p) \ :=\ 1\,, \qquad \text{and} \qquad d(\alpha) \ :=\ \sum_{\beta\lessdot\alpha} d(\beta)\,. \tag{10.15}$$

Schubert discovered this recursion [**129**] and used it to compute the intersection number $d(567) = 462$ when $m = 4$ and $p = 3$. This is the number $\#_{4,3}$ given by the formula,

$$\#_{m,p} \ =\ \frac{1!2!\cdots(m{-}1)!\cdot[mp]!}{p!(p{+}1)!\cdots(m{+}p{-}1)!}\,, \tag{1.5}$$

which is also due to Schubert, as he solved his recursion to obtain a closed formula. This recursion shows that the number $d(\alpha)$ may be interpreted combinatorially as the number of chains in the Bruhat order from its minimum element to α. In fact to each solution we constructed in (10.11), we may associate a distinct path from $1,2,\ldots,p$ to α. Figure 10.1 shows the Bruhat order in this case when $m = 4$ and $p = 3$ and Schubert's recursion for the numbers $d(\alpha)$.



Bruhat order Schubert's recursion

FIGURE 10.1. Schubert's recursion for Gr(3, 7).

In Section 10.1 we demonstrated that Theorem 1.9 is equivalent to Theorem 9.13 when all Schubert conditions are simple (each $\alpha^i = \square$). In fact this case of Theorem 9.13 implies a weak form of the general case, in which we do not require transversality. The main idea is to use the limit (10.13), which we must first reinterpret. The flag $F_\bullet$ is the osculating flag $F_\bullet(t)$ when $t = 0$. In fact, the limit (10.13) still holds if we replace $F_\bullet$ by $F_\bullet(s)$ and 0 by s for any point s of $\mathbb{P}^1$. That is,

$$\lim_{t \to s}\bigl(X_\alpha F_\bullet(s) \cap X_{\square} F_\bullet(t)\bigr) \;=\; \bigcup_{\beta \lessdot \alpha} X_\beta F_\bullet(s)\,. \tag{10.16}$$

This is the limit (10.13) translated by the invertible matrix $M(s)$ with i, j-entry

$$M(s)_{i,j} \;=\; \frac{1}{(i-1)!}\left(\frac{d}{dt}\right)^{i-1} t^{j-1}\Big|_{t=s}\,,$$

as $M(s).F_\bullet(t) = F_\bullet(s+t)$.

THEOREM 10.6. *Suppose that Theorem 9.13 holds for the Schubert problem in which all conditions α^i are simple ($\alpha^i = \square$). Then for any $\alpha^1, \ldots, \alpha^n$ with*

$mp = \sum_i (mp - |\alpha^i|)$ and any distinct $s_1, \ldots, s_n \in \mathbb{RP}^1$, the intersection

$$\bigcap_{i=1}^{n} X_{\alpha^i} F_\bullet(s_i) \tag{10.17}$$

has all points real.

Proof. We prove this by downward induction on the number n of Schubert conditions in (10.17), using the limit (10.16) and the simple idea that a limit of a collection of real points is necessarily a collection of real points.

First, when $n = mp$, each $\alpha^i = \square$ is simple and all points in the intersection (10.17) are real as that is our hypothesis in Theorem 10.6. Suppose that $n < mp$. Then we have $|\alpha^i| < mp - 1$ for some i. Suppose that $|\alpha^n| < mp - 1$ and let $\beta \in \binom{[m+p]}{p}$ be a Schubert condition with $\alpha^n \lessdot \beta$ so that $|\beta| = |\alpha| + 1$. Then

$$\lim_{t \to s_n} \Bigl[\bigcap_{i=1}^{n-1} X_{\alpha^i} F_\bullet(s_i) \Bigr] \cap \Bigl(X_\beta F_\bullet(s_n) \cap X_{\square} F_\bullet(t) \Bigr)$$
$$= \Bigl[\bigcap_{i=1}^{n-1} X_{\alpha^i} F_\bullet(s_i) \Bigr] \cap \Bigl(\bigcup_{\alpha \lessdot \beta} X_\alpha F_\bullet(s_n) \Bigr).$$

The elementary inclusion $\subset$ of the limit in the set on the right is clear from (10.16). The equality of the two sides follows as the intersection on the right is zero-dimensional, and therefore cannot contain any excess intersection. By induction, for general $t \in \mathbb{R}$, every point in the left-hand intersection is real, and so every point in the limit is real. Theorem 10.6 follows as $\alpha^n \lessdot \beta$ and so the intersection (10.17) is a subset of the right-hand side.

10.3. Grassmann duality

In Section 10.2 we showed how the Wronski formulation of the Shapiro Conjecture—m-dimensional spaces of polynomials of degree $m{+}p{-}1$ whose Wronskian has distinct real roots—corresponds to an intersection of hypersurface Schubert varieties in $\mathrm{Gr}(p, m{+}p)$ defined by flags that osculate the rational normal curve at the roots of the Wronskian. The first formulation concerns $\mathrm{Gr}(m, m{+}p)$ while the second concerns $\mathrm{Gr}(p, m{+}p)$. In our proof of this correspondence we considered an m-dimensional space of polynomials as a space of linear forms on $\mathbb{C}^{m+p}$, and associated this to the p-plane annihilated by the linear forms. This gives a natural bijection

$$\mathrm{Gr}(m, (\mathbb{C}^{m+p})^*) \longrightarrow \mathrm{Gr}(p, \mathbb{C}^{m+p}). \tag{10.18}$$

Moreover, the annihilators of the subspaces in a flag $F_\bullet$ in $\mathbb{C}^{m+p}$ form the dual flag $F_\bullet^*$ in $(\mathbb{C}^{m+p})^*$. Under the identification of Grassmannians (10.18), the Schubert variety $X_\alpha F_\bullet$ of $\mathrm{Gr}(p, \mathbb{C}^{m+p})$ is identified with the Schubert variety $X_{\alpha^*} F_\bullet^*$ in $\mathrm{Gr}(m, (\mathbb{C}^{m+p})^*)$, where α^* is the sequence

$$m{+}p{+}1 - \alpha^c_m \;<\; m{+}p{+}1 - \alpha^c_{m-1} \;<\; \cdots \;<\; m{+}p{+}1 - \alpha^c_2 \;<\; m{+}p{+}1 - \alpha^c_1\,,$$

that is, to obtain α^*, first form the complement α^c of α in $[m{+}p]$, then subtract each component from $m{+}p{+}1$, and finally put the result in increasing order. This identification of Schubert varieties is an exercise in combinatorial linear algebra, and the relation $|\alpha| = |\alpha^*|$ is an exercise in combinatorics.

Write $\mathbb{C}_{m+p-1}[t]$ for the space of polynomials of degree at most $m+p-1$, which we identified as the dual space to $\mathbb{C}^{m+p}$. We describe the Schubert subvarieties of $\mathrm{Gr}(m, \mathbb{C}_{m+p-1}[t])$ that correspond to the Schubert varieties $X_\alpha F_\bullet(s)$.

Let $V \subset \mathbb{C}_{m+p-1}[t]$ be an m-dimensional space of polynomials. For any $s \in \mathbb{P}^1$, V has a distinguished basis $f_1, \ldots, f_m$ whose orders of vanishing at the point s are strictly increasing,

$$\mathrm{ord}_s(f_1) \;<\; \mathrm{ord}_s(f_2) \;<\; \cdots \;<\; \mathrm{ord}_s(f_m)\,.$$

This follows by Gaussian elimination in the basis $1, (t-s), (t-s)^2, \ldots, (t-s)^{m+p-1}$ for $\mathbb{C}_{m+p-1}[t]$ applied to any basis $f_1, \ldots, f_m$ of V. Suppose that $0 \leq a_1$ is the minimal order of vanishing at s of some f_i. Reordering the basis, we may assume that $\mathrm{ord}_s(f_1) = a_1$. Subtracting an appropriate multiple of f_1 from the subsequent elements gives a new basis, still written $f_1, \ldots, f_n$, with $a_1 < \mathrm{ord}_s(f_i)$ for $1 < i$. Suppose that $\mathrm{ord}_s(f_2)$ is minimal among the orders of vanishing at s of f_i for $1 < i$ and now subtract appropriate multiples of f_2 from the subsequent elements, and continue. The resulting sequence $a := (a_1, \ldots, a_m) = (\mathrm{ord}_s(f_1), \ldots, \mathrm{ord}_s(f_m))$ is the *ramification sequence* of V at s.

An elementary calculation shows that if V has ramification sequence a at a point $s \in \mathbb{P}^1$, then the Wronskian of V vanishes at s to order

$$|a| \;=\; a_1 - 0 + a_2 - 1 + \;\cdots\; + a_m - (m-1)\,.$$

Define a flag $E_\bullet(s) \subset \mathbb{C}_{m+p-1}[t]$ where $E_i(s)$ is the space of all polynomials that vanish to order at least $m+p-i$ at s. With these definitions, we have the following lemma.

LEMMA 10.7. *A space V of polynomials in $\mathrm{Gr}(m, \mathbb{C}_{m+p-1}[t])$ has ramification sequence a at s if and only if*

$$V \;\in\; X_{a^r} E_\bullet(s)\,,$$

where $a^r \colon m+p-a_m < \cdots < m+p-a_p \in \binom{[m+p]}{m}$.

A polynomial $f(t) \in E_i(s)$ if and only if $(t-s)^{m+p-i}$ divides f, if and only if $f^{(j)}(s) = 0$ for $j = 0, 1, \ldots, m+p-1-i$. If we view f as a linear form on $\mathbb{C}^{m+p}$ so that $f(t) = f \circ \gamma(t)$, we see that $f^{(j)}(t) = f \circ \gamma^{(j)}(t)$, and therefore $f(t) \in E_i(s)$ if and only if f annihilates the osculating subspace $F_{m+p-i}(s)$ to γ at $\gamma(s)$. Thus $E_i(s)^\perp = F_{m+p-i}(s)$, and so $E_\bullet(s)$ is the dual flag to $F_\bullet(s)$. In particular, the Schubert variety $X_\alpha F_\bullet(s)$ corresponds to $X_{\alpha^*} E_\bullet(s)$ under Grassmann duality.

THEOREM 10.8. *The identification of $\mathbb{C}_{m+p-1}[t]$ as the dual space to $\mathbb{C}^{m+p}$ induces an isomorphism of Grassmannians*

$$\mathrm{Gr}(m, (\mathbb{C}^{m+p})^*) \;\longrightarrow\; \mathrm{Gr}(p, \mathbb{C}^{m+p})\,.$$

For any $s \in \mathbb{P}^1$, this restricts to an isomorphism of the Schubert varieties,

$$X_{\alpha^*} E_\bullet(s) \;\longrightarrow\; X_\alpha F_\bullet(s)\,.$$

CHAPTER 11

The Shapiro Conjecture for Rational Functions

We continue our study of the Shapiro Conjecture in the case of $m = 2$ when it becomes a statement about rational functions. Eremenko and Gabrielov [**46**] originally gave a proof in this case using essentially the uniformization theorem from complex analysis. They subsequently found a second, significantly more elementary proof [**42**]. We begin with that second proof, and then discuss a generalization concerning rational functions that are constant on prescribed sets, which leads to a generalization of the Shapiro Conjecture described in Section 13.4.

The key to this elementary proof is the association of a discrete invariant (a *net*) to each real rational function with only real critical points. Then the analytic continuation of rational functions beginning from the rational functions constructed in Theorem 10.1 with only real critical points is unobstructed, when the critical points remain distinct. The reason is simple: if in the continuation a real rational function became complex, it would have to first coincide with the continuation of another real rational function, and therefore two nets would become identical. But this cannot happen, which implies that the continuation is unobstructed.

11.1. Nets of rational functions

The Shapiro Conjecture for $m = 2$ asserts that if $f(t)$ and $g(t)$ are univariate polynomials whose Wronskian

$$\mathrm{Wr}(f,g) \;=\; f'(t)g(t) \;-\; f(t)g'(t)$$

has only real roots, then the complex linear span $\langle f, g\rangle$ is real in that there are real polynomials h and k with $\langle f, g\rangle = \langle h, k\rangle$.

This is very natural for rational functions. The quotient of univariate polynomials f and g defines a rational function $\rho\colon \mathbb{P}^1 \to \mathbb{P}^1$ which on $\mathbb{C} \subset \mathbb{P}^1$ is

$$\rho \,:\, t \;\longmapsto\; f(t)/g(t)\ .$$

The *critical points* of ρ are points where its derivative vanishes. Since

$$\rho'(t) \;=\; \frac{f'(t)g(t) \;-\; f(t)g'(t)}{g(t)^2} \;=\; \frac{\mathrm{Wr}(f,g)}{g(t)^2}\,,$$

if f and g are coprime, the critical points are the roots of their Wronskian.

Rational functions that differ by a fractional linear transformation on the target $\mathbb{P}^1$ are *equivalent*. As a fractional linear transformation on f/g is a change of basis in the linear span $\langle f, g\rangle$, an equivalence class of rational functions is a two-dimensional space of polynomials. An equivalence class is *real* if the corresponding linear space is real. We state the theorem of Eremenko and Gabrielov.

Theorem 11.1. *A rational function $\rho\colon \mathbb{P}^1 \to \mathbb{P}^1$ with only real critical points is equivalent to a real rational function.*

Theorem 10.1 and the Grassmann duality of Theorem 10.8 ensure the existence of a polynomial $\Phi_0(t) \in \mathbb{R}_{2p}[t]$ with only real roots such that every space of polynomials with Wronskian $\Phi_0(t)$ is real, and there are exactly $\#_{2,p} = \frac{1}{p+1}\binom{2p}{p}$ such spaces of polynomials. The elementary proof of Theorem 11.1 analytically continues these $\#_{2,p}$ real spaces of polynomials as the $2p$ distinct real roots of $\Phi_0(t)$ vary. This continuation will produce fewer than $\#_{2,p}$ real spaces of polynomials for some $\Phi(t)$ only if some of the spaces become complex during the continuation. But this can happen only if two spaces of polynomials first become equal.

The proof shows that such a collision cannot occur by associating discrete objects called nets to the real rational functions that are distinct for each of the $\#_{2,p}$ spaces of polynomials with Wronskian $\Phi_0(t)$, and which are preserved under a continuation that varies the roots of $\Phi_0(t)$. Thus no collisions are possible, which implies Theorem 11.1.

A point in the Grassmannian $\mathrm{Gr}(2, \mathbb{R}_{p+1}[t]) \simeq \mathrm{Gr}_{\mathbb{R}}(2, p{+}2)$ is a two-dimensional space of real univariate polynomials of degree at most $p{+}1$. Each point gives an equivalence class of rational functions $\rho\colon \mathbb{P}^1 \to \mathbb{P}^1$ of degree $p{+}1{-}d$, where d is the number of common roots in $\mathbb{P}^1$ of the polynomials in this space. Working with this equivalence is awkward, so we will instead use the real *Stiefel manifold,* $\mathrm{St}_{\mathbb{R}}(2, p{+}1)$, which is a $GL(2, \mathbb{R})$-fiber bundle over $\mathrm{Gr}(2, \mathbb{R}_{p+1}[t])$.

The points of $\mathrm{St}_{\mathbb{R}}(2, p{+}1)$ are pairs of nonproportional real univariate polynomials of degree at most $p{+}1$. Hence $\mathrm{St}_{\mathbb{R}}(2, p{+}1)$ is an open subset of $\mathbb{R}^{2p+4}$, with coordinates the coefficients of the polynomials f and g. We give $\mathrm{St}_{\mathbb{R}}(2, p{+}1)$ the subspace topology. The association $\mathrm{St}_{\mathbb{R}}(2, p{+}1) \ni (f, g) \mapsto f/g$ defines a map from $\mathrm{St}_{\mathbb{R}}(2, p{+}1)$ to the space of rational functions. While this is not a continuous map of spaces, it does have the weak continuity property given in Lemma 11.2 below.

Let $Z \subset \mathrm{St}_{\mathbb{R}}(2, p{+}1)$ be the locus of pairs (f, g) with either

$$\deg(\mathrm{gcf}(f, g)) > 0 \qquad \text{or} \qquad \deg(f), \deg(g) < p{+}1.$$

That is, f and g either have a common root in $\mathbb{P}^1$, and thus define a rational function f/g of degree less than $p{+}1$.

LEMMA 11.2. *Let $\{(f_i, g_i) \mid i \in \mathbb{N}\} \subset \mathrm{St}_{\mathbb{R}}(2, p{+}1) \setminus Z$ be a sequence that converges to $(f, g) \in Z$. Let $z_1, \ldots, z_k$ be the common roots of f and g (including ∞ if $\deg(f)$ and $\deg(g)$ are both less than $p{+}1$). Then the sequence of rational functions $\{f_i/g_i \mid j \in \mathbb{N}\}$ converges to f/g uniformly on compact subsets of $\mathbb{P}^1 \setminus \{z_1, \ldots, z_k\}$.*

PROOF. Let $K \subset \mathbb{P}^1$ be a compact subset disjoint from the common roots $\{z_1, \ldots, z_k\}$ of f and g. We may cover $\mathbb{P}^1$ by the standard affine charts $\mathbb{C}_0$ and $\mathbb{C}_\infty$ whose coordinates are t and $1/t$, respectively. Then we may write $K = K_0 \cup K_\infty$, where $K_0 \subset \mathbb{C}_0$ and $K_\infty \subset \mathbb{C}_\infty$ are compact subsets of the two affine charts. It suffices to show that the sequence of functions $\{f_i/g_i \mid i \in \mathbb{N}\}$ converges uniformly to f/g on each set K_0 and K_∞.

Now K_0 is itself covered by compact sets K_0^f and K_0^g, where K_0^f contains no roots of f and K_0^g contains no roots of g. Removing finitely many members of the sequence $\{(f_i, g_i) \mid i \in \mathbb{N}\}$, we may assume that no f_i has a root in K_0^f and no g_i has a root in K_0^g. As (f_i, g_i) converges to (f, g) in $\mathrm{St}_{\mathbb{R}}(2, p{+}1)$, and no g_i has a root in K_0^g, both sequences of functions

$$\{f_i(t) \mid i \in \mathbb{N}\} \qquad \text{and} \qquad \{(g_i(t))^{-1} \mid i \in \mathbb{N}\}$$

are uniformly bounded in K_0^g. Therefore, the sequence of functions

$$\frac{f_i(t)}{g_i(t)} \;:\; K_0^g \;\longrightarrow\; \mathbb{C} \;\xrightarrow{\sim}\; \mathbb{C}_0 \;\subset\; \mathbb{P}^1$$

is uniformly bounded and converges pointwise to $f(t)/g(t)$ on the compact set K_0^g. This implies that the convergence is uniform on K_0^g. The same arguments for K_0^f and K_∞ complete the proof.

This lemma is half of the engine of this proof of Eremenko and Gabrielov. The other half is the asymptotic proof of the Shapiro Conjecture, Theorem 10.1.

We associate an embedded graph with distinguished vertices to each real rational function. Let $\mathcal{R}_{p+1}$ be the set of nonconstant real rational functions of degree at most $p+1$, all of whose critical points are real. If $\rho \in \mathcal{R}_{p+1}$, then $\rho^{-1}(\mathbb{RP}^1) \subset \mathbb{P}^1$ defines an embedded (multi-) graph Γ with the following properties:

(i) Γ is stable under complex conjugation and $\mathbb{RP}^1 \subset \Gamma$.

Any edge in $\Gamma \setminus \mathbb{RP}^1$ is an *interior edge*. These occur in complex conjugate pairs.

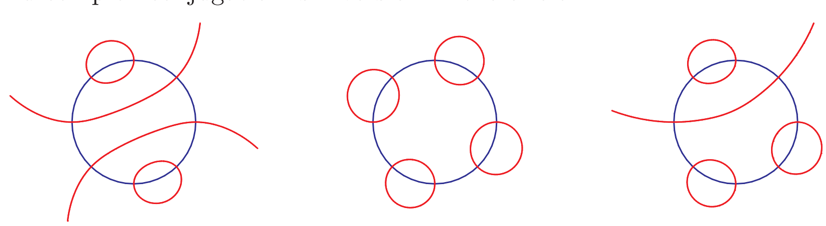
(ii) The vertices of Γ lie on $\mathbb{RP}^1$ and are the critical points of ρ. The valence of a vertex is even and it equals twice the order of ramification of ρ at the critical point, which we call the *local degree* of Γ at the vertex.

The set-theoretic difference $\mathbb{P}^1 \setminus \Gamma$ is a union of $2d$ cells, where d is the degree of ρ. The closure of each cell is homeomorphic to a disc, and the boundary of each cell maps homeomorphically onto $\mathbb{RP}^1$. This is because the cells (and their closures) are the inverse images of one of the two discs in $\mathbb{P}^1 \setminus \mathbb{RP}^1$ (or their closures), and there are no critical points in the interior of any cell. We deduce the following additional property of these multi-graphs.

(iii) No interior edge of Γ can begin and end at the same vertex.

Indeed, if an interior edge e begins and ends at the same vertex, then $\rho(e) = \mathbb{RP}^1$ as v is the only critical point on e. But then e must be the boundary of any cell adjacent to e, which implies that Γ consists of only two cells and one edge e and so ρ has degree 1 and therefore no critical points, and in fact $e = \mathbb{RP}^1$ was not an interior edge after all.

Example 11.3. Below are three pictures of such embedded (multi-) graphs for quintic rational functions. We draw $\mathbb{RP}^1$ as a circle with the upper half plane in its interior. The point $\sqrt{-1}$ is at the center of the circle, $-\sqrt{-1}$ is the point at infinity, and complex conjugation is inversion in the circle.

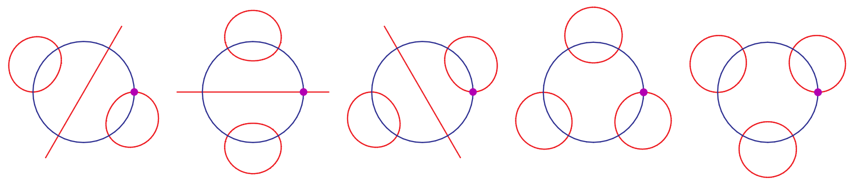


We seek to analytically continue rational functions whose Wronskians lie in a curve of polynomials $\{\Phi_z(t) \mid z \in [0,1]\}$ where each $\Phi_z(t)$ has degree $2p$ with distinct real roots $s_1(z), s_2(z), \ldots, s_{2p}(z)$, and where each s_i is a continuous function of z. The vertices of the graph $\rho^{-1}(\mathbb{RP}^1)$ associated to a rational function

ρ with Wronskian $\Phi_z(t)$ are labeled by these roots, or equivalently by the numbers $1, 2, \ldots, 2p$. Since the relative order of these roots $s_1(z), s_2(z), \ldots, s_{2p}(z)$ does not change as z varies (because each polynomial $\Phi_z(t)$ has distinct roots), we may capture this information by labeling only one root, say $s_1(z)$ (which is a vertex of the corresponding graph), and assuming that the roots are ordered in a manner consistent with a fixed orientation of $\mathbb{RP}^1$. It is these labeled graphs that we wish to consider up to isotopy (deformation in $\mathbb{P}^1$).

DEFINITION 11.4. A *net* is an (isotopy) equivalence class of such embedded multi-graphs in $\mathbb{P}^1$ satisfying (i), (ii), and (iii), with a distinguished vertex.

EXAMPLE 11.5. There are five nets with six vertices, each with local degree two. These correspond to rational functions of degree four with simple ramification.



The number $\#_{2,p} = \frac{1}{p+1}\binom{2p}{p}$ is a Catalan number, and it is a pleasing exercise to show that there are $\#_{2,p}$ nets with $2p$ vertices each with local degree 2.

The uniform convergence of Lemma 11.2 implies a certain continuity of nets. Two subsets $X, Y \subset \mathbb{P}^1$ lie within *Hausdorff distance* ϵ of each other if every point of X lies within a distance ϵ of Y and vice-versa. This gives the *Hausdorff metric* on subsets of $\mathbb{P}^1$.

LEMMA 11.6. *Let* $\{(f_i, g_i) \mid i \in \mathbb{N}\} \subset \mathrm{St}_{\mathbb{R}}(2, p{+}1)$ *be a convergent sequence with limit* (f, g). *Then the sets* $\{(f_i/g_i)^{-1}(\mathbb{RP}^1)\}$ *converge in the Hausdorff metric to the set* $\{(f/g)^{-1}(\mathbb{RP}^1)\}$.

We deduce two corollaries from this lemma.

COROLLARY 11.7. *Suppose that* $\{\rho_z \mid z \in [0,1]\}$ *is a continuous path in* $\mathcal{R}_{p+1}$ *where each* ρ_z *has the same number of critical points. Let* $v_1(z)$ *be a continuous function of* z *which is equal to a critical point of* ρ_z, *for each* z. *Then the net of the pair*

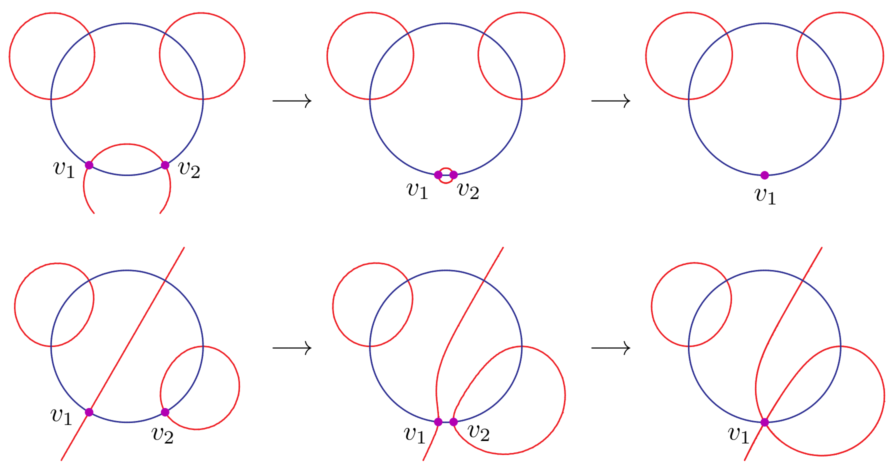
$$\left(\rho_z^{-1}(\mathbb{RP}^1),\ v_1(z)\right)$$

does not depend upon z.

This corollary holds as a continuous family of nets can only change isotopy class if two vertices collide.

COROLLARY 11.8. *Suppose that* $\{(f_z, g_z) \mid z \in [0,1]\}$ *is a continuous path in* $\mathrm{St}_{\mathbb{R}}(2, p{+}1)$ *such that* f_z/g_z *is a rational function with* n *distinct critical points for* $z > 0$, *but two critical points collide in the limit as* $z \to 0$, *while the rest remain distinct. These two critical points are continuous functions* $v_1(z)$ *and* $v_2(z)$ *of* $z \in [0,1]$ *which are distinct for* $z > 0$ *but have* $v_1(0) = v_2(0)$. *Then the degree of* f_z/g_z *is constant for* $z \in (0,1]$, *and* $\deg(f_0/g_0) < \deg(f_1/g_1)$ *if and only if the net of* f_1/g_1 *has an interior edge between the two critical points* $v_1(1)$ *and* $v_2(1)$.

Example 11.9. Consider two nets for the quartic rational functions from Example 11.5 as two of their vertices collide.



In the first row there is an interior edge of $\Gamma \setminus \mathbb{RP}^1$ between the vertices v_2 and v_1. This edge collapses in the limit as v_2 approaches v_1, eliminating two regions of $\mathbb{P}^1 \setminus \Gamma$. With two fewer regions, the limiting net corresponds to a rational function of degree three. There is no such edge in the nets of the second row, and the limiting net still has eight regions and thus its rational function still has degree four.

Proof of Corollary 11.8. The degree of a rational function $\rho \in \mathcal{R}_{p+1}$ is one-half the number of cells in the complement $\mathbb{P}^1 \setminus \rho^{-1}(\mathbb{RP}^1)$ of the net of ρ. Set $\rho_z := f_z/g_z$. The only way for the number of cells in the complement of the net of ρ_z to change at some $z_0 \in [0,1]$ would be if some edge of $\rho_z^{-1}(\mathbb{RP}^1)$ disappeared as $z \to z_0$. By the continuity of nets, this can only occur in the neighborhood of a vertex. Since the vertices of $\rho_z^{-1}(\mathbb{RP}^1)$ are the critical points $v_1(z), \ldots, v_n(z)$, which are distinct for $z \in (0,1]$, we see that the degree of ρ_z is constant for $z \in (0,1]$.

If the degree of ρ_0 is less than that of ρ_1, then some cell must disappear in the limit as $z \to 0$. Therefore that cell is bounded by an edge between $v_1(z)$ and $v_2(z)$ which collapses, as they are the only critical points which collide in the limit as $z \to 0$. By (iii), such a cell must be bounded by more than one edge which implies that there was an edge between $v_1(1)$ and $v_2(1)$ outside of $\mathbb{RP}^1$. This shows the necessity of an interior edge between $v_1(1)$ and $v_2(1)$ for the degree to drop.

For sufficiency, note that if there is an interior edge between $v_1(1)$ and $v_2(1)$, then it must collapse in the limit as $z \to 0$ for otherwise condition (iii) for nets would be violated.

11.2. Schubert induction for rational functions and nets

The proof of Theorem 10.1 used Schubert induction to construct a sequence of numbers $s_1, \ldots, s_{mp} \in \mathbb{R}$ and sufficiently many real points in each Schubert variety $X_\alpha F_\bullet(0)$ which also lie in $X_{\square} F_\bullet(s_i)$ for $i = 1, \ldots, |\alpha|$. Without re-running that proof, we will describe what that construction gives for rational functions.

The construction of Theorem 10.1 relevant for rational functions was in the Grassmannian $\mathrm{Gr}(p, p{+}2)$. Under the Grassmann duality of Theorem 10.8, this becomes a construction in $\mathrm{Gr}(2, \mathbb{C}_{p+1}[t])$ and involves a Schubert variety $X_\alpha E_\bullet(t)$

with $\alpha \in \binom{[p+2]}{2}$. In fact, the statements become identical after replacing $F_\bullet(t)$ by $E_\bullet(t)$. We will briefly recall them in this setting.

A point in the Schubert cell $X^\circ_\alpha E_\bullet(s)$ for $\alpha\colon \alpha_1 < \alpha_2$ is a two-dimensional subspace of polynomials of degree $p{+}1$ with a basis $\langle f, g\rangle$ where

$$\text{(11.1)} \qquad \operatorname{ord}_s(f) \;=\; p{+}2{-}\alpha_2 \qquad \text{and} \qquad \operatorname{ord}_s(g) \;=\; p{+}2{-}\alpha_1\,.$$

In particular,

$$(t-s)^{p+2-\alpha_2} \,\|\, f \qquad \text{and} \qquad (t-s)^{p+2-\alpha_1} \,\|\, g\,.$$

(Here, $a^k \,\|\, b$ means that a^k divides b, but a^{k+1} does not divide b.)

Given $\langle f, g\rangle \in X^\circ_\alpha E_\bullet(s)$, a consequence of (11.1) is that its Wronskian,

$$\operatorname{Wr}(f,g) \;=\; f'(t)g(t) - f(t)g'(t)\,,$$

vanishes to order $p{+}1{-}\alpha_2{+}p{+}2{-}\alpha_1 = 2p{-}|\alpha|$ at s.

By Lemma 10.3, there exist numbers $s_1, \ldots, s_{2p} \in \mathbb{R}$ (in fact they are ordered $s_1 > \cdots > s_{2p} > 0$) such that for all $\alpha \in \binom{[p+2]}{2}$, the intersection

$$\text{(11.2)} \qquad X_\alpha E_\bullet(0) \;\cap\; \bigcap_{i=1}^{|\alpha|} X_{\square} E_\bullet(s_i)$$

is transverse, and it consists of $d(\alpha)$ real points. Any point $\langle f, g\rangle$ in the intersection (11.2) will have Wronskian

$$f'(t)g(t) - f(t)g'(t) \;=\; \text{constant} \cdot t^{2p-|\alpha|} \cdot \prod_{i=1}^{|\alpha|}(t - s_i)$$

As noted in Remark 10.5, the proof of Lemma 10.3 did much more. Suppose that $|\alpha| > 0$, and define

$$\beta^1 \;:=\; \alpha_1{-}1 < \alpha_2 \qquad \text{and} \qquad \beta^2 \;:=\; \alpha_1 < \alpha_2{-}1\,,$$

when possible. (β^1 is only defined if $1 < \alpha_1$ and β^2 is only defined if $\alpha_1{+}1 < \alpha_2$) Then the proof constructed $d(\alpha) = d(\beta^1) + d(\beta^2)$ continuous families $\{\langle f_z, g_z\rangle \mid z \in [0, s_{|\alpha|}]\}$ of polynomials such that

(1) For $z \neq 0$, $\langle f_z, g_z\rangle \in X^\circ_\alpha E_\bullet(0)$.

(2) $f_z'(t)g_z(t) - f_z(t)g_z'(t) \;=\; \text{constant} \cdot t^{2p-|\alpha|} \cdot \Big(\prod_{i=1}^{|\alpha|-1}(t - s_i)\Big) \cdot (t - z)$.

(3) Exactly $d(\beta^i)$ of these families began in $X_{\beta^i} E_\bullet(0)$. That is, for $d(\beta^i)$ of these families, we have $\langle f_0, g_0\rangle \in X_{\beta^i} E_\bullet(0)$.

The main idea in the proof of Theorem 11.1 is that each of the rational functions constructed in Lemma 10.3 has different nets.

Theorem 11.10. *Each of the $d(\alpha)$ rational functions in $X^\circ_\alpha E_\bullet(0)$ constructed in Lemma 10.3 have different nets.*

Proof. Suppose that $\langle f, g\rangle$ is a point in the intersection (11.2) where f and g satisfy (11.1) for $s = 0$. Then its Wronskian vanishes to order $2p{-}|\alpha|$ at 0 and to order 1 at the points $s_1, \ldots, s_{|\alpha|}$. In particular, 0 is the only common zero of f and g. Removing the common factor $t^{p+2-\alpha_2}$ from both f and g gives relatively prime polynomials of degree at most $\alpha_2{-}1$. Indeed, if f and g had a common root s, then a linear combination would vanish to order at least 2 at s and so their Wronskian would vanish to order at least 2 at s. It follows that the rational function $\rho := f/g$ has degree $\alpha_2{-}1$ with Wronskian

$$\text{constant} \cdot t^{\alpha_2-\alpha_1-1} \cdot \prod_{i=1}^{|\alpha|}(t-s_i)\,.$$

The point $\langle f, g\rangle$ corresponds to a unique path in the Bruhat order from $(1,2)$ to α in the Bruhat order. This path may be recovered from the net $\rho^{-1}(\mathbb{RP}^1)$ of ρ.

Indeed, consider the ith step in the construction, when the critical point s_i was created. By Corollary 11.8, the interior edge from s_i has its other endpoint 0 if the degree of the rational function increased at the ith step, and if its degree did not increase, then the other endpoint of that edge is at some critical point s_k with $s_k > s_i$ and so $k < i$. Subsequent steps in the construction will not affect an edge from s_i to s_k with $k < i$, but an edge between 0 and s_i will be moved to an edge between s_i and s_j for some $j > i$.

Thus the degree of the rational function increased at step i if and only if the other endpoint of an interior edge from s_i is at s_j with $j > i$. If $\beta \lessdot \beta'$ is the ith step in the chain corresponding to our rational function ρ, then either

(1) $\beta_2{+}1 = \beta'_2$, so the degree of the rational function increased, if the interior edge from s_i has endpoint s_j with $j > i$ (so $s_j < s_i$), or
(2) $\beta_1 + 1 = \beta'_1$, so the degree of the rational function did not increase, if the interior edge from s_i has endpoint s_k with $k < i$ (so $s_k > s_i$).

This completes the proof.

Figure 11.1 illustrates the formation of the nets during the Schubert induction for quartic rational functions, as well as the recursion for $d(\alpha)$.

We complete the proof of the Shapiro Conjecture for rational curves.

THEOREM 11.11. *Let $\Phi(t)$ be a real polynomial of degree $2p$ all of whose roots are real. Then there are exactly $d(p{+}1, p{+}2)$ real equivalence classes of rational functions with Wronskian $\Phi(t)$.*

PROOF. Let $s_1 > s_2 > \cdots > s_{2p} \in \mathbb{R}$ be numbers such that the intersection

$$\bigcap_{i=1}^{2p} X_{\square} E_{\bullet}(s_i)$$

transverse with all points real. Each point in the intersection is an equivalence class of rational functions with Wronskian

$$\Phi_0(t) \;=\; \prod_{i=1}^{2p}(t-s_i)\,.$$

Let $\{\Phi_z \mid z \in [0,1]\}$ be a continuous family of polynomials of degree $2p$ all with distinct real roots and with $\Phi_1(t) = \Phi(t)$. We attempt to analytically continue each point in the fiber $\mathrm{Wr}^{-1}(\Phi_z)$ from $z = 0$ to $z = 1$. The only way this continuation could fail would be if it encountered a fiber $\mathrm{Wr}^{-1}(\Phi_z)$ containing a multiple point, so that some of the rational functions in this fiber coincide. In particular, two would have the same net, (where we have labeled the nets by the root of the $\Phi_z(t)$ corresponding to s_1). This implies that two of the original rational functions in $\mathrm{Wr}^{-1}(\Phi_0)$ have the same net, by Corollary 11.7. But this contradicts Theorem 11.10.

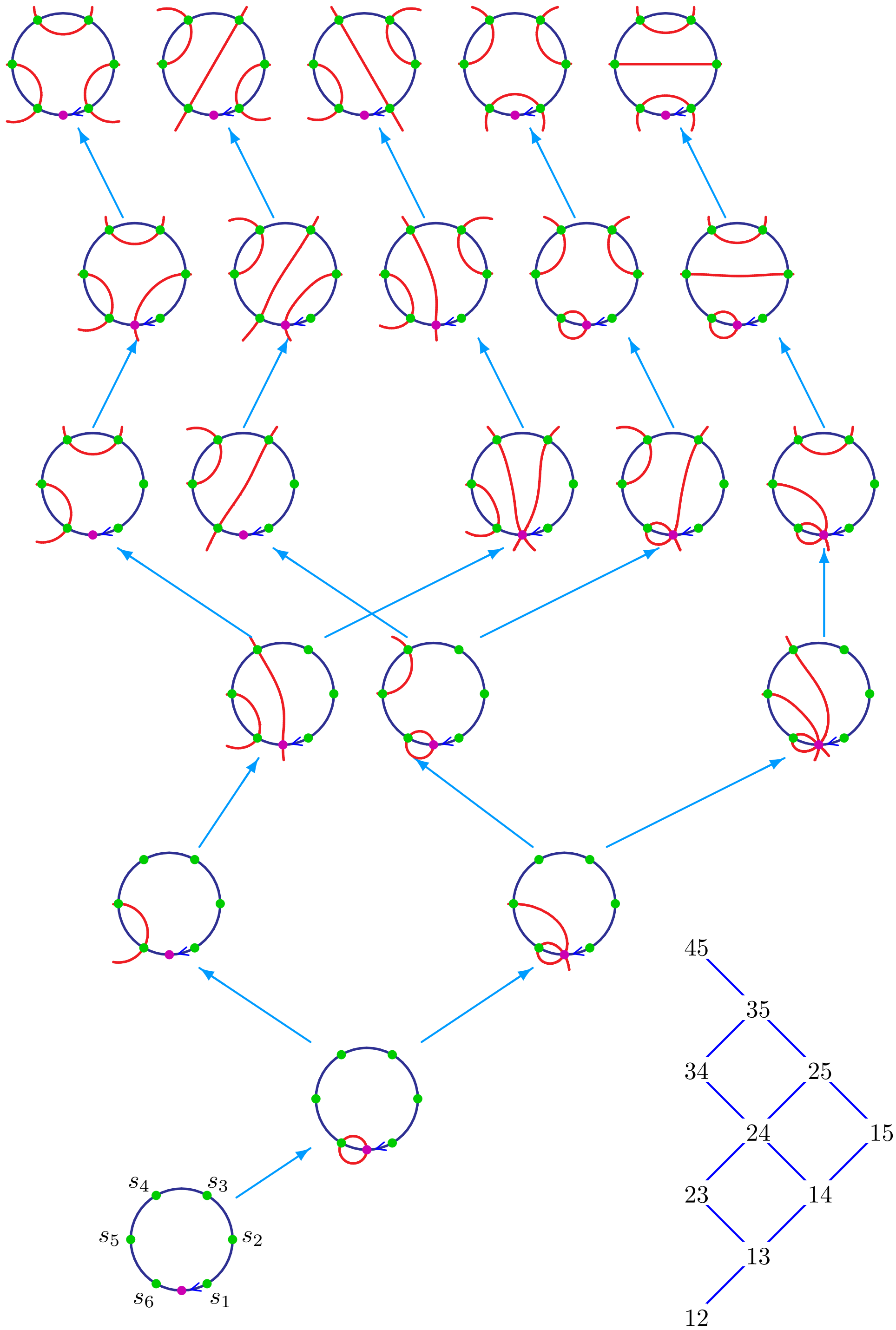


FIGURE 11.1. Formation of nets during Schubert induction.

11.3. Rational functions with prescribed coincidences

The results of Sections 11.1 and 11.2 can be used to prove a result about real rational functions that satisfy a certain interpolation condition. This is due to Eremenko, Gabrielov, Shapiro, and Vainstein [**47**], and may be interpreted in the Grassmannian $\mathrm{Gr}(p, p+2)$ as an appealing generalization of the Shapiro Conjecture. We will discuss this generalization, the Secant Conjecture, in Section 13.4.

Let $A_1, \ldots, A_n$ be disjoint finite subsets of $\mathbb{P}^1$ where the set A_i has $1 + a_i$ elements with $1 \leq a_i \leq p$ and $a_1 + \cdots + a_n = 2p$. Write $\mathbf{a}$ for this sequence $(a_1, \ldots, a_n)$ of numbers. The interpolation problem is to determine the equivalence classes of rational functions ρ of degree $p+1$ that satisfy

$$\rho|_{A_i} \text{ is constant for } i = 1, \ldots, n\,.$$

There are in fact finitely many such equivalence classes of rational functions when the sets A_i are general. We will later prove this finiteness and show that the number of equivalence classes is a Kostka number $K_{\mathbf{a}}$ [**53**, p.25],[**96**, I,6]. We expect finitely many equivalence classes because the condition that a rational function is constant on a set of $1+a$ elements gives a equations.

A collection of sets $A_i \subset \mathbb{RP}^1$ for $i = 1, \ldots, n$ is *separated* if there exist disjoint intervals $I_1, \ldots, I_n$ of $\mathbb{RP}^1$ with $A_i \subset I_i$ for $i = 1, \ldots, n$. We state the Theorem of Eremenko, Gabrielov, Shapiro, and Vainstein [**47**].

THEOREM 11.12. *Let $\mathbf{a} = (a_1, \ldots, a_n)$ with $1 \leq a_i \leq p$ and $a_1 + \cdots + a_n = 2p$. For general separated subsets of $\mathbb{RP}^1$, $A_1, \ldots, A_n$ with $|A_1| = 1+a_i$, there are exactly $K_{\mathbf{a}}$ real equivalence classes of rational functions ρ such that*

$$\rho|_{A_i} \text{ is constant for } i = 1, \ldots, n\,. \tag{11.3}$$

Given general separated subsets $A_1, \ldots, A_n$ of $\mathbb{RP}^1$, we construct a real rational function satisfying (11.3) for every net with a certain property, described below (11.5). We next relate this interpolation problem to a problem in the Schubert Calculus with $K_{\mathbf{a}}$ solutions, and finally show that $K_{\mathbf{a}}$ is the number of nets with the property (11.5). This implies that we have constructed all the solutions.

Theorem 11.12 generalizes Theorem 11.1. Suppose that we have a family of subsets $\{A_z \mid z \in (0,1]\}$ of $\mathbb{RP}^1$ depending continuously on z, each of cardinality $a+1$, whose limit as $z \to 0$ consists of a single point,

$$\lim_{z \to 0} A_z \;=\; \{s\}\,.$$

Suppose further that we have a family $\{\rho_z \mid z \in [0,1]\}$ of rational functions that depend continuously on z, and such that for $z > 0$, ρ_z is constant on A_z. Then ρ_0 will have a critical point at s of order at least a.

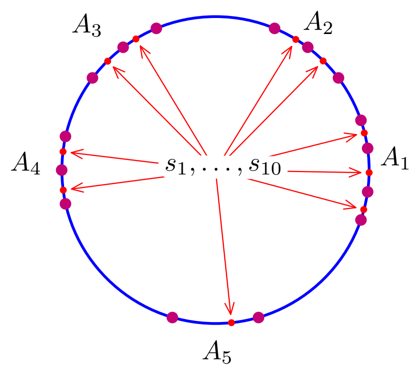
In this way, Theorem 11.12 implies Theorem 11.1 by simply considering the limit as each set A_i collapses to a point. In fact, this analysis will enable us to deduce a stronger form of Theorem 11.1.

THEOREM 11.13. *Let $a_1, \ldots, a_n$ be integers with $1 \leq a_i \leq p$ and $a_1 + \cdots + a_n = 2p$. Then every rational function of degree $p+1$ with n real critical points of multiplicities $a_1, \ldots, a_n$ is real. There are exactly $K_{\mathbf{a}}$ classes of such rational functions, and the corresponding Schubert varieties meet transversally.*

Let $\mathcal{R}_{p+1}$ be the set of real rational functions of degree $p+1$ with exactly $2p$ real critical points. We use two consequences of the work in Sections 11.1 and 11.2.

(1) If $\rho_1, \rho_2 \in \mathcal{R}_{p+1}$ have the same critical points and isotopic nets (distinguishing the same vertex each net), then ρ_1 is equivalent to ρ_2.
(2) For every net $\Gamma \subset \mathbb{P}^1$ with a given set V of $2p$ vertices (and distinguished vertex v_1), there is a unique equivalence class of rational functions in $\mathcal{R}_{p+1}$ with critical set V and net (with distinguished vertex v_1) isotopic to Γ.

Proof of Theorems 11.12 and 11.13. Fix separated subsets $A_1, \ldots, A_n$ of $\mathbb{RP}^1$ satisfying the hypotheses. Choose $2p$ additional points $s_1, \ldots, s_{2p}$ where, for each i, a_i of the points interlace the $a_i{+}1$ points of A_i, and therefore lie in intervals bounded by points of A_i. Write $[x_j, y_j]$ for the interval that contains s_j and note that $x_j, y_j \in A_i$, for some i. We show an example when $p = 5$ and $\mathbf{a} = (3, 2, 2, 2, 1)$.

(11.4)


Consider nets with vertices $s_1, \ldots, s_{2p}$, each of local degree 2, that satisfy the additional hypothesis:

(11.5) There are no edges between points interlacing the same set A_i.

We show the five nets satisfying (11.5) for the points A_i of (11.4) (we only draw the edges in the upper half plane, which is the interior of the circle).

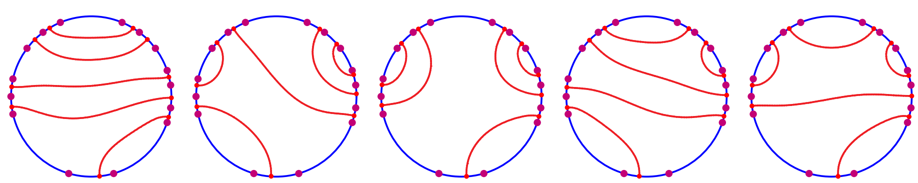


Figure 11.2. Nets for the points A_i of (11.4).

Fix an isotopy class Γ of nets satisfying (11.5) and suppose that we have chosen points $s_1, \ldots, \widehat{s_j}, \ldots, s_{2p}$ (s_j is omitted) where $s_i \in (x_i, y_i)$ is fixed but arbitrary for each $i = 1, \ldots, 2p$ with $i \neq j$. For each $s \in [x_j, y_j]$, let $\rho_s \in \mathcal{R}_{p+1}$ be a rational function with the critical points $s_1, \ldots, \widehat{s_j}, \ldots, s_{2p}, s$ and net Γ. We may suppose that $\{\rho_s \mid s \in [x_j, y_j]\}$ is a continuous family.

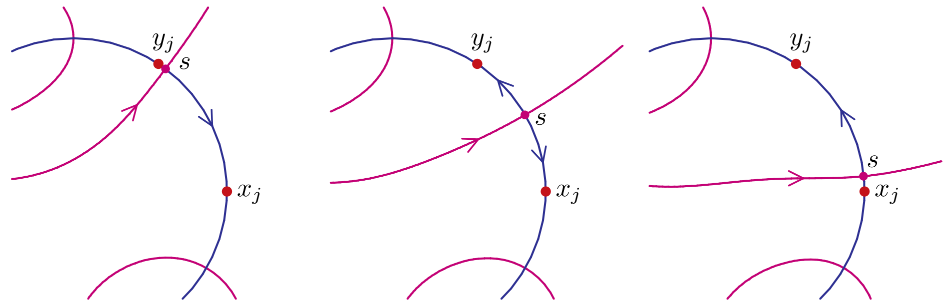
Lemma 11.14. *There exists a point $s \in (x_j, y_j)$ such that $\rho_s(x_j) = \rho_s(y_j)$.*

Proof. We assume that ρ_s is normalized so that $\rho_s(s) = 0$ and ρ_s maps the interior edge of Γ terminating at s to $[-\infty, 0] \subset \mathbb{RP}^1$. Then the difference

$$\rho_s(x_j) \; - \; \rho_s(y_j)$$

is positive when s is near y_j and negative when s is near x_j, so it takes the value zero at some point in $[x_j, y_j]$.

We illustrate this argument.



$\rho_s(x_j) - \rho_s(y_j) > 0$ $\qquad$ $\rho_s(x_j) - \rho_s(y_j) = 0$ $\qquad$ $\rho_s(x_j) - \rho_s(y_j) < 0$

The arrows point in the direction of increase of $\rho_s(t)$ for t in the net.

LEMMA 11.15. *If Γ is a net satisfying* (11.5)*, then there is a choice of critical points $s_1, \ldots, s_{2p}$ interlacing the points of the sets A_i such that every rational function ρ of degree $p+1$ with the net Γ and critical points s_j satisfies*

$$\rho|_{A_i} \text{ is constant for } i = 1, \ldots, n\,. \tag{11.6}$$

PROOF. The set of possible critical points $\mathbf{s} = (s_1, \ldots, s_{2p}) \in (\mathbb{RP}^1)^{2p}$ interlacing the sets A_i forms the interior of a closed cube

$$Q \;:=\; [x_1, y_1] \times [x_2, y_2] \times \cdots \times [x_{2p}, y_{2p}]\,.$$

By the arguments in the proof of Lemma 11.14, for every j, the continuous function $\varphi_j(\mathbf{s}) := \rho_{\mathbf{s}}(x_j) - \rho_{\mathbf{s}}(y_j)$ (defined as described in the proof of Lemma 11.14) is positive on the face $s_j = y_j$ and negative on the face $s_j = x_j$. Thus the map

$$\varphi \;:=\; (\varphi_1, \ldots, \varphi_{2p}) \;:\; Q \;\longrightarrow\; \mathbb{R}^{2p}$$

maps the boundary of Q to a set which encloses the origin. Therefore the origin lies in the image of φ. That is, there is a point $\mathbf{s}$ in the interior of Q where $\varphi_j(\mathbf{s}) = 0$ for all j, that is $\rho_{\mathbf{s}}(x_j) = \rho_{\mathbf{s}}(y_j)$ for all j. Since these intervals interlace the sets A_i, this implies (11.6).

The next step in the proof of Theorem 11.12 is to show that the number of nets satisfying (11.5) for sets $A_1, \ldots, A_n$ where A_i has $1+a_i$ members and $a_1+\cdots+a_n = 2p$ is the Kostka number $K_{\mathbf{a}}$. This Kostka number is the number of Young tableaux of shape $2 \times p$ and content $\mathbf{a}$ [**53**, p.25]. These are arrays consisting of two rows of integers, each of length p such that the integers increase weakly across each row and strictly down each column, and there are a_i occurrences of i for each $i = 1, \ldots, n$. For example, here are the five Young tableaux of shape 2×5 and content $(3, 2, 2, 2, 1)$, showing that $K_{(3,2,2,2,1)} = 5$.

(11.7)

1	1	1	2	2
3	3	4	4	5

1	1	1	3	4
2	2	3	4	5

1	1	1	3	3
2	2	4	4	5

1	1	1	2	4
2	3	3	4	5

1	1	1	2	3
2	3	4	4	5

We only describe the map from nets to Young tableaux, the interested reader may fill in the details necessary to show that it is a bijection. Given a net satisfying (11.5), we successively place integers into a left-justified two-rowed array while traversing $\mathbb{RP}^1$. This starts from the first (in the canonical ordering on $\mathbb{RP}^1$) point in A_1 and begins with an empty array. When a critical point s interlacing points of A_i is encountered, there will be an interior edge of the net with endpoint s. Place the integer i in the second row if the other endpoint of that edge has already been encountered, and in the first row if it has not been encountered. For example, the tableaux in (11.7) correspond, in order, to the nets in Figure (11.2). (There, the order on $\mathbb{RP}^1$ is counterclockwise on the circles.)

This bijection shows that we have constructed $K_{\mathbf{a}}$ equivalence classes of rational functions satisfying (11.3). To complete the proof of Theorem 11.12, we first show that the Kostka number $K_{\mathbf{a}}$ is the expected number of equivalence classes of rational functions satisfying (11.3), and then that there is some choice of the sets A_i for which there are exactly $K_{\mathbf{a}}$ equivalence classes of rational functions. This last step will also prove Theorem 11.13.

Recall that a polynomial f of degree $p{+}1$ corresponds to a linear map on $\mathbb{C}^{p+2}$ so that the composition with the rational normal curve $\gamma(t)\colon \mathbb{C} \to \mathbb{C}^{p+2}$ gives the polynomial $f(t)$. We used this to relate ramification to osculating flags in Section 10.3. A two-dimensional space $\langle f, g\rangle$ of polynomials gives a map

$$\mathbb{C} \xrightarrow{\ \gamma(t)\ } \mathbb{C}^{p+2} \xrightarrow{\ (f,g)\ } \mathbb{C}^2\,.$$

The kernel H of the map $\mathbb{C}^{p+2} \xrightarrow{(f,g)} \mathbb{C}^2$ corresponds to $\langle f, g\rangle$ under Grassmann duality.

Suppose that the rational function $\rho = f/g$ is constant on a set A. Then the line $(f(a), g(a)) \subset \mathbb{C}^2$ is constant for $a \in A$. Thus in $\mathbb{C}^{p+2}$ we have

$$\langle H, \gamma(a)\rangle \;=\; \langle H, \gamma(b)\rangle \;\subsetneq\; \mathbb{C}^{p+2}\,,$$

for any $a, b \in A$. In particular, H has exceptional position with respect to the $|A|$-plane $S(A) := \langle \gamma(a) \mid a \in A\rangle$ in that the two subspaces do not span $\mathbb{C}^{p+2}$.

Thus, the equivalence classes of rational functions ρ of degree $p+1$ that satisfy (11.6) correspond to the p-planes H in $\mathbb{C}^{p+2}$ such that

$$\operatorname{span}(H, S(A_i)) \;\neq\; \mathbb{C}^{p+2} \qquad \text{for } i = 1, \ldots, n\,. \tag{11.8}$$

Those H which satisfy (11.8) are an intersection of Schubert varieties. Let $\alpha(a) := (2, \ldots, a{+}1,\ a{+}3, \ldots, p{+}2) \in \binom{[p+2]}{p}$. Then $X_{\alpha(a)}F_\bullet$ consists of the $H \in \mathrm{Gr}(p, p{+}2)$ such that

$$\operatorname{span}(H, F_{a+1}) \;\neq\; \mathbb{C}^{p+2}\,.$$

We will also write $X_{\alpha(a)}F_{a+1}$ for this Schubert variety, which has dimension $|\alpha(a)| = 2p - a$. Thus the solutions to the interpolation problem (11.6) correspond to the intersection of Schubert varieties

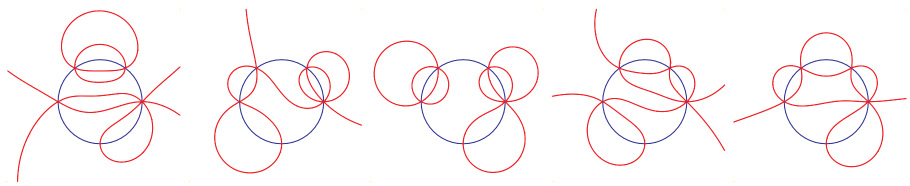
$$X_{\alpha(a_1)}S(A_1) \,\cap\, X_{\alpha(a_2)}S(A_2) \,\cap\, \cdots \,\cap\, X_{\alpha(a_n)}S(A_n)\,,$$

which is expected to be zero-dimensional. These are *special Schubert varieties*, so the expected number of points in this intersection may be computed using the Pieri formula, and it is the Kostka number $K_{\mathbf{a}}$ [**54**, p.25].

All that remains to show is that there is some choice of the sets A_i for which there are finitely many equivalence classes of rational functions satisfying the interpolation condition (11.6). We show that indirectly, by passing to the limit as each set A_i collapses to a single point, s_i. If we consider the rational functions for a given net in this limit, then the limiting rational function still has degree $p{+}1$, by Corollary 11.8 as no interior edges were collapsed in the limit, by Condition (11.5).

The limiting rational function has a critical point at each s_i of multiplicity a_i, and is necessarily real.

There are still $K_{\mathbf{a}}$ nets with critical points of multiplicity a_i at points s_i—the same bijection works. For example, here are the nets of rational functions of degree five with indicated critical points having multiplicities $(3, 2, 2, 2, 1)$.



Moreover, the corresponding intersection of Schubert varieties is expected to have $K_{\mathbf{a}}$ points. For the same reason as in Remark (10.4), the Plücker formula [**115**] for rational curves implies that the intersection of Schubert varieties is zero-dimensional. More elementarily, if there were a positive-dimensional component in the intersection, there would be rational functions of degree $p+1$ whose total ramification exceeded $2p$. This implies that we have constructed the expected number of real rational functions with the desired critical points. This completes the proofs of Theorems 11.12 and 11.13.

CHAPTER 12

Proof of the Shapiro Conjecture for Grassmannians

The Shapiro Conjecture for Grassmannians was proven by Mukhin, Tarasov, and Varchenko [**104**]. Like the proofs when $m = 2$ by Eremenko and Gabrielov [**46, 42**] (see in Chapter 11) the proof in the general case did not use much algebraic geometry. Instead it used results from mathematical physics, specifically the theory of integrable systems, with some representation theory. This chapter contains a sketch of some of the main ideas in their proof, but it by no means complete, and we recommend that the serious reader go to the original sources. The *coup-de-grâce* of the proof, the fundamental fact forcing reality, is that the eigenvalues and eigenvectors of a symmetric matrix are real. The genius and depth of this proof lies in its reducing the Shapiro Conjecture to this elementary fact of linear algebra. An account of the Shapiro Conjecture and its proof appeared in the Bulletin of the AMS [**146**]. What follows is an expanded version of Sections 2, 3, and 4 of that article.

By Theorem 10.6, the general case of the Shapiro Conjecture follows from the special case when all the Schubert conditions are equal to $\square$, and this case is equivalent to the Wronski formulation of Theorem 1.9. A further reduction is possible: as the Wronski map $\mathrm{Wr}\colon \mathrm{Gr}(m, m{+}p) \to \mathbb{P}^{mp}$ is a finite map in that it has finite fibers, a standard limiting argument (given, for example, in Section 1.3 of [**104**] or Remark 3.4 of [**144**]) shows that it suffices to prove Theorem 1.9 when the Wronskian has distinct real roots that are sufficiently general. Since $\#_{m,p}$ is the upper bound for the number of spaces of polynomials with a given Wronskian, it suffices to construct this number of distinct spaces of real polynomials with a given Wronskian, when the Wronskian has distinct real roots that are sufficiently general. In fact, this is exactly what Mukhin, Tarasov, and Varchenko do [**104**].

Theorem 1.9′. *If $s_1, \ldots, s_{mp}$ are generic real numbers, there are $\#_{m,p}$ real spaces of polynomials in $\mathrm{Gr}(m, \mathbb{C}_{m+p-1}[t])$ whose Wronskian has roots $s_1, \ldots, s_{mp}$.*

The proof first constructs $\#_{m,p}$ distinct spaces of polynomials with a given Wronskian having generic complex roots, which we describe in Section 12.1. This uses a Fuchsian differential equation given by the critical points of a remarkable symmetric function, called the master function. The next step uses the Bethe Ansatz in a certain representation V of $\mathfrak{sl}_m\mathbb{C}$: each critical point of the master function gives a Bethe eigenvector of the Gaudin Hamiltonians which turns out to be a highest weight vector for an irreducible submodule of V. This is described in Section 12.2, where the eigenvalues of the Gaudin Hamiltonians on a Bethe vector are shown to be the coefficients of the Fuchsian differential equation giving the corresponding spaces of polynomials. This is the germ of the new, deep connection between representation theory and Schubert Calculus that led to the proof of the

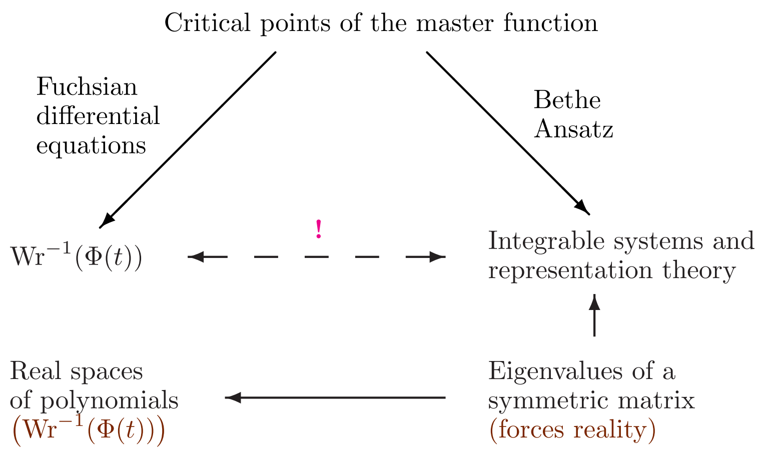


FIGURE 12.1. Schematic of proof of the Shapiro Conjecture for Grassmannians.

full statement of Theorem 9.13 (reality and transversality). Finally, the Gaudin Hamiltonians are real symmetric operators when the Wronskian has only real roots, so their eigenvalues are real, and thus the Fuchsian differential equation has real coefficients and the corresponding space of polynomials is also real. Figure 12.1 presents a schematic of this extraordinary proof, where Φ is a generic Wronski polynomial with all roots real.

12.1. Spaces of polynomials with given Wronskian

The construction of $\#_{m,p}$ spaces of polynomials with a given Wronskian begins with the critical points of a symmetric rational master function that arose in the study of hypergeometric solutions to the Knizhnik-Zamolodchikov equations [**125**] and the Bethe Ansatz for the Gaudin model.

The master function depends upon parameters $\mathbf{s} := (s_1, \ldots, s_{mp})$, which are the roots of our Wronskian Φ, and an additional $\binom{m}{2} \cdot p$ variables

$$\mathbf{x} \ := \ (x_1^{(1)}, \ldots, x_p^{(1)}, x_1^{(2)}, \ldots, x_{2p}^{(2)}, \ \ldots \ , x_1^{(m-1)}, \ldots, x_{(m-1)p}^{(m-1)}) .$$

Each set of variables $\mathbf{x}^{(i)} := (x_1^{(i)}, \ldots, x_{ip}^{(i)})$ will turn out to be the roots of certain intermediate Wronskians.

Define the *master function* $\Xi(\mathbf{x}; \mathbf{s})$ by the (rather formidable) formula

$$\Xi(\mathbf{x};\mathbf{s}) \ := \ \frac{\displaystyle\prod_{i=1}^{m-1} \prod_{1\leq j<k\leq ip} (x_j^{(i)} - x_k^{(i)})^2 \ \cdot \prod_{1\leq j<k<mp} (s_j - s_k)^2}{\displaystyle\prod_{i=1}^{m-2} \prod_{j=1}^{ip} \prod_{k=1}^{(i+1)p} (x_j^{(i)} - x_k^{(i+1)}) \ \cdot \ \prod_{j=1}^{(m-1)p} \prod_{k=1}^{mp} (x_j^{(m-1)} - s_k)} . \tag{12.1}$$

This is separately symmetric in each set of variables $\mathbf{x}^{(i)}$ and in the parameters $\mathbf{s}$. The Cartan matrix for $\mathfrak{sl}_m$ appears in the exponents of the factors $(x_*^{(i)} - x_*^{(j)})$ in (12.1). This hints at the relation of these master functions to Lie theory, which we do not discuss. It is a master function in the sense of Chapters 6 and 9 for a highly structured arrangement of hyperplanes.

Example 12.1. Consider this in the first nontrivial case of $m = p = 2$. Then there is one set of two variables, $\mathbf{x}^{(1)} = (x, y)$, and four parameters $\mathbf{s} = (s_1, s_2, s_3, s_4)$, and the master function is

$$\Xi_{2,2}(x, y; \mathbf{s}) \ := \ \frac{(x-y)^2}{(x-s_1)(y-s_1)\ \cdots\ (x-s_4)(y-s_4)} \cdot \prod_{1\le i<j\le 4} (s_j - s_k)^2 \,.$$

Critical points of the master function are solutions to the system of equations

$$\partial_{x_j^{(i)}} \log\bigl(\Xi(\mathbf{x}; \mathbf{s})\bigr) \ = \ \frac{1}{\Xi}\frac{\partial}{\partial x_j^{(i)}}\Xi(\mathbf{x}; \mathbf{s}) \ = \ 0\,, \tag{12.2}$$

for $i = 1, \ldots, m{-}1$ and $j = 1, \ldots, ip$. When the parameters $\mathbf{s}$ are generic, these *Bethe Ansatz equations* have finitely many solutions. This follows from Theorem 9.8 and the remarks following its proof. All solutions to the critical point equations are real and simple, and their number is the number of bounded chambers of the hyperplane complement when the exponents are positive. For nonpositive exponents but general parameters, there will still be the same number of solutions to the critical point equations, but they will not necessarily be real.

The master function is invariant under the group

$$S \ := \ S_p \times S_{2p} \times \ \cdots \ \times S_{(m-1)p}\,,$$

where S_N is the group of permutations of $\{1, \ldots, N\}$, and the factor S_{ip} permutes the variables in $\mathbf{x}^{(i)}$. Thus S acts on the critical points. The invariants of this action are polynomials whose roots are the coordinates of the critical points.

Example 12.2. For the master function $\Xi_{2,2}$, the Bethe Ansatz equations are

$$\begin{aligned}
\partial_x \log\, \Xi_{2,2}(x, y; \mathbf{s}) \quad &= \quad \frac{2}{x-y} - \sum_{i=1}^{4} \frac{1}{x-s_i}\,,\\
\partial_y \log\, \Xi_{2,2}(x, y; \mathbf{s}) \quad &= \quad -\frac{2}{x-y} - \sum_{i=1}^{4} \frac{1}{y-s_i}\,.
\end{aligned}$$

Clearing denominators and writing $W(z) = \prod_{i=1}^{4}(z - s_i)$, this becomes

$$2W(x) - (x-y)W'(x) \ = \ -2W(y) - (x-y)W'(y) \ = \ 0\,.$$

If we add these two equations, they become

$$\begin{aligned}
&2(W(x) - W(y)) - (x-y)(W'(x) + W'(y))\\
&\qquad \overset{!}{=} \ (x-y)^3(2(x+y) - (s_1+s_2+s_3+s_4)) \ = \ 0\,.
\end{aligned} \tag{12.3}$$

Subtracting the two Bethe Ansatz equations gives

$$2(W(x) + W(y)) - (x-y)(W'(x) - W'(y))\,.$$

Writing in terms of $a := x + y$ and $b := xy$, this becomes

$$12b^2 \ - \ (12a^2 - 6e_1a + 4e_2)b \ + \ 2a^4 - e_1a^3 - 2e_3a - 4e_4 \ = \ 0\,, \tag{12.4}$$

where e_1, e_2, e_3, and e_4 are the elementary symmetric polynomials in $s_1, \ldots, s_4$, which are the coefficients of the polynomial W. Assuming $x \neq y$, we solve (12.3) to get $a = x + y = e_1/2$, which we substitute into (12.4) to obtain the quadratic polynomial in b,

$$12b^2 \ - \ 4e_2b \ + \ e_1e_3 \ - \ 4e_4 \ = \ 0\,. \tag{12.5}$$

Given a critical point $\mathbf{x}$, define monic polynomials $\mathbf{g_x} := (g_1, \dots, g_{m-1})$ where the components $\mathbf{x}^{(i)}$ of $\mathbf{x}$ are the roots of g_i,

$$g_i \;:=\; \prod_{j=1}^{ip} (t - x_j^{(i)}) \qquad \text{for} \quad i = 1, \dots, m{-}1\,. \tag{12.6}$$

Also write g_m for the Wronskian Φ, the monic polynomial with roots $\mathbf{s}$. The master function is greatly simplified by this notation. The discriminant $\mathrm{Discr}(f)$ of a polynomial f is the square of the product of differences of its roots and the resultant $\mathrm{Res}(f,h)$ is the product of all differences (root of f $-$ root of h) [**31**]. Then the formula for the master function (12.1) becomes

$$\Xi(\mathbf{x};\mathbf{s}) \;=\; \prod_{i=1}^{m} \mathrm{Discr}(g_i) \Big/ \prod_{i=1}^{m-1} \mathrm{Res}(g_i, g_{i+1})\ . \tag{12.7}$$

The connection between the critical points of $\Xi(\mathbf{x};\mathbf{s})$ and spaces of polynomials with Wronskian Φ is through a Fuchsian differential equation (every singular point is regular). Given (an orbit of) a critical point $\mathbf{x}$ represented by the list of polynomials $\mathbf{g_x}$, define the *fundamental differential operator $D_{\mathbf{x}}$ of the critical point $\mathbf{x}$* by

$$\Big(\frac{d}{dt} - \mathrm{dlog}\Big(\frac{\Phi}{g_{m-1}}\Big)\Big)\ \cdots\ \Big(\frac{d}{dt} - \mathrm{dlog}\Big(\frac{g_2}{g_1}\Big)\Big)\Big(\frac{d}{dt} - \mathrm{dlog}(g_1)\Big)\,, \tag{12.8}$$

where $\mathrm{dlog}(f) := \frac{d}{dt}\ln f$. The kernel $V_{\mathbf{x}}$ of $D_{\mathbf{x}}$ is the *fundamental space of the critical point $\mathbf{x}$*.

EXAMPLE 12.3. Since

$$\Big(\frac{d}{dt} \;-\; \mathrm{dlog}(f)\Big)\ f \;=\; \Big(\frac{d}{dt} \;-\; \frac{f'}{f}\Big)\ f \;=\; f' - \frac{f'}{f}f \;=\; 0\,,$$

we see that g_1 is a solution of $D_{\mathbf{x}}$. Consider $D_{\mathbf{x}}$ and $V_{\mathbf{x}}$ when $m = 2$. Suppose that g is a solution to $D_{\mathbf{x}}$ that is linearly independent from g_1. Then

$$0 \;=\; \Big(\frac{d}{dt} - \mathrm{dlog}\Big(\frac{\Phi}{g_1}\Big)\Big)\Big(\frac{d}{dt} - \mathrm{dlog}(g_1)\Big)\ g \;=\; \Big(\frac{d}{dt} - \mathrm{dlog}\Big(\frac{\Phi}{g_1}\Big)\Big)\big(g' - \frac{g_1'}{g_1}g\big)\,.$$

This implies that

$$\frac{\Phi}{g_1} \;=\; g' - \frac{g_1'}{g_1}g \;=\; \frac{1}{g_1}\big(g_1 g' - g_1' g\big)\,,$$

so $\Phi = \mathrm{Wr}(g_1, g)$, and the kernel of $D_{\mathbf{x}}$ is a two-dimensional space of functions with Wronskian Φ.

What we just saw is always the case. The following result is due to Scherbak and Varchenko [**127**] for $m = 2$ and to Mukhin and Varchenko [**107**, §5] for all m.

THEOREM 12.4. *Suppose that $V_{\mathbf{x}}$ is the fundamental space of a critical point $\mathbf{x}$ of the master function Ξ with generic parameters $\mathbf{s}$ which are the roots of Φ.*

(1) *Then $V_{\mathbf{x}} \in \mathrm{Gr}(m, \mathbb{C}_{m+p-1}[t])$ has Wronskian Φ.*
(2) *The critical point $\mathbf{x}$ is recovered from $V_{\mathbf{x}}$ in some cases as follows. Suppose that $f_1, \dots, f_m$ are monic polynomials in $V_{\mathbf{x}}$ with $\deg f_i = p{-}1{+}i$, each f_i is square-free, and that the pairs f_i and f_{i+1} are relatively prime. Then, up to scalar multiples, the polynomials $g_1, \dots, g_{m-1}$ in $\mathbf{g_x}$ are*

$$f_1\,,\ \mathrm{Wr}(f_1, f_2)\,,\ \mathrm{Wr}(f_1, f_2, f_3)\,,\ \dots\,,\ \mathrm{Wr}(f_1, \dots, f_m)\,.$$

REMARK 12.5. Statement (2) includes a general result about factoring a linear differential operator into differential operators of degree 1. Linearly independent C^∞ functions $f_1, \ldots, f_m$ span the kernel of the differential operator of degree m,

$$\det\begin{pmatrix} f_1 & f_2 & \cdots & f_m & 1 \\ f_1' & f_2' & \cdots & f_m' & \frac{d}{dt} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_1^{(m)} & f_2^{(m)} & \cdots & f_m^{(m)} & \frac{d^m}{dt^m} \end{pmatrix}.$$

If we set $g_i := \mathrm{Wr}(f_1, \ldots, f_i)$, then (12.8) is a factorization over $\mathbb{C}(t)$ of this determinant into differential operators of degree 1. This follows from some interesting identities among Wronskians shown in the Appendix of [**107**].

To get an idea of this, suppose that g_1, g_2, g_3 are continuous functions and f_1, f_2, f_3 are linearly independent functions such that

$$\begin{aligned} 0 \;=\; \Big(\frac{d}{dt} \;-\; \mathrm{dlog}(g_1)\Big)\, f_1 \;&=\; \Big(\frac{d}{dt} \;-\; \mathrm{dlog}\Big(\frac{g_2}{g_1}\Big)\Big)\Big(\frac{d}{dt} \;-\; \mathrm{dlog}(g_1)\Big)\, f_2 \\ &= \Big(\frac{d}{dt} \;-\; \mathrm{dlog}\Big(\frac{g_3}{g_2}\Big)\Big)\Big(\frac{d}{dt} \;-\; \mathrm{dlog}\Big(\frac{g_2}{g_1}\Big)\Big)\Big(\frac{d}{dt} \;-\; \mathrm{dlog}(g_1)\Big)\, f_3 \end{aligned}$$

Then, by what we have seen,

$$g_1 \;=\; f_1 \qquad \text{and} \qquad g_2 \;=\; \mathrm{Wr}(f_1, f_2)\,,$$

by the first two equations. We substitute these into the third equation and evaluate the last factor on f_3 to get

$$0 \;=\; \Big(\frac{d}{dt} \;-\; \mathrm{dlog}\Big(\frac{g_3}{\mathrm{Wr}(f_1,f_2)}\Big)\Big)\Big(\frac{d}{dt} \;-\; \mathrm{dlog}\Big(\frac{\mathrm{Wr}(f_1,f_2)}{f_1}\Big)\Big)\cdot\frac{\mathrm{Wr}(f_1,f_3)}{f_1}\,. \tag{12.9}$$

Applying the second factor to $\mathrm{Wr}(f_1, f_3)/f_1$ gives

$$\begin{aligned} &\frac{\mathrm{Wr}(f_1,f_3)'}{f_1} \;-\; \frac{\mathrm{Wr}(f_1,f_3)}{f_1^2}f_1' \;-\; \frac{\mathrm{Wr}(f_1,f_2)'}{\mathrm{Wr}(f_1,f_2)}f_1' \;+\; \frac{\mathrm{Wr}(f_1,f_2)}{f_1^2}f_1' \\ &\quad=\; \frac{1}{f_1\,\mathrm{Wr}(f_1,f_2)}\cdot\Big(\mathrm{Wr}(f_1,f_3)'\,\mathrm{Wr}(f_1,f_2) \;-\; \mathrm{Wr}(f_1,f_3)\,\mathrm{Wr}(f_1,f_2)'\Big) \\ &\overset{!}{=}\; \frac{1}{f_1\,\mathrm{Wr}(f_1,f_2)}\cdot f_1\cdot\mathrm{Wr}(f_1,f_2,f_3) \;=\; \frac{\mathrm{Wr}(f_1,f_2,f_3)}{\mathrm{Wr}(f_1,f_2)}\,. \end{aligned}$$

And so (12.9) becomes

$$0 \;=\; \Big(\frac{d}{dt} \;-\; \mathrm{dlog}\Big(\frac{g_3}{\mathrm{Wr}(f_1,f_2)}\Big)\Big)\frac{\mathrm{Wr}(f_1,f_2,f_3)}{\mathrm{Wr}(f_1,f_2)}\;,$$

which implies that $g_3 = \mathrm{Wr}(f_1, f_2, f_3)$.

Theorem 12.4 is deeper than this curious fact. When the polynomials $g_1, \ldots, g_m$ and Φ are square-free, consecutive pairs are relatively prime, and $\mathbf{s}$ is generic, Theorem 12.4 implies that the kernel of an operator of the form (12.8) is a space of polynomials with Wronskian Φ having roots $\mathbf{s}$ if and only if the polynomials $g_1, \ldots, g_m$ come from the critical points of the master function (12.1) corresponding to Φ.

This gives an injection from S-orbits of critical points of the master function Ξ with parameters $\mathbf{s}$ to spaces of polynomials whose Wronskian has roots $\mathbf{s}$. Mukhin and Varchenko showed that this is a bijection when $\mathbf{s}$ is generic.

THEOREM 12.6 (Theorem 6.1 in [**108**]). *For generic complex numbers* $\mathbf{s}$, *the master function* Ξ *has nondegenerate critical points that form* $\#_{m,p}$ *distinct orbits.*

The structure of their proof is remarkably similar to the structure of the proof of Theorem 10.1 using Schubert induction; they allow the parameters to collide one-by-one, and study how the orbits of critical points behave. Ultimately, they obtain the same recursion as in (10.15), which mimics the Pieri formula for the branching rule for tensor products of representations of $\mathfrak{sl}_m$ with its last fundamental representation $V_{\omega_{m-1}}$. This same structure is also found in the main argument in [**45**]. In fact, this is the same recursion in α that Schubert established for intersection numbers $d(\alpha)$, and then solved to obtain the formula (1.5) in [**129**]. Thus Theorem 12.6 uses a coincidence of numbers: $\#_{m,p}$ counts solutions to Schubert problems, orbits of critical points of master functions, *and* the multiplicity of the trivial module in a certain tensor product of representations of $\mathfrak{sl}_m$.

12.2. The Gaudin model

The (periodic) Gaudin model is a quantum integrable system consisting of a family of commuting operators called the Gaudin Hamiltonians that act on representations V of $\mathfrak{sl}_m\mathbb{C}$, commuting with $\mathfrak{sl}_m\mathbb{C}$. The Bethe Ansatz is a general (conjectural) method to find pure states, called *Bethe vectors*, of quantum integrable systems. For the Gaudin model, the Bethe vectors turn out to be highest weight vectors generating irreducible submodules of V, and so this also gives a method for decomposing V into irreducible submodules. We explain the essentials of the (periodic) Gaudin model and discuss the Bethe Ansatz in the next section.

The Lie algebra $\mathfrak{sl}_m\mathbb{C}$ (or simply $\mathfrak{sl}_m$) is the space of m by m matrices with trace zero. It has a decomposition

$$\mathfrak{sl}_m \;=\; \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+\,,$$

where $\mathfrak{n}_+$ ($\mathfrak{n}_-$) are the strictly upper (lower) triangular matrices, and $\mathfrak{h}$ consists of the diagonal matrices with trace zero. The universal enveloping algebra $U\mathfrak{sl}_m$ of $\mathfrak{sl}_m$ is the associative algebra generated by $\mathfrak{sl}_m$ subject to the relations $uv - vu = [u,v]$ for $u, v \in \mathfrak{sl}_m$, where $[u,v]$ is the Lie bracket in $\mathfrak{sl}_m$,

$$U\mathfrak{sl}_m \;:=\; \bigoplus_{n\geq 0}(\mathfrak{sl}_m)^{\otimes n}\Big/\big\langle u\otimes v - v\otimes u - [u,v] \mid u,v\in\mathfrak{sl}_m\big\rangle\,.$$

We consider only finite-dimensional representations (modules) of $\mathfrak{sl}_m$ (equivalently, of $U\mathfrak{sl}_m$). For a more complete treatment, see [**55**]. Any module V of $\mathfrak{sl}_m$ decomposes into joint eigenspaces of $\mathfrak{h}$, called *weight spaces*,

$$V \;=\; \bigoplus_{\mu\in\mathfrak{h}^*} V[\mu]\,,$$

where, for $v \in V[\mu]$ and $h \in \mathfrak{h}$, we have $h.v = \mu(h)v$. The possible weights μ of modules lie in the integral *weight lattice*. This has a distinguished basis of *fundamental weights* $\omega_1, \ldots, \omega_{m-1}$ that generate the cone of *dominant weights*.

The *roots* of $\mathfrak{sl}_m$ are the weights of $\mathfrak{n}_- \oplus \mathfrak{n}_+$. For $i \neq j$, let $E_{i,j} \in \mathfrak{sl}_m$ be the elementary matrix with all entries 0 except for a 1 in position (i,j) (row i and column j). If $h = \mathrm{diag}(h_1, \ldots, h_n) \in \mathfrak{h}$, then

$$hE_{i,j} - E_{i,j}h \;=\; (h_i - h_j)E_{i,j}\,.$$

Thus $E_{i,j}$ spans a weight space with weight $\alpha_{i,j}$, where $\alpha_{i,j}(h) = h_i - h_j$, which is $L_i(h) - L_j(h)$ where $L_1, \ldots, L_m$ is the standard basis of the dual of the diagonal matrices [**55**, Ch. 15]. On $\mathfrak{sl}_m$, $0 = L_1 + \cdots + L_m$. Thus $\{\alpha_{i,j} \mid 1 \leq i \neq j \leq m\}$ are the roots of $\mathfrak{sl}_m$. The simple (positive) roots are $\alpha_{1,2}, \alpha_{2,3}, \ldots, \alpha_{m-1,m}$, which we write as $\alpha_1, \ldots, \alpha_{m-1}$. The fundamental weights are $\omega_i = L_1 + \cdots + L_i$.

An irreducible module V has a unique one-dimensional weight space that is annihilated by the nilpotent subalgebra $\mathfrak{n}_+$ of $\mathfrak{sl}_m$. The associated weight μ is dominant, and it is called the *highest weight* of V. Any nonzero vector with this weight is a highest weight vector of V, and it generates V as a $\mathfrak{sl}_m$ or $U\mathfrak{sl}_m$-module. Furthermore, any two irreducible modules with the same highest weight are isomorphic. Write V_μ for the *highest weight module* with highest weight μ. Lastly, there is one highest weight module for each dominant weight.

More generally, if V is any module for $\mathfrak{sl}_m$ and μ is a weight, then the *singular vectors* in V of weight μ, written $\operatorname{sing}(V[\mu])$, are the vectors in $V[\mu]$ annihilated by $\mathfrak{n}_+$. If $v \in \operatorname{sing}(V[\mu])$ is nonzero, then the submodule $U\mathfrak{sl}_m.v$ it generates is isomorphic to the highest weight module V_μ. Thus V decomposes as a direct sum of submodules generated by the singular vectors,

$$V \;=\; \bigoplus_{\mu} U\mathfrak{sl}_m.\operatorname{sing}(V[\mu])\,, \tag{12.10}$$

so that the multiplicity of the highest weight module V_μ in V is simply the dimension of its space of singular vectors of weight μ.

When V is a tensor product of highest weight modules, the Littlewood-Richardson rule [**54**] gives formulas for the dimensions of the spaces of singular vectors. Since this is the same rule for the number of points in an intersection (9.6) of Schubert varieties from a Schubert problem, these geometric intersection numbers are equal to the dimensions of spaces of singular vectors. In particular, if $V_{\omega_1} \simeq \mathbb{C}^m$ is the defining representation of $\mathfrak{sl}_m$ and $V_{\omega_{m-1}} = \bigwedge^{m-1} V_{\omega_1} = V_{\omega_1}^*$ (these are the first and last fundamental representations of $\mathfrak{sl}_m$), then

$$\dim(\operatorname{sing}(V_{\omega_{m-1}}^{\otimes mp}[0])) \;=\; \#_{m,p}\,, \tag{12.11}$$

as $\#_{m,p}$ is the multiplicity of the trivial module in $V_{\omega_{m-1}}^{\otimes mp}$, as we remarked following Theorem 12.6. This equality of numbers is purely formal, in that the same formula governs both numbers. A direct connection remains to be found.

The Gaudin Hamiltonians act on $V_{\omega_{m-1}}^{\otimes n}$ and depend upon n distinct complex numbers $s_1, \ldots, s_n$ and a complex variable t. Let $\mathfrak{gl}_m$ be the Lie algebra of m by m complex matrices. For each $i, j = 1, \ldots, m$, let $E_{i,j} \in \mathfrak{gl}_m$ be the matrix whose only nonzero entry is a 1 in row i and column j. These include the elementary matrices $E_{i,j} \in \mathfrak{sl}_m$, but also the diagonal matrices E_{ii}, which do not lie in $\mathfrak{sl}_m$. Consider the differential operator $X_{i,j}(t)$ acting on $V_{\omega_{m-1}}^{\otimes n}$-valued functions of t,

$$X_{i,j}(t) \;:=\; \delta_{i,j}\frac{d}{dt} \;-\; \sum_{k=1}^{n} \frac{E_{j,i}^{(k)}}{t - s_k}\;,$$

where $E_{j,i}^{(k)}$ acts on tensors in $V_{\omega_{m-1}}^{\otimes n}$ by $E_{j,i}$ in the kth factor and by the identity in other factors. This reversal of the order of indices in $E_{j,i}$ is intentional. Define a differential operator acting on $V_{\omega_{m-1}}^{\otimes n}$-valued functions of t,

$$\mathbf{M} \;:= \sum \operatorname{sign}(w)\; X_{1,w(1)}(t)\; X_{2,w(2)}(t)\; \cdots\; X_{m,w(m)}(t)\;,$$

the sum over all permutations m of $\{1,\dots,m\}$ where $\mathrm{sign}(w)=\pm$ is the sign of w.

Write $\mathbf{M}$ in standard form

$$\mathbf{M} \;=\; \frac{d^m}{dt^m} \;+\; M_1(t)\frac{d^{m-1}}{dt^{m-1}} \;+\; \cdots \;+\; \frac{d}{dt}M_{m-1}(t) \;+\; M_m(t)\,.$$

These coefficients $M_1(t),\dots,M_m(t)$ are called the (higher) *Gaudin Hamiltonians*. They are linear operators that depend rationally on t and act on $V_{\omega_{m-1}}^{\otimes n}$.

EXAMPLE 12.7. When $m=p=2$, we have

$$\mathbf{M} \quad=\quad \det\begin{pmatrix} \frac{d}{dt}-\sum_{k=1}^{4}\frac{E_{11}^{(k)}}{t-s_k} & -\sum_{k=1}^{4}\frac{E_{21}^{(k)}}{t-s_k} \\ -\sum_{k=1}^{4}\frac{E_{12}^{(k)}}{t-s_k} & \frac{d}{dt}-\sum_{k=1}^{4}\frac{E_{22}^{(k)}}{t-s_k}\end{pmatrix}$$

$$(12.12)\qquad =\quad \frac{d^2}{dt^2} \;-\; \Bigl(\sum_{k=1}^{4}\frac{E_{11}^{(k)}+E_{22}^{(k)}}{t-s_k}\Bigr)\frac{d}{dt}$$

$$(12.13)\qquad +\;\sum_{k<j}\frac{1}{t-s_k}\,\frac{1}{t-s_j}\,\Bigl(E_{11}^{(k)}E_{22}^{(j)}+E_{22}^{(k)}E_{11}^{(j)}-E_{21}^{(k)}E_{12}^{(j)}-E_{12}^{(k)}E_{21}^{(j)}\Bigr)\;.$$

As $E_{11}^{(k)}+E_{22}^{(k)}$ is the identity on the kth factor, $M_1(t)$ is a rational multiple of the identity acting on $V_{\omega_1}^{\otimes 4}$, while $M_2(t)$ is more interesting.

We collect together some properties of the Gaudin Hamiltonians.

THEOREM 12.8. *Suppose that $s_1,\dots,s_n$ are distinct complex numbers. Then*

1. *The Gaudin Hamiltonians commute, that is, $[M_i(u),M_j(v)]=0$ for all $i,j=1,\dots,m$ and $u,v\in\mathbb{C}$.*
2. *The Gaudin Hamiltonians commute with the action of $\mathfrak{sl}_m$ on $V_{\omega_{m-1}}^{\otimes n}$.*

Proofs of Theorem 12.8 are given in [**89**], as well as Propositions 7.2 and 8.3 in [**103**], and are based on results of Talalaev [**157**]. A consequence of the second assertion is that the Gaudin Hamiltonians preserve the weight space decomposition of the singular vectors of $V_{\omega_{m-1}}^{\otimes n}$. Since they commute, the singular vectors of $V_{\omega_{m-1}}^{\otimes n}$ have a basis of common eigenvectors of the Gaudin Hamiltonians.

12.3. The Bethe Ansatz for the Gaudin model

The Bethe Ansatz is a (conjectural) method to obtain a complete set of eigenvectors for the integrable system on $V:=V_{\omega_{m-1}}^{\otimes n}$ given by the Gaudin Hamiltonians. Since these Gaudin Hamiltonians commute with $\mathfrak{sl}_m$, the Bethe Ansatz also gives an explicit basis for $\mathrm{sing}(V[\mu])$, thus explicitly giving the decomposition (12.10).

This begins with a rational function taking values in a weight space $V_{\omega_{m-1}}^{\otimes n}[\mu]$,

$$\mathbf{v}\;:\;\mathbb{C}^l\times\mathbb{C}^n\;\;--\to\;\;V_{\omega_{m-1}}^{\otimes n}[\mu]\;.$$

Schechtman and Varchenko introduced [**125**] this *universal weight function* to solve the Knizhnik-Zamolodchikov equations with values in $V_{\omega_{m-1}}^{\otimes n}[\mu]$. When $(\mathbf{x},\mathbf{s})$ is a critical point of a master function, the vector $\mathbf{v}(\mathbf{x},\mathbf{s})$ is both singular and an eigenvector of the Gaudin Hamiltonians. (This master function is a generalization of (12.1).) The Bethe Ansatz Conjecture for the periodic Gaudin model asserts that the vectors $\mathbf{v}(\mathbf{x},\mathbf{s})$ for critical points $(\mathbf{x},\mathbf{s})$ form a basis for the space of singular vectors.

We fix some notation to describe the universal weight function. In the standard basis $\mathbf{e}_1,\ldots,\mathbf{e}_m$ of the defining representation V_{ω_1} of $\mathfrak{sl}_m$, the vector $\mathbf{e}_i$ is $(0,\ldots,0,1,0\ldots,0)$ with the 1 in position i. We have $E_{i,j}.\mathbf{e}_k=\delta_{j,k}\mathbf{e}_i$ and $\mathbf{e}_m$ is the highest weight vector. The last fundamental representation $V_{\omega_{m-1}}$ has a basis

$$\mathbf{v}_1 \ :=\ \mathbf{e}_2\wedge\mathbf{e}_3\wedge\cdots\wedge\mathbf{e}_m\,,\ \mathbf{v}_2\ :=\ \mathbf{e}_1\wedge\mathbf{e}_3\wedge\cdots\wedge\mathbf{e}_m\,,\ \ldots\,,\ \mathbf{v}_m\ :=\ \mathbf{e}_1\wedge\mathbf{e}_2\wedge\cdots\wedge\mathbf{e}_{m-1}\,.$$

Then $E_{i,j}.v_k=(-1)^{j-i-1}\delta_{k,i}v_j$ and the highest weight vector is $\mathbf{v}_m\in V_{\omega_{m-1}}[\omega_{m-1}]$. In particular $E_{i+1,i}.v_k$ is zero unless $k=i{+}1$, and then it equals v_i.

Note that $\mathbf{v}_m^{\otimes n}$ generates $V_{\omega_{m-1}}^{\otimes n}$ as a $U\mathfrak{sl}_m{}^{\otimes n}$-module. In particular, any vector in $V_{\omega_{m-1}}^{\otimes n}$ is a linear combination of vectors that are obtained from $v_m^{\otimes n}$ by applying a sequence of operators $E^{(k)}_{i+1,i}$, for $1\leq k\leq n$ and $1\leq i\leq m{-}1$. The universal weight function is a linear combination of such vectors of weight μ.

When $n=mp$, $l=p\binom{m}{2}$, and $\mu=0$, the universal weight function is a map

$$v\ :\ \mathbb{C}^{p\binom{m}{2}}\times\mathbb{C}^{mp}\ \longrightarrow\ V_{\omega_{m-1}}^{\otimes mp}[0]\,.$$

To describe it, note that a vector $E_{a+1,a}E_{b+1,b}\cdots E_{c+1,c}.\mathbf{v}_m$ is nonzero only if

$$(a,b,\ldots,c)\ =\ (a,a{+}1,\ldots,m{-}2,m{-}1)\,,$$

and then it is the vector $\mathbf{v}_a$. Thus only some sequences of operators $E^{(k)}_{i+1,i}$ applied to $\mathbf{v}_m^{\otimes mp}$ give a nonzero vector. These sequences are completely determined once we know the weight of the result. The operator $E^{(k)}_{i+1,i}$ lowers the weight of a weight vector by the root α_i. Since

$$m\omega_{m-1}\ =\ \alpha_1+2\alpha_2+\cdots+(m{-}1)\alpha_{m-1}\,, \tag{12.14}$$

there are ip occurrences of $E^{(k)}_{i+1,i}$, which is the number of variables in $\mathbf{x}^{(i)}$. To see (12.14), recall that $\alpha_i=L_i-L_{i+1}$, so that the right hand side is

$$\begin{aligned}L_1-L_2+2L_2-2L_3+\cdots+(m{-}1)L_{m-1}-(m{-}1)L_m&\\ =\ L_1+L_2+\cdots+L_{m-1}-(m{-}1)L_m\,.&\end{aligned}$$

Add $0=(m{-}1)(L_1+\cdots+L_m)$ to this to get

$$m(L_1+L_2+\cdots+L_{m-1})\ =\ m\omega_{m-1}\,,$$

as $\omega_{m-1}=L_1+\cdots+L_{m-1}$, which is the weight of $v_m=\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_{m-1}$.

Let $\mathcal{B}$ be the set of all sequences $(b_1,b_2,\ldots,b_{mp})$ of integers $1\leq b_k\leq m$ where each integer $1\leq i\leq m$ occurs exactly p times. Given a sequence B in $\mathcal{B}$, define

$$\begin{aligned}\mathbf{v}_B\ &:=\ \mathbf{v}_{b_1}\otimes\mathbf{v}_{b_2}\otimes\cdots\otimes\mathbf{v}_{b_{mp}}\\ &=\ \bigotimes_{k=1}^{mp}\big(E^{(k)}_{b_k+1,b_k}\cdots E^{(k)}_{m-1,m-2}\cdot E^{(k)}_{m,m-1}\big).\mathbf{v}_m\,,\end{aligned}$$

where the operator $E^{(k)}_{b_k+1,b_k}\cdots E^{(k)}_{m-1,m-2}\cdot E^{(k)}_{m,m-1}$ is the identity if $b_k=m$. Then $\mathbf{v}_B$ is a vector of weight 0, by (12.14). The universal weight function is a linear combination of these vectors $\mathbf{v}_B$,

$$\mathbf{v}(\mathbf{x};\mathbf{s})\ =\ \sum_{B\in\mathcal{B}}w_B(\mathbf{x};\mathbf{s})\cdot\mathbf{v}_B\,, \tag{12.15}$$

where the function $w_B(\mathbf{x}, \mathbf{s})$ is separately symmetric in each set of variables $\mathbf{x}^{(i)}$.

To describe $w_B(\mathbf{x};\mathbf{s})$, suppose that

$$\mathbf{z} \;=\; (\mathbf{z}^{(1)}, \mathbf{z}^{(2)}, \ldots, \mathbf{z}^{(mp)})$$

is a partition of the variables $\mathbf{x}$ into mp sets of variables where the kth set $\mathbf{z}^{(k)}$ of variables has exactly one variable from each set $\mathbf{x}^{(i)}$ with $b_k \le i$ (and is empty when $b_k = m$). That is, if $b_k \le m-1$, then

$$\mathbf{z}^{(k)} \;=\; (x^{(b_k)}_{c_{b_k}}, x^{(b_k+1)}_{c_{b_k+1}}, \ldots, x^{(m-1)}_{c_{m-1}})\,, \tag{12.16}$$

for some indices $c_{b_k}, \ldots, c_{m-1}$. If $b_k = m$, set $w_k(\mathbf{z}) := 1$, and otherwise

$$w_k(\mathbf{z};\mathbf{s}) \;:=\; \frac{1}{x^{(b_k)}_{c_{b_k}} - x^{(b_k+1)}_{c_{b_k+1}}} \cdots \frac{1}{x^{(m-2)}_{c_{m-2}} - x^{(m-1)}_{c_{m-1}}} \cdot \frac{1}{x^{(m-1)}_{c_{m-1}} - s_k}\,,$$

in the notation (12.16). Then we set

$$w(\mathbf{z};\mathbf{s}) \;:=\; \prod_{k=1}^{mp} w_k(\mathbf{z};\mathbf{s})\,.$$

Finally, $w_B(\mathbf{x};\mathbf{s})$ is the sum of the rational functions $w(\mathbf{z};\mathbf{s})$ over all such partitions $\mathbf{z}$ of the variables $\mathbf{x}$. (Equivalently, it is the symmetrization of any single $w(\mathbf{z};\mathbf{s})$.)

While $\mathbf{v}(\mathbf{x}, \mathbf{s})$ (12.15) is a rational function of $\mathbf{x}$ and hence not globally defined, if the coordinates of $\mathbf{s}$ are distinct and $\mathbf{x}$ is a critical point of the master function Ξ (12.1), then the vector $\mathbf{v}(\mathbf{x}, \mathbf{s}) \in V^{\otimes mp}_{\omega_{m-1}}[0]$ is well-defined, nonzero and it is in fact a singular vector (Lemma 2.1 of [**108**]). Such a vector $\mathbf{v}(\mathbf{x}, \mathbf{s})$ when $\mathbf{x}$ is a critical point of the master function is called a *Bethe vector*. Mukhin and Varchenko also prove the following, which is the second part of Theorem 6.1 in [**108**].

THEOREM 12.9. *When $\mathbf{s} \in \mathbb{C}^{mp}$ is general, the Bethe vectors form a basis of the space* $\operatorname{sing}\bigl(V^{\otimes mp}_{\omega_{m-1}}[0]\bigr)$.

These Bethe vectors are the joint eigenvectors of the Gaudin Hamiltonians.

THEOREM 12.10 (Theorem 9.2 in [**103**]). *For any critical point $\mathbf{x}$ of the master function Ξ (12.1), the Bethe vector $\mathbf{v}(\mathbf{x}, \mathbf{s})$ is a joint eigenvector of the Gaudin Hamiltonians $M_1(t)$, $M_2(t)$, $\ldots$, $M_m(t)$. Its eigenvalues $\mu_1(t), \ldots, \mu_m(t)$ are given by the formula*

$$\frac{d^m}{dt^m} \;+\; \mu_1(t)\frac{d^{m-1}}{dt^{m-1}} \;+\; \cdots \;+\; \mu_{m-1}(t)\frac{d}{dt} \;+\; \mu_m(t) \;= \tag{12.17}$$
$$\Bigl(\frac{d}{dt}+\mathrm{dlog}(g_1)\Bigr)\Bigl(\frac{d}{dt}+\mathrm{dlog}\Bigl(\frac{g_2}{g_1}\Bigr)\Bigr) \;\cdots\; \Bigl(\frac{d}{dt}+\mathrm{dlog}\Bigl(\frac{g_{m-1}}{g_{m-2}}\Bigr)\Bigr)\Bigl(\frac{d}{dt}+\mathrm{dlog}\Bigl(\frac{\Phi}{g_{m-1}}\Bigr)\Bigr)\,,$$

where $g_1(t), \ldots, g_{m-1}(t)$ are the polynomials (12.6) *associated to the critical point $\mathbf{x}$ and $\Phi(t)$ is the polynomial with roots $\mathbf{s}$.*

Observe that (12.17) is similar to the formula (12.8) for the differential operator $D_{\mathbf{x}}$ of the critical point $\mathbf{x}$. This similarity is made more precise if we replace the Gaudin Hamiltonians by a different set of operators. Consider the differential operator formally conjugate to $(-1)^m M$,

$$\begin{aligned} K \;&:=\; \frac{d^m}{dt^m} \;-\; \frac{d^{m-1}}{dt^{m-1}}M_1(t) \;+\; \cdots \;+\; (-1)^{m-1}\frac{d}{dt}M_{m-1}(t) \;+\; (-1)^m M_m(t) \\ &=\; \frac{d^m}{dt^m} \;+\; K_1(t)\frac{d^{m-1}}{dt^{m-1}} \;+\; \cdots \;+\; K_{m-1}(t)\frac{d}{dt} \;+\; K_m(t)\,. \end{aligned}$$

These coefficients $K_i(t)$ are operators on $V_{\omega_{m-1}}^{\otimes mp}$ that depend rationally on t and are also called the Gaudin Hamiltonians. Here are the first three,

$$K_1(t) \;=\; -M_1(t)\,, \qquad\qquad K_2(t) \;=\; M_2(t) \;-\; (m{-}1)M_1'(t)\,,$$
$$K_3(t) \;=\; -M_3(t) \;+\; (m{-}2)M_2'(t) \;-\; \binom{m-1}{2}M_1''(t)\,,$$

and in general $K_i(t)$ is a differential polynomial in $M_1(t),\ldots,M_i(t)$.

Like the $M_i(t)$, these operators commute with each other and with $\mathfrak{sl}_m$, and the Bethe vector $\mathbf{v}(\mathbf{x},\mathbf{s})$ is a joint eigenvector of these new Gaudin Hamiltonians $K_i(t)$. The corresponding eigenvalues $\lambda_1(t),\ldots,\lambda_m(t)$ are given by the formula

$$\frac{d^m}{dt^m} \;+\; \lambda_1(t)\frac{d^{m-1}}{dt^{m-1}} \;+\; \cdots \;+\; \lambda_{m-1}(t)\frac{d}{dt} \;+\; \lambda_m(t) \;= \tag{12.18}$$
$$\Bigl(\frac{d}{dt} - \mathrm{dlog}\Bigl(\frac{\Phi}{g_{m-1}}\Bigr)\Bigr)\Bigl(\frac{d}{dt} - \mathrm{dlog}\Bigl(\frac{g_{m-1}}{g_{m-2}}\Bigr)\Bigr)\;\cdots\;\Bigl(\frac{d}{dt} - \mathrm{dlog}\Bigl(\frac{g_2}{g_1}\Bigr)\Bigr)\Bigl(\frac{d}{dt} - \mathrm{dlog}(g_1)\Bigr)\,,$$

which is (!) the fundamental differential operator $D_{\mathbf{x}}$ of the critical point $\mathbf{x}$.

COROLLARY 12.11. *Suppose that $\mathbf{s}\in\mathbb{C}^{mp}$ is generic.*

1. *The Bethe vectors form an eigenbasis of* $\mathrm{sing}(V_{\omega_{m-1}}^{\otimes mp}[0])$ *for the Gaudin Hamiltonians $K_1(t),\ldots,K_m(t)$.*
2. *The Gaudin Hamiltonians $K_1(t),\ldots,K_m(t)$ have simple spectrum in that their eigenvalues separate the basis of eigenvectors.*

Statement (1) follows from Theorems 12.9 and 12.10. For Statement (2), suppose that two Bethe vectors $\mathbf{v}(\mathbf{x},\mathbf{s})$ and $\mathbf{v}(\mathbf{x}',\mathbf{s})$ have the same eigenvalues. By (12.18), the corresponding fundamental differential operators would be equal, $D_{\mathbf{x}} = D_{\mathbf{x}'}$. But this implies that the fundamental spaces coincide, $V_{\mathbf{x}} = V_{\mathbf{x}'}$. By Theorem 12.4 the fundamental space determines the orbit of critical points, so the critical points $\mathbf{x}$ and $\mathbf{x}'$ lie in the same orbit, which implies that $\mathbf{v}(\mathbf{x},\mathbf{s}) = \mathbf{v}(\mathbf{x}',\mathbf{s})$.

12.4. Shapovalov form and the proof of the Shapiro Conjecture

The last step in the proof of Theorem 1.9 is to show that if $\mathbf{s}\in\mathbb{R}^{mp}$ is generic and $\mathbf{x}$ is a critical point of the master function (12.1), then the fundamental space $V_{\mathbf{x}}$ of the critical point $\mathbf{x}$ has a basis of real polynomials. The ultimate reason for this reality is that the eigenvectors and eigenvalues of a symmetric matrix are real.

We begin with the Shapovalov form. The map $\tau\colon E_{i,j} \mapsto E_{j,i}$ induces an antiautomorphism on $\mathfrak{sl}_m$. Given a highest weight module V_μ and a highest weight vector $\mathbf{v}\in V_\mu[\mu]$, the *Shapovalov form* $\langle\cdot,\cdot\rangle$ on V_μ is defined recursively by

$$\langle \mathbf{v},\mathbf{v}\rangle \;=\; 1 \qquad\text{and}\qquad \langle g.\mathbf{v},\mathbf{w}\rangle \;=\; \langle \mathbf{v},\tau(g).\mathbf{w}\rangle\,,$$

for $g\in\mathfrak{sl}_m$ and $\mathbf{v},\mathbf{w}\in V$. In general, this Shapovalov form is nondegenerate on V_μ and positive definite on the real part of V_μ.

The Shapovalov form on $V_{\omega_{m-1}}$ is $\langle \mathbf{v}_i,\mathbf{v}_j\rangle = \delta_{i,j}$, in the basis $\mathbf{v}_1,\ldots,\mathbf{v}_m$ of Section 12.3. This is the standard positive definite Euclidean inner product on the real part of $V_{\omega_{m-1}}$. This induces the symmetric (tensor) Shapovalov form on the tensor product $V_{\omega_{m-1}}^{\otimes mp}$, which is positive definite on the real part of $V_{\omega_{m-1}}^{\otimes mp}$.

THEOREM 12.12 (Proposition 9.1 in [**103**]). *The Gaudin Hamiltonians are symmetric with respect to the tensor Shapovalov form,*

$$\langle K_i(t).\mathbf{v},\, \mathbf{w}\rangle \;=\; \langle \mathbf{v},\, K_i(t).\mathbf{w}\rangle\,,$$

for all $i = 1, \dots, m$, $t \in \mathbb{C}$, *and* $\mathbf{v}, \mathbf{w} \in V_{\omega_{m-1}}^{\otimes mp}$.

EXAMPLE 12.13. We examine Theorem 12.12 when $m = p = 2$. The defining representation V of $\mathfrak{sl}_2$ is spanned by vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ with weights -1 and 1, respectively. Write $\mathbf{e}_{abcd}$ for the basis element $\mathbf{e}_a \otimes \mathbf{e}_b \otimes \mathbf{e}_c \otimes \mathbf{e}_d \in V^{\otimes 4}$. Then the generator E_{12} of $\mathfrak{n}_+$ sends $\mathbf{e}_{abcd}$ to the sum

$$\delta_{a,1}\mathbf{e}_{2bcd} \;+\; \delta_{b,1}\mathbf{e}_{a2cd} \;+\; \delta_{c,1}\mathbf{e}_{ab2d} \;+\; \delta_{d,1}\mathbf{e}_{abc2}\,, \tag{12.19}$$

where δ_{ij} is the Kronecker δ-function.

We determine $\operatorname{sing}(V^{\otimes 4}[0])$. There are six vectors spanning $V^{\otimes 4}[0]$,

$$\mathbf{e}_{1122}\,,\ \mathbf{e}_{1212}\,,\ \mathbf{e}_{1221}\,,\ \mathbf{e}_{2112}\,,\ \mathbf{e}_{2121}\,,\ \mathbf{e}_{2211}\,,$$

which are orthonormal with respect to the tensor Shapovalov form, $\langle\cdot,\cdot\rangle$. Using (12.19), we see that $\operatorname{sing}(V^{\otimes 4}[0])$ is two-dimensional and is spanned by

$$\mathbf{v} \;:=\; \mathbf{e}_{1122} + \mathbf{e}_{2211} - \mathbf{e}_{1221} - \mathbf{e}_{2112} \qquad\text{and}\qquad \mathbf{w} \;:=\; \mathbf{e}_{1221} + \mathbf{e}_{2112} - \mathbf{e}_{1212} - \mathbf{e}_{2121}\,.$$

We have $\langle \mathbf{v}, \mathbf{v}\rangle = \langle \mathbf{w}, \mathbf{w}\rangle = 4$ and $\langle \mathbf{v}, \mathbf{w}\rangle = -2$.

The Gaudin Hamiltonians are $K_1(t) = -M_1(t)$ and $K_2(t) = -M_2(t) - M_1'(t)$, where $M_1(t)$ is the coefficient of $\frac{d}{dt}$ in (12.12) and $M_2(t)$ is given by (12.13). Thus $K_2(t)$ is

$$\sum_{k<j} \frac{E_{21}^{(k)}E_{12}^{(j)} + E_{12}^{(k)}E_{21}^{(j)} - E_{11}^{(k)}E_{22}^{(j)} - E_{22}^{(k)}E_{11}^{(j)}}{(t-s_k)(t-s_j)} \;-\; \sum_{k=1}^{4} \frac{1}{(t-s_k)^2}\;.$$

Since $K_1(t)$ is a scalar multiple of the identity, it suffices consider $K_2(t)$. We give a sketch as the full calculation is tedious. First, note that when $k = 1$ and $j = 2$, the constant operator $E_{21}^{(k)}E_{12}^{(j)} + E_{12}^{(k)}E_{21}^{(j)} - E_{11}^{(k)}E_{22}^{(j)} - E_{22}^{(k)}E_{11}^{(j)}$ in the sum above annihilates $\mathbf{e}_{1122}$ and $\mathbf{e}_{2211}$, but its action on the other basis elements of $V^{\otimes 4}[0]$ is

$$\begin{array}{lll} \mathbf{e}_{1212} \longmapsto \mathbf{e}_{2112} - \mathbf{e}_{1212} & \qquad & \mathbf{e}_{2121} \longmapsto \mathbf{e}_{1221} - \mathbf{e}_{2121} \\ \mathbf{e}_{1221} \longmapsto \mathbf{e}_{2121} - \mathbf{e}_{1221} & \qquad & \mathbf{e}_{2112} \longmapsto \mathbf{e}_{1212} - \mathbf{e}_{2112} \end{array}$$

Thus we see that it sends $\mathbf{v} \mapsto \mathbf{w}$ and $\mathbf{w} \mapsto -2\mathbf{w}$. Similar calculations for the other five pairs of indices $k < j$ shows that $K_1(t)$ and $K_2(t)$ act on $\operatorname{sing}(V^{\otimes 4}[0])$.

Set $\tau_{k,j} := (t-s_k)^{-1}(t-s_j)^{-1}$ and $\sigma := \tau_{11} + \tau_{22} + \tau_{33} + \tau_{44}$, then the matrix

$$\begin{pmatrix} -2\tau_{13} - 2\tau_{24} - \tau_{14} - \tau_{23} - \sigma & \tau_{13} + \tau_{24} - \tau_{14} - \tau_{23} \\ \tau_{12} + \tau_{34} - \tau_{14} - \tau_{23} & -2\tau_{12} - 2\tau_{34} - \tau_{14} - \tau_{23} - \sigma \end{pmatrix} \tag{12.20}$$

gives the action of $K_2(t)$ on the basis $\{\mathbf{v}, \mathbf{w}\}$ of $\operatorname{sing}(V^{\otimes 4}[0])$. Then $\langle K_2(t).\mathbf{v}, \mathbf{w}\rangle$ is

$$\begin{aligned} &\langle(-2\tau_{13} - 2\tau_{24} - \tau_{14} - \tau_{23} - \sigma)\mathbf{v} + (\tau_{12} + \tau_{34} - \tau_{14} - \tau_{23})\mathbf{w},\ \mathbf{w}\rangle \\ &\qquad = \ 4\tau_{12} + 4\tau_{34} + 4\tau_{13} + 4\tau_{24} - 2\tau_{14} - 2\tau_{23} - 2\sigma \\ &\qquad = \ \langle \mathbf{v},\ (\tau_{13} + \tau_{24} - \tau_{14} - \tau_{23})\mathbf{v} + (-2\tau_{12} - 2\tau_{34} - \tau_{14} - \tau_{23} - \sigma)\mathbf{w}\rangle \\ &\qquad = \ \langle \mathbf{v}, K_2(t).\mathbf{w}\rangle\,. \end{aligned}$$

Thus $K_2(t)$ is a symmetric operator on $\operatorname{sing}(V^{\otimes 4}[0])$.

We give the most important consequence of this result for our story.

COROLLARY 12.14. *When the parameters* $\mathbf{s}$ *and variable* t *are real, the Gaudin Hamiltonians* $K_1(t), \ldots, K_m(t)$ *are real linear operators with real spectrum.*

PROOF. The Gaudin Hamiltonians $M_1(t), \ldots, M_m(t)$ are real linear operators which act on the real part of $V_{\omega_{m-1}}^{\otimes mp}$, by their definition. The same is then also true of the Gaudin Hamiltonians $K_1(t), \ldots, K_m(t)$. But these are symmetric with respect to the Shapovalov form and thus have real spectrum.

PROOF OF THEOREM 1.9. Suppose that $\mathbf{s} \in \mathbb{R}^{mp}$ is general. The Gaudin Hamiltonians for $t \in \mathbb{R}$ acting on $\operatorname{sing}(V_{\omega_{m-1}}^{mp}[0])$ are symmetric operators on a Euclidean space, and so have real eigenvalues, By Corollary 12.14. The Bethe vectors $\mathbf{v}(\mathbf{x}, \mathbf{s})$ for critical points $\mathbf{x}$ of the master function with parameters $\mathbf{s}$ form an eigenbasis for the Gaudin Hamiltonians. As $\mathbf{s}$ is general, the eigenvalues are distinct by Corollary 12.11 (2), and so the Bethe vectors must be real.

Given a critical point $\mathbf{x}$, the eigenvalues $\lambda_1(t), \ldots, \lambda_m(t)$ of the Bethe vectors are then real rational functions, and so the fundamental differential operator $D_{\mathbf{x}}$ has real coefficients. But then the fundamental space $V_{\mathbf{x}}$ of polynomials is real. Thus each of the $\#_{m,p}$ spaces of polynomials $V_{\mathbf{x}}$ whose Wronskian has roots $\mathbf{s}$ that were constructed in Section 12.1 is in fact real. This proves Theorem 1.9.

We conclude this chapter with the following (obvious) remark. While this work of Mukhin, Tarasov, and Varchenko [**104**] establishes the Shapiro Conjecture for Grassmannians, it does not necessarily illuminate it, and it remains an open problem to give a more elementary proof, as was done by Eremenko and Gabrielov [**42**] and described in Sections 11.1 and 11.2.

CHAPTER 13

Beyond the Shapiro Conjecture for the Grassmannian

The result of Mukhin, Tarasov, and Varchenko [**104**] settles the reality portion of the original conjecture by Boris Shapiro and Michael Shapiro in the case of Grassmannians. The original conjecture however was broader—it included all (type A) flag manifolds. Early study [**143**] suggested that for Grassmannians, intersections of Schubert varieties for real osculating flags were transverse, and that while the conjecture failed for flag varieties beyond the Grassmannian, it could be repaired and generalized [**123**]. Currently, much more is known and there are refinements and extensions of the original conjecture. We conclude this book with a survey of the landscape that is emerging beyond this work of Mukhin, Tarasov, and Varchenko. This chapter will tell this story for Grassmannians and the next will treat other flag manifolds.

13.1. Transversality and the Discriminant Conjecture

The second proof of the Shapiro Conjecture by Mukhin, Tarasov, and Varchenko [**106**] showed that the intersection of Schubert varieties in a Grassmannian given by real osculating flags was transverse (as well as real)—this extended an earlier transversality result for $\mathrm{Gr}(2, p{+}2)$ of Eremenko and Gabrielov [**46**]. Transversality is mathematically appealing, and it appears to be fundamental to the Shapiro Conjecture and its generalizations, which is why we start with it.

Chapter 11 presented Eremenko and Gabrielov's elementary proof [**42**] of the Shapiro Conjecture for rational functions. Its main point was that there is no obstruction to analytically continuing the rational functions that were constructed in Theorem 10.1 to give rational functions with any given Wronskian having distinct real zeroes. The key to this was the association of a net to each rational function with only real critical points.

A consequence of analytic continuation being unobstructed is the statement that when $\min(m, p) = 2$, the Wronski map

$$\mathrm{Wr}\ :\ \mathrm{Gr}(p, \mathbb{C}^{m+p})\ \longrightarrow\ \mathbb{CP}^{mp}$$

is unramified over the locus of polynomials with distinct real roots. This is in fact true for all Grassmannians, as Mukhin, Tarasov, and Varchenko showed [**106**].

Theorem 13.1. *The Wronski map is unramified over the locus of polynomials with distinct real roots, for any m and p.*

We give a simple proof of this due to Eremenko and Gabrielov, showing that the original reality result of Mukhin, Tarasov, and Varchenko [**104**] implies Theorem 13.1. We next discuss what we conjecture is behind this transversality, that the discriminant of these Schubert problems is a sum of squares.

THEOREM 13.2 (Eremenko and Gabrielov). *Let $F: X \to Y$ be a real analytic map between real complex analytic manifolds of the same dimension whose differential DF is not identically zero. If $U \subset Y_{\mathbb{R}}$ is an open set with $F^{-1}(U) \subset X_{\mathbb{R}}$ then F is unramified over U.*

By "real", we mean defined by real analytic functions. This can be derived from the following observation which is sometimes called Rellich's Theorem.

LEMMA 13.3. *Let f be a germ at 0 of a nonconstant real analytic function of one complex variable with $f(0) = 0$. If for every real t in a neighborhood of 0, the full preimage $f^{-1}(t)$ is real, then $f'(0) \neq 0$.*

PROOF. Suppose that $f'(0) = 0$. Since $f(0) = 0$, $f(z)$ has has a power series expansion $f(z) = cz^m + \cdots$ with $c \in \mathbb{R}^*$ and $m \geq 2$. Solving $w = f(z)$ for z as a Puiseaux series in w, we obtain $z = (w/c)^{1/m} + \cdots$. Thus for all small real w some preimages are not real, which completes the proof.

Lemma 13.3 also holds for vector-valued functions—simply apply it to one coordinate of the function that is not constant.

PROOF OF THEOREM 13.2. Because the result is local, we may assume that $X = Y = \mathbb{C}^n$. Let J be the Jacobian determinant of F. Then J is not identically equal to zero. Suppose that $J(a) = 0$ for some $a \in F^{-1}(U)$. This means that the Jacobi matrix $DF(a)$ is singular, so we can choose a vector $\mathbf{v} \in \ker(DF(a))$. As $DF(a)$ is a real matrix, we can choose $\mathbf{v}$ to be real. Now $f(t) := F(a + \mathbf{v}t) - F(a)$ is a real analytic function of one variable t with $f'(0) = 0$. For all sufficiently small real t, the full preimage $f^{-1}(t)$ is real because $a + \mathbf{v}t$ belongs to U. If f is not identically equal to zero, then Lemma 13.3 implies a contradiction.

If f is identically equal to zero, we replace $a + \mathbf{v}t$ by a real holomorphic curve ϕ from a neighborhood of 0 in $\mathbb{C}$ to X with $\phi(0) = a$, $\phi'(0) = \mathbf{v}$ such that $f(t) := F(a + \phi(t)) - F(a)$ is not identically equal to zero.

Theorem 13.1 is simply Theorem 13.2 when F is the Wronski map and U is the set of polynomials with distinct real zeros.

The Shapiro Conjecture more generally concerned intersections of the form

$$X_{\alpha^1} F_\bullet(s_1) \cap X_{\alpha^2} F_\bullet(s_2) \cap \cdots \cap X_{\alpha^n} F_\bullet(s_n)\,, \tag{13.1}$$

where $\alpha^1, \alpha^2, \ldots, \alpha^n$ form a Schubert problem and $s_1, s_2, \ldots, s_n \in \mathbb{P}^1$. A consequence of the second proof [**106**] of Mukhin, Tarasov, and Varchenko is the strengthening of Theorem 13.1.

TRANSVERSALITY THEOREM 13.4. *If the points $s_1, s_2, \ldots, s_n$ are real and distinct, then the intersection* (13.1) *is transverse.*

This Transversality Theorem may be rephrased in terms of discriminants, which leads to a conjectural strengthening in terms of real algebra.

DEFINITION 13.5. The *discriminant* of the Wronski map is the locus in $\mathbb{P}^{mp}$ of its critical values (points over which it is ramified). It is an algebraic hypersurface and defined by a single polynomial, also called the discriminant.

The Transversality Theorem asserts that the discriminant does not meet the set of polynomials with distinct roots.

We consider a simultaneous reparametrization and generalization of this Wronski discriminant. Suppose that $\alpha^1, \alpha^2, \ldots, \alpha^n$ form a Schubert problem and consider the family of all intersections of the form (13.1). Then the discriminant is the set of points $(s_1, s_2, \ldots, s_n) \in (\mathbb{P}^1)^n$ for which this intersection is not transverse. Again, this discriminant is a polynomial $\Delta(s_1, s_2, \ldots, s_n)$ in the parameters $(s_1, s_2, \ldots, s_n) \in (\mathbb{P}^1)^n$, and the Transversality Theorem asserts that Δ does not vanish when the parameters are real and distinct, that is, when $s_i \neq s_j$, for all i, j. We conjecture something much stronger.

Conjecture 13.6 (Discriminant Conjecture). *The discriminant polynomial* Δ *is a positive sum of monomials in the squared differences* $(s_i - s_j)^2$.

There is some fascinating evidence for this conjecture.

Example 13.7. In the problem of four lines from Example 1.10, suppose that each line is real and tangent to the rational normal curve. In local coordinates $\mathbf{x} = (x_{13}, x_{14}, x_{23}, x_{24})$ for the Grassmannian $\mathrm{Gr}(2,4)$, the Schubert variety $X_{\square}(s)$ is given by the single polynomial

$$f(s;\mathbf{x}) \ := \ \det \begin{pmatrix} 1 & s & s^2 & s^3 \\ 0 & 1 & 2s & 3s^2 \\ 1 & 0 & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \end{pmatrix} \ ,$$

and so the intersection $X_{\square}F_\bullet(s_1) \cap X_{\square}F_\bullet(s_2) \cap X_{\square}F_\bullet(s_3) \cap X_{\square}F_\bullet(s_4)$ is modeled by the system of equations

$$f(s_1;\mathbf{x}) \ = \ f(s_2;\mathbf{x}) \ = \ f(s_3;\mathbf{x}) \ = \ f(s_4;\mathbf{x}) \ = \ 0\,,$$

which reduces to

$$x_{23} \ = \ \tfrac{1}{2}e_1\,, \qquad x_{14} \ = \ -\tfrac{1}{2}e_3\,, \qquad x_{13} \ = \ \tfrac{1}{3}(x_{24} - e_2)\,, \quad \text{and}$$

$$4x_{24}^2 \ - \ 4e_2x_{24} \ + \ 3e_1e_3 \ - \ 12e_4 \ = \ 0\,, \tag{13.2}$$

where e_1, e_2, e_3, e_4 are the elementary symmetric polynomials in the parameters s_1, s_2, s_3, s_4. The quadratic's (13.2) discriminant is the symmetric sum of squares

$$8\big(\,(s_1{-}s_2)^2(s_3{-}s_4)^2 \ + \ (s_1{-}s_3)^2(s_2{-}s_4)^2 \ + \ (s_1{-}s_4)^2(s_2{-}s_3)^2\,\big)\,.$$

Example 13.8. For $\mathrm{Gr}(2,5)$, the discriminant of the Schubert intersection

$$X_{\square}F_\bullet(0) \ \cap \ X_{\square}F_\bullet(s_1) \ \cap \ X_{\square}F_\bullet(s_2) \ \cap \ X_{\square}F_\bullet(s_3) \ \cap \ X_{\square}F_\bullet(s_4) \ \cap \ X_{\square}F_\bullet(\infty)\,,$$

has degree 20 in the parameters s_1, s_2, s_3, s_4, and it has 711 different terms. Ad hoc methods [**143**] showed that it was a sum of squares. This was quite surprising, for Hilbert [**69**] showed that not every polynomial of even degree greater than 2 in four variables that is nonnegative can be written as a sum of squares. Several other discriminants for small Schubert problems were also computed in [**143**] and shown to be sums of squares of the form in the Discriminant Conjecture.

A promising approach to the Discriminant Conjecture is through the work of Mukhin, Tarasov, and Varchenko [**104, 106**], for the discriminant of a real symmetric matrix (the discriminant of its characteristic polynomial) is canonically a sum of squares [**20, 74**]. The first proof of the Shapiro Conjecture [**104**] crucially involved the eigenvalues of the Gaudin Hamiltonians, which are symmetric linear operators on the span of Bethe vectors, and the second proof [**106**] showed an isomorphism between the action of a commutative algebra (the Bethe algebra) on the span of the Bethe vectors and the action of the the coordinate ring of a big cell of the Grassmannian on the coordinate ring of the intersection (13.1). We give two intriguing hints about this approach to the Discriminant Conjecture.

EXAMPLE 13.9. In Example 12.2, we showed how the Bethe Ansatz equations for the master function $\Xi_{2,2}(x,y;\mathbf{s})$ reduce to

$$x+y \;=\; \tfrac{1}{2}e_1 \qquad\text{and}\qquad 12(xy)^2 \;-\; 4e_2(xy) \;+\; e_1e_3 \;-\; 4e_4 \;=\; 0\,.$$

The substitution $xy = x_{24}/3$ converts this quadratic equation into (13.2), showing that the discriminant of the Bethe Ansatz equations for $\Xi_{2,2}$ coincides with the discriminant for the problem of four lines of Example (13.7).

Now consider the matrix (12.20) of Example 12.13, which gives the action of the Gaudin Hamiltonian $K_2(t)$ on the basis $\{\mathbf{v},\mathbf{w}\}$ of $\operatorname{sing}(V^{\otimes 4}[0])$. Dividing its discriminant by $2(t-s_1)^6(t-s_2)^6(t-s_3)^6(t-s_4)^6$ we obtain,

$$(\,(s_1-s_2)^2(s_3-s_4)^2 \;+\; (s_1-s_3)^2(s_2-s_4)^2 \;+\; (s_1-s_4)^2(s_2-s_3)^2\,)\,,$$

which is the discriminant for the problem of four lines of Example (13.7).

13.2. Maximally inflected curves

A list $f_1(t),\dots,f_k(t)$ of degree d polynomials defines a map $\varphi\colon \mathbb{P}^1 \to \mathbb{P}^{k-1}$,

$$\varphi \;:\; \mathbb{P}^1 \ni t \;\longrightarrow\; [f_1(t) : f_2(t) : \cdots : f_k(t)] \in \mathbb{P}^{k-1}\,,$$

which is a parametrized rational curve of degree d. The image of the curve is convex at a point $\varphi(t)$ if and only if the first $k{-}1$ derivatives of $\varphi(t)$ are linearly independent. The failure to be convex is measured exactly by the vanishing of the Wronskian of the polynomials $f_1(t),\dots,f_k(t)$. The curve that φ is *ramified* at a point t if this Wronskian vanishes at t. Another geometric term is that the curve φ has an *inflection point* (or flex) at t. This corresponds to the usual notion of an inflection point for plane curves. A rational curve φ of degree d in $\mathbb{P}^{k-1}$ has $k(d{+}1{-}k)$ flexes, counted with multiplicity.

The connection between the Schubert Calculus and rational curves in projective space (linear series on $\mathbb{P}^1$) originated in work of Castelnuovo [**27**] on g-nodal rational curves. This led to the use of Schubert Calculus in Brill-Noether theory (see Chapter 5 of [**65**] for an elaboration). In turn, the theory of limit linear series of Eisenbud and Harris [**40, 41**] provided essential tools to show reality of the special Schubert Calculus [**141**]. (That result on the special Schubert Calculus was a generalization of Theorem 10.1 of Chapter 10.)

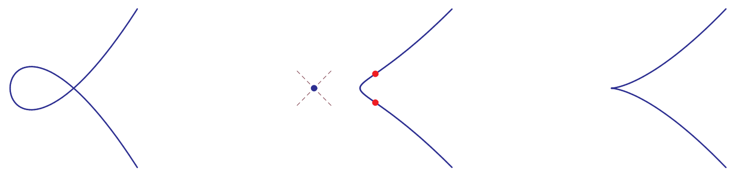
MUKHIN, TARASOV, AND VARCHENKO THEOREM FOR RATIONAL CURVES. *If a rational curve in $\mathbb{P}^{k-1}$ has all of its flexes real, then it must be real.*

A real rational curve with all of its flexes real is *maximally inflected*. The Theorem of Mukhin, Tarasov, and Varchenko asserts that there are many (maximally many, in fact) maximally inflected curves. Let us examine them in the plane $\mathbb{RP}^2$.

Up to projective transformation and reparameterization, there are only three real rational plane cubic curves. These are represented by the equations

$$y^2 \;=\; x^3 + x^2\,, \qquad y^2 \;=\; x^3 - x^2\,, \qquad\text{and}\qquad y^2 \;=\; x^3\,,$$

and they have the shapes shown below.



All three have a real flex at infinity and are singular at the origin. The first has a node and no other real flexes, the second has a solitary point and two real flexes at $(\frac{4}{3}, \pm\frac{4}{3\sqrt{3}})$ (we indicate these with dots and the complex conjugate tangents at the solitary point with dashed lines), and the third has a cusp. The last two are maximally inflected, while the first is not.

By the Schubert Calculus, there will be five rational quartics with six given points of inflection and Figure 13.1 shows five maximally inflected curves with flexes at $\{-3, -1, 0, 1, 3, \infty\}$. (Each nodal curve has two flexes at its node, which is a consequence of the symmetry.) We indicate the differences in the parameterizations of these curves, labeling the flex at -3 by the larger dot and the flex at -1 by the

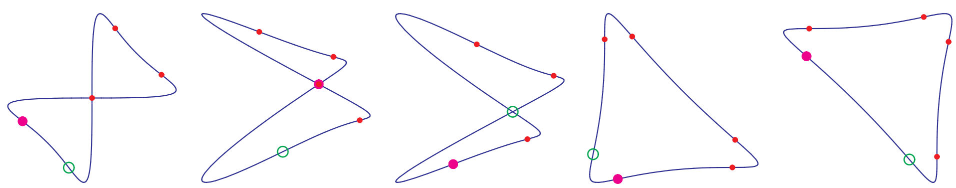


FIGURE 13.1. The five curves with flexes at $\{-3, -1, 0, 1, 3, \infty\}$.

circle. The solitary points are not drawn. The first three curves have two solitary points, while the last two have three solitary points.

Figure 13.2 shows eight smooth maximally inflected quintics and four that are singular. The flexes are indicated (the symmetric curves have one additional flex at infinity), but we do not draw the solitary points. Also, the open circles represent two flexes which have merged into a planar point. None of these curves has many nodes. There are three types of ordinary double points of a real curve; nodes (both branches at the singular point have real tangents), solitary points (the tangents are complex conjugate), and the third type is a pair of complex conjugate double points. Since plane rational curves of degree d have arithmetic genus $\binom{d-1}{2}$, rational quartics without cusps have three ordinary double points and quintics without cusps have six, but none of the quartics of Figure 13.1 have more than one node, and none of the quintics in Figure 13.2 have more then three nodes.

More generally, consider a maximally inflected curve with only flexes and cusps, and whose other singularities are ordinary double points. Let ι be its number of flexes and κ be its number of cusps. Then by the Plücker [**115**] formula, we have $\iota + 2\kappa = 3(d-2)$. By the genus formula, it has $\binom{d-1}{2} - \kappa$ double points. The following theorem is an easy consequence of the Klein [**87**] formula.

THEOREM 13.10 (Topological Restrictions [**82**]). *A maximally inflected curve with only flexes, cusps, and ordinary double points has at least $d-2-\kappa$ solitary points and at most $\binom{d-2}{2}$ nodes, where κ is its number of cusps.*

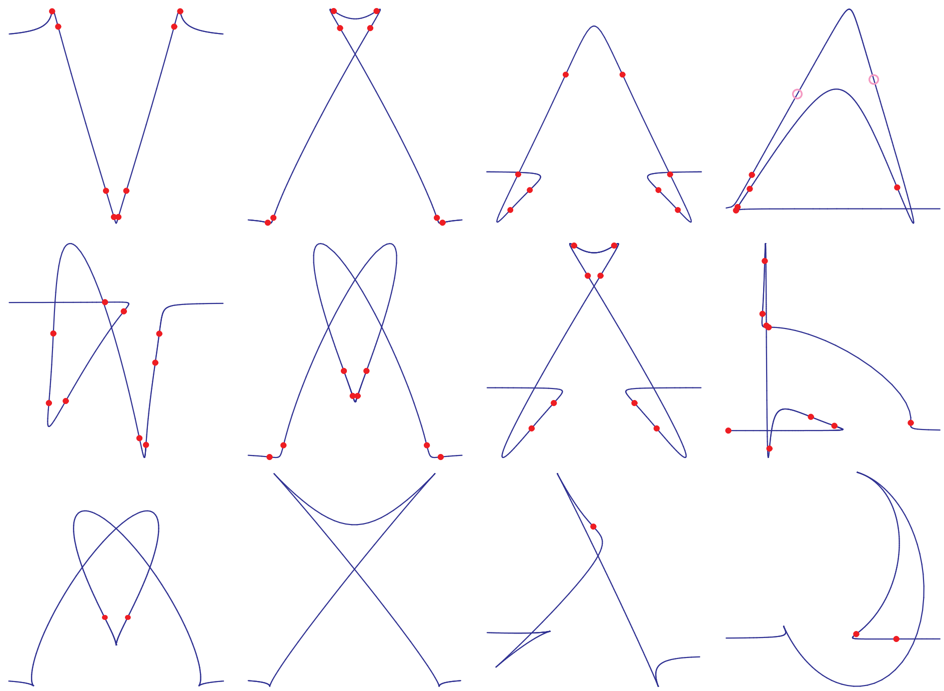


FIGURE 13.2. Some maximally inflected plane quintics.

Thus maximally inflected cubics have at most $\binom{3-2}{2} = 0$ nodes, quartics have at most $\binom{4-2}{2} = 1$ node, and quintics have at most $\binom{5-2}{2} = 3$ nodes, which we have seen.

The existence of curves satisfying the hypotheses of Theorem 13.10 is not guaranteed, even given the Theorem of Mukhin, Tarasov, and Varchenko. Also, the construction in Theorem 4 of Chapter 10 says nothing about the ordinary double points of a maximally inflected curve. There are however two constructions which guarantee curves having only double points. The first uses Shustin's patchworking of singular curves [**133**] to obtain degenerate Harnack curves with $\binom{d-1}{2}$ solitary points. The second perturbs $d-2$ lines tangent to a conic to obtain maximally inflected curves of degree d with minimally many solitary points (and up to $d-2$ cusps). Figure 13.3 shows these constructions when $d = 5$.

The topological classification of maximally inflected plane quintics is open. For example, computations [**82**, § 6.2] suggest that the number of solitary points is a deformation invariant of maximally inflected quintics. We do not know which combinations of nodes, solitary points, and cusps are possible for quintics (see Table 6.1 of [**82**]). We also do not know which embeddings of $\mathbb{RP}^1$ into $\mathbb{RP}^2$ (up to isotopy) are possible for maximally inflected quintics. For example, the curves of Figure 13.2 realize the seven different embeddings we have observed for quintics without cusps, and while we have ruled out some other possibilities, we do not know if the two embeddings illustrated in Figure 13.4 occur.

Lastly, maximally inflected curves in higher-dimensional space have not been investigated. For example, which knot types can occur for maximally inflected space

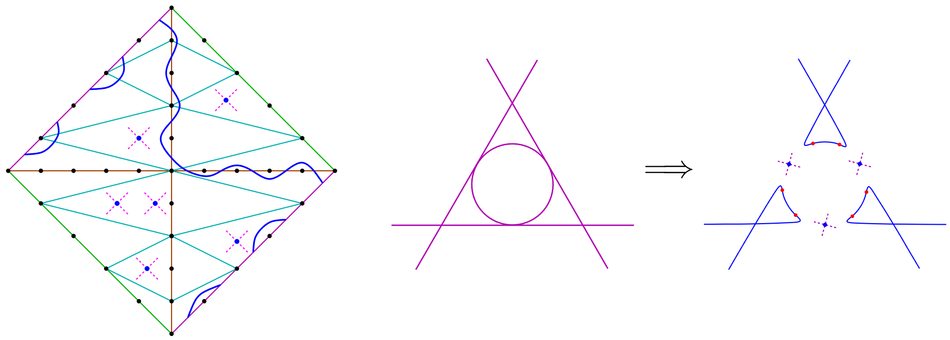


FIGURE 13.3. Constructions of quintics.

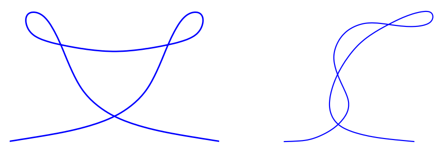


FIGURE 13.4. Embeddings of quintics that have not been observed.

curves? Another possibility, analogous to nodes, is whether there are restrictions on the number of real quadrisecants of a maximally inflected space curve.

13.3. Degree of Wronski maps and beyond

Recall from Sections 1.5 and 8.2 that Eremenko and Gabrielov computed the degree of oriented double cover of the real Wronski map

$$\mathrm{Wr}_{\mathbb{R}} \;:\; \mathrm{Gr}(m, \mathbb{R}_{m+p-1}[t]) \;\longrightarrow\; \mathbb{P}(\mathbb{R}_{mp}[t]) \;\simeq\; \mathbb{RP}^{mp}\,,$$

They showed (Theorem 1.14) that if $m+p$ is even then this map has degree zero, and if $m+p$ is odd then it has degree $\sigma_{m,p}$, which is

$$\frac{1!2!\cdots(m-1)!(p-1)!(p-2)!\cdots(p-m+1)!(\frac{mp}{2})!}{(p-m+2)!(p-m+4)!\cdots(p+m-2)!\,\big(\frac{p-m+1}{2}\big)!\,\big(\frac{p-m+3}{2}\big)!\cdots\big(\frac{p+m-1}{2}\big)!}\,. \tag{13.3}$$

We assume in (13.3) that $m \leq p$; Otherwise set $\sigma_{m,p} := \sigma_{p,m}$, and set $\sigma_{m,p} = 0$ when $m+p$ is even. Then, for any regular value $\Phi(t) \in \mathbb{R}_{mp}[t]$ of the complex Wronski map, the fiber $\mathrm{Wr}_{\mathbb{R}}^{-1}(\Phi(t))$ consists of at least $\sigma_{m,p}$ real points. For example, $\sigma_{2,5} = 2$, so that there will be at least two real points of $\mathrm{Gr}(2, \mathbb{C}_6[t])$ with a general real Wronskian $\Phi(t) \in \mathbb{R}_{10}[t]$, out of the the $\#_{2,5} = 42$ complex points.

The Shapiro Conjecture for Grassmannians states that if $\Phi(t) \in \mathbb{R}_{mp}[t]$ has distinct real roots, then each of the $\#_{m,p}$ points of $\mathrm{Gr}(m, \mathbb{C}_{m+p-1}[t])$ with Wronskian $\Phi(t)$ will be real. Theorem 1.14 implies that some (at least $\sigma_{m,p}$) reality remains if we only require that the *set* of roots be real (that is, they are the roots of a real polynomial and so are stable under complex conjugation).

When $m = 2$, Eremenko and Gabrielov [**43, 44**] showed that this lower bound $\sigma_{2,p}$ is attained. The question remains whether or not $\sigma_{m,p}$ is the tight lower bound when $\min(m,p) > 2$. Likely this is not the case as $\sigma_{3,3} = 0$, but computations (see Table 13.4) suggest that the lower bound is 2.

Fibers of the Wronski map over a general real polynomial $\Phi(t)$ have the form

$$X_{\square}F_\bullet(s_1) \cap X_{\square}F_\bullet(s_2) \cap \cdots \cap X_{\square}F_\bullet(s_{mp})\,, \tag{13.4}$$

where $s_1, \ldots, s_{mp}$ are the roots of $\Phi(t)$. Let r be the number of real roots of $\Phi(t)$ and c the number of complex conjugate pairs of roots. Then the pair (r, c) (or just c if r is understood) determines the *type* of $\{s_1, \ldots, s_{mp}\}$ as a real zero scheme. When $c = 0$, we are in the case of the Shapiro Conjecture and the intersection (13.4) has only real points. For $0 < c \leq mp/2$, the result of Eremenko and Gabrielov shows there are at least $\sigma_{m,p}$ real points in the intersection (13.4). It is natural to ask what is the lower bound on the number of real points in the intersection (13.4), as a function of c. This should be quite interesting, as we can see in Table 13.1, which displays the result of an experiment [**66**] which tested 100,000 instances of

TABLE 13.1. Lower bounds for $\square^8 = 14$ on $\mathrm{Gr}(2,6)$.

	Number of real solutions							
c	0	2	4	6	8	10	12	14
1			35613	16702	14707	6754	1900	24324
2		31317	27221	20417	11343	3407	1121	5174
3	9449	37204	17382	25600	6880	1538	496	1451
4				88180	9511	1207	290	812

the intersection (13.4) with $(m, p) = (2, 4)$ for each possible value 1,2,3, and 4 of $c \neq 0$, and used 5.2 gigahertz-hours of computing. Each cell records the number of instances with a given value of c and number of real solutions. Empty cells indicate that no instances were observed. These computations suggest an apparent lower bound which is different for different values of c.

The caption of Table 13.1 introduces a shorthand notation for Schubert problems. This problem involves eight simple conditions, $\square$, and it has 14 solutions, so we write $\square^8 = 14$.

More generally, if we are given a Schubert problem $\alpha^1, \ldots, \alpha^n$ where some of the conditions coincide, we can consider an intersection

$$X_{\alpha^1}F_\bullet(s_1) \cap X_{\alpha^2}F_\bullet(s_2) \cap \cdots \cap X_{\alpha^n}F_\bullet(s_n)\,, \tag{13.5}$$

where the numbers s_i are real, except in some cases when $\alpha^i = \alpha^j$, we may have $\overline{s_i} = s_j$, for then the intersection (13.5) defines a real variety. For a given pattern of pairs of complex conjugates, we can ask about lower bounds for the number of real points in the intersection. For example, in $\mathrm{Gr}(3,7)$, if we set $\square\square := (3,6,7)$, then $|\square\square| = 10$. Since $\dim(\mathrm{Gr}(3,7)) = 12$, six conditions $\square\square$ give a Schubert problem which has 16 solutions, and we may consider intersections of the form

$$X_{\square\square}F_\bullet(s_1) \cap X_{\square\square}F_\bullet(s_2) \cap \ \cdots\ \cap X_{\square\square}F_\bullet(s_5) \cap X_{\square\square}F_\bullet(s_6)\,,$$

where $\{s_1, \ldots, s_6\} = \{\overline{s_1}, \ldots, \overline{s_6}\}$. As before, let c be the number of complex conjugate pairs in $\{s_1, \ldots, s_6\}$. Table 13.2 displays the results of an experiment testing 30,000 instances of this intersection for each of the different values of c. From this computation, we suspect that there is a lower bound of $2c$ on the number of real solutions to this problem, when the set of points $\{s_1, \ldots, s_6\}$ is real with $c > 0$ complex conjugate pairs. This computation took 48.8 Gigahertz-days.

TABLE 13.2. Lower bounds for $\square\!\square^6 = 16$ on $\mathrm{Gr}(3,7)$.

	Number of real solutions								
c	0	2	4	6	8	10	12	14	16
1		4051	4395	9097	4959	2236	647	689	3926
2			17397	6880	2693	1332	454	324	920
3				22537	2390	4302	95	196	480

In addition to lower bounds that are dependent upon the number of complex conjugate pairs, there appear to be other interesting phenomena. Consider instances of the Schubert problem $\boxplus \cdot \square^7 = 20$ in $\mathrm{Gr}(4,8)$,

$$X_{\boxplus}F_\bullet(\infty) \cap X_{\square}F_\bullet(s_1) \cap \cdots \cap X_{\square}F_\bullet(s_7)\,,$$

where $\{s_1,\ldots,s_7\} = \{\overline{s_1},\ldots,\overline{s_7}\}$ with $c = 1, 2$, or 3. Table 13.3 displays the result of an experiment computing 200,000 instances of this problem for different values

TABLE 13.3. Lower bounds and gaps for $\boxplus \cdot \square^7 = 20$ on $\mathrm{Gr}(4,8)$.

	Number of real solutions										
c	0	2	4	6	8	10	12	14	16	18	20
1					163874						36126
2			117572		73117						9311
3	49316		106851		39708						4125

of c. It took 8.5 gigahertz-hours. This computation suggests both an interesting restriction modulo 4 on the numbers of real solutions, as well as a gap (as in Section 8.3). A similar restriction modulo 4 is seen in computations for the problem $\square^9 = 42$ on $\mathrm{Gr}(3,6)$ as given in Table 13.4. This has an apparent lower bound of

TABLE 13.4. Lower bounds and gaps for $\square^9 = 42$ on $\mathrm{Gr}(3,6)$.

	Number of real solutions										
c	0	2	4	6	8	10	12	14	16	18	20
1		1099		7975		42235		9081		6102	
2		24495		30089		25992		5054		3632	
3		39371		35022		15924		3150		1990	
4				76117		14481		3754		1375	

	Number of real solutions										
c	22	24	26	28	30	32	34	36	38	40	42
1	8827		1597		4207		1343		172		17362
2	4114		955		1586		832		63		3188
3	2183		494		622		367		35		842
4	2925		271		364		204		32		477

2, despite $\sigma_{3,3} = 0$. This computation used 6.9 gigahertz-days.

We close with a brief remark on how these tables were generated. Using a version of local coordinates (10.9) valid for the full Grassmannian, the condition that a point of the Grassmannian lie in a given Schubert variety $X_\alpha F_\bullet(s)$ may be formulated as a system of polynomial equations, similar to (10.8). A given intersection (13.5) may be formulated as a system of equations, and Algorithm 2.9 may be used to determine its numbers of real solutions. This may be partially or wholly automated to compute hundreds to billions of instances of Schubert problems. For an explanation of this, see [**70**].

13.4. The Secant Conjecture

In Section 11.3, we gave the result of Eremenko, Gabrielov, Shapiro, and Vainshtein [**47**] (Theorem 11.12) about real rational functions with prescribed coincidences. This suggests a generalization of the Shapiro Conjecture which replaces flags osculating a rational normal curve by flags that are secant to the rational normal curve, and which satisfy a condition analogous to the separated condition of Theorem 11.12. Let γ be a real rational normal curve and I an interval (arc) of γ. A flag $F_\bullet$ is *secant along I* if each subspace of $F_\bullet$ is spanned by its intersections with I. Such a flag is a *secant flag*. A collection of flags that are secant to γ is *separated* if they are secant along pairwise disjoint intervals of γ. We offer a generalization of the Shapiro Conjecture.

CONJECTURE 13.11 (Secant Conjecture). *Let $\alpha^1, \alpha^2, \ldots, \alpha^n$ be a Schubert problem. Then for any separated secant flags $F^1_\bullet, F^2_\bullet, \ldots, F^n_\bullet$, the intersection*

$$X_{\alpha^1} F^1_\bullet \cap X_{\alpha^2} F^2_\bullet \cap \cdots \cap X_{\alpha^n} F^n_\bullet$$

is transverse with all points real.

Theorem 11.12 establishes the Secant Conjecture for $\mathrm{Gr}(m, m{+}2)$, giving evidence for its validity. In the same way that Theorem 11.1 is a limiting case of Theorem 11.12, the Shapiro Conjecture is a limiting case of the Secant Conjecture. The osculating subspace $F_i(s)$ is the unique i-plane having maximal order of contact with the rational normal curve γ at the point $\gamma(s)$. This and compactness of $\mathbb{RP}^1$ (or a direct calculation) implies that it is a limit of secant flags.

LEMMA 13.12. *Let $\{s_1^{(j)}, \ldots, s_i^{(j)}\}$ for $j = 1, 2, \ldots$ be a sequence of lists of i distinct complex numbers with the property that for each $p = 1, \ldots, i$, we have*

$$\lim_{j\to\infty} s_p^{(j)} \;=\; s\,,$$

for some number s. Then

$$\lim_{j\to\infty} \mathrm{span}\{\gamma(s_1^{(j)}), \ldots, \gamma(s_i^{(j)})\} \;=\; F_i(s)\,.$$

We may deduce that the Shapiro Conjecture is the limiting case of the Secant Conjecture by a standard limiting argument.

THEOREM 13.13. *Let $\alpha^1, \ldots, \alpha^n$ be a Schubert problem and $s_1, \ldots, s_n$ be distinct points of the rational normal curve γ. Then there exists a number $\epsilon > 0$ such that if for each $i = 1, \ldots, n$, $F^i_\bullet$ is a flag secant to γ along an interval of length ϵ containing t_i, then the intersection*

$$X_{\alpha^1} F^1_\bullet \;\cap\; X_{\alpha^2} F^2_\bullet \;\cap\; \cdots \;\cap\; X_{\alpha^n} F^n_\bullet$$

is transverse with all points real.

It is instructive to interpret the Secant Conjecture in the problem of four lines. Figure 13.5 shows three lines secant to the rational normal curve γ in three-space,

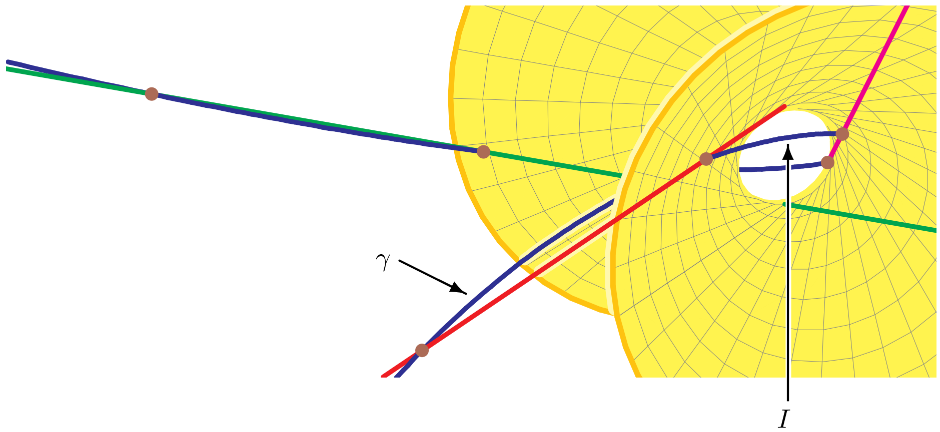


FIGURE 13.5. The problem of four secant lines.

and the hyperboloid of one sheet that they lie upon. The three lines are secant along disjoint intervals, which are disjoint to the indicated interval I. Any line secant along I will meet the hyperboloid in two points, giving two real solutions to this instance of the Secant Conjecture, as the four lines are separated. This always occurs, as the problem of four lines is one of the cases of the Secant Conjecture implied by the result of Eremenko, Gabrielov, Shapiro, and Vainshtein [**47**].

Further evidence for the Secant condition comes from work of Mukhin, Tarasov, and Varchenko [**105**] which implies the following for the Schubert problem $\square^{mp} = \#_{m,p}$. If the points of secancy defining each flag form an arithmetic sequence with the same step size for all flags, then the intersection is transverse with all points real. (See either their paper, or the discussion [**59**, § 3.1.1] for more details.)

Some of the strongest evidence for the Secant Conjecture comes from a computer experiment studying it and related phenomena [**70, 59**]. This considered 703 different Schubert problems on 13 different Grassmannians, verifying the Secant Conjecture in each of the $448,381,157$ instances it computed. In all it determined the number of real solutions in $1,855,810,000$ instances of these Schubert problems and used 1.065 teraHertz-years of computing, mostly on computers in instructional labs at Texas A&M University which moonlight as a supercomputer outside of teaching hours. This work was done using the symbolic methods of Algorithm 2.9.

The remaining $1,407,428,843$ instances involved flags that were not secant along disjoint intervals, but had some overlap in their intervals of secancy. This overlap is measured by a statistic called the *overlap number*, which is zero if and only if the flags are separated. Table 13.5 shows part of the data obtained for the Schubert problem $\yng(2,1) \cdot \square^6 = 16$ on $\mathrm{Gr}(3,6)$. There were $2,500,000$ computed instances of this Schubert problem, all involving flags secant to the rational normal curve. This computation took 16.327 gigahertz-years. The rows are labeled with the even integers from 0 to 16, for the number of observed real solutions. The first column with overlap number 0 represents tests of the Secant Conjecture. Since the only entries are in the row for 16 real solutions, the Secant Conjecture was verified in $560,827$ instances. The column labeled overlap number 1 is empty because flags for

TABLE 13.5. Real solutions vs. overlap number for $⊞ \cdot \square^6 = 16$.

Overlap Number

Real Solutions \	0	1	2	3	4	5	6	7	···	Total
0								3	···	566
2							10	32	···	7452
4				406	1699	176	191	411	···	51416
6				1926	5233	958	662	1184	···	160629
8				2821	7382	1691	1130	1975	···	321827
10				2484	6500	2591	1116	2026	···	430179
12				3288	6185	3296	1320	2250	···	417358
14				1610	2832	2346	767	1376	···	244259
16	560827		19741	61429	50832	17096	8527	9674	···	866314
Total	560827		19741	73964	80663	28154	13723	18931	···	2500000

this problem cannot have overlap number 1. Perhaps the most interesting feature is that for overlap number 2, all computed solutions were real, while for overlap numbers 3, 4, and 5, at least four of the 16 solutions were always real. This inner border, which indicates that some solutions are forced to be real when there is low overlap or that there is a lower bound on the number of real solutions for small overlap number, is found on many of the other problems that we investigated and is a new phenomenon that we do not understand.

In addition to the symbolic computation of this experiment, the Secant Conjecture was also verified in 25,000 instances of the Schubert problem $\square\square^8 = 126$ on $\mathrm{Gr}(4, 8)$, using numerical methods, as this problem is beyond the capabilities of the symbolic software we used for the other computations. This verification used the Bertini package [**6**] (based on numerical homotopy continuation [**137**]) to compute the solutions, whose reality was certified using Smale's α-theory [**135**] as implemented in the package alphaCertified [**67**].

CHAPTER 14

The Shapiro Conjecture Beyond the Grassmannian

There is a version of the Shapiro Conjecture for any flag manifold. However, the conjecture typically fails in this generality. In some cases this failure is quite interesting and the Shapiro Conjecture may be repaired. We will discuss the Shapiro Conjecture for flag manifolds that are similar to the Grassmannian. The Shapiro Conjecture is true for the orthogonal Grassmannian, by a theorem of Purbhoo [**116**], but the Shapiro Conjecture for the Lagrangian Grassmannian is false: while transversality appears to hold, no points of intersection are real. Failure for the classical flag manifold is more subtle, but the Shapiro Conjecture can be repaired through the Monotone Conjecture for which there is compelling evidence. We close with an appealing common generalization of the Secant and Monotone Conjectures.

14.1. The Shapiro Conjecture for the orthogonal Grassmannian

The orthogonal Grassmannian is a flag manifold for the orthogonal group whose properties are quite similar to those of the classical Grassmannian. Fix a positive integer m and suppose that $\mathbb{C}^{2m+1}$ is equipped with a nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ that is split over $\mathbb{R}$. Then $\mathbb{R}^{2m+1}$ has a basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{2m+1}$ for which this form is anti-diagonal,

$$\langle \mathbf{e}_i\,,\ \mathbf{e}_{2m+2-j}\rangle \;=\; \delta_{i,j}\,. \tag{14.1}$$

A subspace $V \subset \mathbb{C}^{2m+1}$ is *isotropic* if $\langle V, V\rangle \equiv 0$ (equivalently, $V \subset V^{\perp}$) and so an isotropic subspace V may have any dimension up to m. The existence of real isotropic subspaces of this dimension, such as the span of $\mathbf{e}_1, \ldots, \mathbf{e}_m$, is equivalent to the form (14.1) being split. The *orthogonal Grassmannian* $\mathrm{OG}(m)$ is the set of all maximal (m-dimensional) isotropic subspaces. It is a manifold of dimension $\binom{m+1}{2}$. A source for further information about isotropic flag manifolds is the book [**56**].

The Shapiro Conjecture for $\mathrm{OG}(m)$ requires a rational normal curve γ whose geometry is related to that of $\mathrm{OG}(m)$. Let γ be the curve with parametrization

$$\begin{aligned} t \;\longmapsto\; &\mathbf{e}_1 \;+\; t\mathbf{e}_2 \;+\; \frac{t^2}{2}\mathbf{e}_3 \;+\; \cdots \;+\; \frac{t^m}{m!}\mathbf{e}_{m+1} \\ &-\; \frac{t^{m+1}}{(m{+}1)!}\mathbf{e}_{m+2} \;+\; \frac{t^{m+2}}{(m{+}2)!}\mathbf{e}_{m+3} \;-\; \cdots \;+\; (-1)^n\frac{t^{2m}}{(2m)!}\mathbf{e}_{2m+1}\,. \end{aligned} \tag{14.2}$$

The flag $F_\bullet(t)$ osculating γ at a point $\gamma(t)$ is *isotropic* in that

$$\langle\, F_i(t)\,,\, F_{2m+1-i}(t)\rangle \;\equiv\; 0 \quad \text{for all } i = 1, \ldots, 2m\,, \tag{14.3}$$

so that $F_i(t)^{\perp} = F_{2m+1-i}(t)$. In general, an isotropic flag $F_\bullet$ of $\mathbb{C}^{2m+1}$ is one with $F_i^{\perp} = F_{2m+1-i}$ for all i. While it is a pleasant exercise to verify (14.3), a more intrinsic understanding of the curve γ (14.2) and its osculating flags comes from the theory of algebraic groups [**21**], which also explains how the Shapiro Conjecture may be posed in any flag manifold. We discuss this in Section 14.3.

Schubert varieties of $\mathrm{OG}(m)$ are defined with respect to an isotropic flag $F_\bullet$. In fact, they are the intersections of certain Schubert varieties $X_\alpha F_\bullet$ of $\mathrm{Gr}(m, 2m{+}1)$ with $\mathrm{OG}(m)$. Schubert varieties of $\mathrm{OG}(m)$ are indexed by subsets κ of $[m] := \{1, 2, \ldots, m\}$. Given $\kappa \subset [m]$, let $\lambda := [m] \setminus \kappa$ be its complement and define $\alpha(\kappa) \in \binom{2m+1}{m}$ to be the Schubert condition

$$(m{+}1{-}\lambda_1, m{+}1{-}\lambda_2, \ldots, m{+}1{-}\lambda_l,\ m{+}1{+}\kappa_k, \ldots, m{+}1{+}\kappa_2, m{+}1{+}\kappa_1)\,, \tag{14.4}$$

where $\lambda\colon \lambda_1 > \lambda_2 > \cdots > \lambda_l$ and $\kappa\colon \kappa_1 > \kappa_2 > \cdots > \kappa_k$. For example, when $m = 6$ and $\kappa = \{5, 3\}$, we have $\lambda = \{6, 4, 2, 1\}$ and $\alpha(\kappa) = (1, 3, 5, 6, 10, 12)$.

Given an isotropic flag $F_\bullet$ and a subset $\kappa \subset [m]$,

$$O_\kappa F_\bullet \ := \ \mathrm{OG}(m) \cap X_{\alpha(\kappa)} F_\bullet$$

is the Schubert subvariety of $\mathrm{OG}(m)$ indexed by κ. This has dimension $\|\kappa\| := \kappa_1 + \cdots + \kappa_k$ and codimension $\binom{m+1}{2} - \|\kappa\| = \|\lambda\|$. The Kleiman-Bertini Theorem [**86**] implies that if $\kappa^1, \ldots, \kappa^n$ are subsets of $[m]$ that satisfy

$$\sum_{i=1}^{n} \left(\binom{m+1}{m} - \|\kappa^i\|\right) \ = \ \sum_{i=1}^{n} \|\lambda^i\| \ = \ \dim(\mathrm{OG}(m)) \ = \ \binom{m+1}{m}\,, \tag{14.5}$$

(such a list $\kappa^1, \ldots, \kappa^n$ is a *Schubert problem* for $\mathrm{OG}(m)$), then for general isotropic flags $F^1_\bullet, \ldots, F^n_\bullet$, the intersection

$$O_{\kappa^1} F^1_\bullet \cap O_{\kappa^2} F^2_\bullet \cap \cdots \cap O_{\kappa^n} F^n_\bullet \tag{14.6}$$

is transverse (and hence also zero-dimensional).

Purbhoo established the Shapiro Conjecture for $\mathrm{OG}(m)$ [**116**].

THEOREM 14.1. *Let $\kappa^1, \ldots, \kappa^n$ be a Schubert problem for $\mathrm{OG}(m)$. Then for every choice of n points $s_1, \ldots, s_n \in \mathbb{RP}^1$, the intersection*

$$O_{\kappa^1} F_\bullet(s_1) \,\cap\, O_{\kappa^2} F_\bullet(s_2) \,\cap\, \cdots \,\cap\, O_{\kappa^n} F_\bullet(s_n)$$

is transverse with all points real.

We deduce this from the Theorem of Mukhin, Tarasov, and Varchenko [**106**]. The key observation is that $\kappa^1, \ldots, \kappa^n$ form a Schubert problem for $\mathrm{OG}(m)$ if and only if the conditions $\alpha(\kappa^1), \ldots, \alpha(\kappa^n)$ form a Schubert problem for $\mathrm{Gr}(m, 2m{+}1)$. To see this, we compare the dimensions and codimensions of the Schubert varieties $O_\kappa F_\bullet$ and $X_{\alpha(\kappa)} F_\bullet$, for an isotropic flag $F_\bullet$. By (14.4), $|\alpha(\kappa)|$ is

$$\begin{aligned} &= \ m{+}1{-}\lambda_1 + \cdots + m{+}1{-}\lambda_l \ + \ m{+}1{+}\kappa_k + \cdots + m{+}1{+}\kappa_1 \ - 1 - \cdots - m \\ &= \ m(m{+}1) - \|\lambda\| + \|\kappa\| - \binom{m+1}{2} \ = \ m(m{+}1) - 2\|\lambda\| \ = \ 2\|\kappa\|\,. \end{aligned}$$

Since $\dim(\mathrm{OG}(m)) = \binom{m+1}{2} = \frac{1}{2}m(m{+}1)$ and $\dim(\mathrm{Gr}(m, 2m{+}1)) = m(m{+}1)$, the condition (14.5) holds for $\kappa^1, \ldots, \kappa^n$ if and only if

$$\sum_{i=1}^{n} \bigl(m(m{+}1) - |\alpha(\kappa^i)|\bigr) \ = \ m(m{+}1) \ = \ \dim(\mathrm{Gr}(m, 2m{+}1))\,,$$

which establishes this observation.

Proof of Theorem 14.1. Suppose that $\kappa^1,\dots,\kappa^n$ forms a Schubert problem for $\mathrm{OG}(m)$. Then $\alpha(\kappa^1),\dots,\alpha(\kappa^n)$ is a Schubert problem for $\mathrm{Gr}(m,2m+1)$. For any $s_1,\dots,s_n\in\mathbb{RP}^1$, the isotropic flags $F_\bullet(s_1),\dots,F_\bullet(s_n)$ are real flags which osculate the rational normal curve γ (14.2). By the Theorem of Mukhin, Tarasov, and Varchenko [**106**] (their second proof of the Shapiro Conjecture), the intersection

$$X_{\alpha(\kappa_1)}F_\bullet(s_1)\,\cap\, X_{\alpha(\kappa_2)}F_\bullet(s_2)\,\cap\,\cdots\cap\, X_{\alpha(\kappa_n)}F_\bullet(s_n)$$

is transverse with all points real. Thus its intersection with $\mathrm{OG}(m)$, which is

$$O_{\kappa^1}F_\bullet(s_1)\,\cap\, O_{\kappa^2}F_\bullet(s_2)\,\cap\,\cdots\cap\, O_{\kappa^n}F_\bullet(s_n)\,,$$

consists only of real points. But this intersection is transverse in $\mathrm{OG}(m)$, by our observation about the dimensions and codimensions of the Schubert varieties $O_\kappa F_\bullet$ and $X_{\alpha(\kappa)}F_\bullet$.

14.2. The Shapiro Conjecture for the Lagrangian Grassmannian

The Lagrangian Grassmannian is remarkably similar to the orthogonal Grassmannian, having identical Schubert decomposition, and closely related cohomology. However, it behaves dramatically different with respect to the Shapiro Conjecture. Fix a positive integer $m\geq 1$ and let $\langle\cdot,\cdot\rangle$ be a nondegenerate skew-symmetric bilinear form on $\mathbb{C}^{2m}$. Then $\mathbb{C}^{2m}$ has a basis $\mathbf{e}_1,\dots,\mathbf{e}_{2m}$ such that

$$\langle \mathbf{e}_i\,,\,\mathbf{e}_{2m+1-j}\rangle \;=\; \begin{cases} \delta_{ij} & \text{if } i\leq m\\ -\delta_{ij} & \text{if } i>m\end{cases}\;.$$

A subspace $V\subset\mathbb{C}^{2m}$ is *isotropic* if $\langle V,V\rangle\equiv 0$ (equivalently, $V\subset V^\perp$) and so an isotropic subspace V may have any dimension up to m. A subspace L is *Lagrangian* if it is isotropic with maximal dimension m. The *Lagrangian Grassmannian* $\mathrm{LG}(m)$ is the set of all Lagrangian subspaces of $\mathbb{C}^{2m}$. It is a manifold of dimension $\binom{m+1}{2}$ and is naturally a subvariety of $\mathrm{Gr}(m,2m)$.

Schubert varieties of $\mathrm{LG}(m)$ are indexed by subsets $\kappa\subset[m]$ and are the intersection of certain Schubert subvarieties of $\mathrm{Gr}(m,2m)$ with $\mathrm{LG}(m)$. Given $\kappa\subset[m]$, let $\lambda:=[m]\setminus\kappa$ be its complement and define $\beta(\kappa)\in\binom{2m}{m}$ to be the Schubert condition

$$(m{+}1{-}\lambda_1, m{+}1{-}\lambda_2,\dots,m{+}1{-}\lambda_l,\; m{+}\kappa_k,\dots,m{+}\kappa_2,m{+}\kappa_1)\,,$$

where $\lambda\colon \lambda_1>\lambda_2>\cdots>\lambda_l$ and $\kappa\colon\kappa_1>\kappa_2>\cdots>\kappa_k$. For example, when $m=6$ and $\kappa=\{5,3\}$, we have $\lambda=\{6,4,2,1\}$ and $\beta(\kappa)=(1,3,5,6,9,11)$.

A flag $F_\bullet$ in $\mathbb{C}^{2m}$ is *isotropic* if

$$F_i^\perp \;=\; F_{2m-i}\qquad \text{for } i=1,\dots,2m{-}1\,.$$

The Schubert variety $L_\kappa F_\bullet$ of $\mathrm{LG}(n)$ where $\kappa\subset[m]$ and $F_\bullet$ is an isotropic flag is

$$L_\kappa F_\bullet \;:=\; \mathrm{LG}(m)\cap X_{\beta(\kappa)}F_\bullet\,.$$

The dimension of $L_\kappa F_\bullet$ is $\|\kappa\|$ and its codimension in $\mathrm{LG}(m)$ is $\|\lambda\|$.

The special rational normal curve in the Shapiro Conjecture for $LG(m)$ is

$$\gamma\,:\,t\;\longmapsto\;\mathbf{e}_1\;+\;t\mathbf{e}_2\;+\;\frac{t^2}{2}\mathbf{e}_3\;+\;\cdots\;+\;\frac{t^m}{m!}\mathbf{e}_{m+1}-\;\frac{t^{m+1}}{(m{+}1)!}\mathbf{e}_{m+2}\;+\;\frac{t^{m+2}}{(m{+}2)!}\mathbf{e}_{m+2}\;-\;\cdots\;+\;(-1)^{m-1}\frac{t^{2m-1}}{(2m{-}1)!}\mathbf{e}_{2m}\,. \tag{14.7}$$

The flag $F_\bullet(s)$ osculating this rational normal curve is isotropic, $F_i(t)^\perp = F_{2m-i}(t)$ for all t and $i = 1, \ldots, 2m-1$.

The natural extension of the Shapiro Conjecture for $\mathrm{LG}(m)$ (which was not posed by Boris Shapiro or by Michael Shapiro) considers intersections of the form

$$L_{\kappa^1} F_\bullet(s_1) \cap L_{\kappa^2} F_\bullet(s_2) \cap \cdots \cap L_{\kappa^n} F_\bullet(s_n)\,, \tag{14.8}$$

where $\kappa^1, \cdots, \kappa^n$ form a Schubert problem for $\mathrm{LG}(m)$, and posits that this intersection is transverse with all points real. This turns out to be false, but in a very interesting way.

Example 14.2. Consider $\mathrm{LG}(2)$, which has dimension $1+2=3$. Its one interesting Schubert problem involves Lagrangian 2-planes that meet each of three fixed Lagrangian 2-planes nontrivially. We solve this problem when the three 2-planes osculate the rational normal curve γ of (14.7). In the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ of $\mathbb{C}^4$, this curve is

$$\gamma(t) \;:=\; (1,\, t,\, \tfrac{1}{2}t^2,\, -\tfrac{1}{6}t^3)\,.$$

The osculating 2-plane $F_2(t)$ is the linear span of $\{\gamma(t), \gamma'(t)\}$. A general Lagrangian plane in $\mathbb{C}^4$ (point in $\mathrm{LG}(2)$) is the row space L of the 2 by 4 matrix

$$L \;:=\; \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & a \end{pmatrix}.$$

Then the equation for L to meet $F_2(t)$ is $f(a,b,c;t) = 0$, where

$$f(a,b,c;t) \;:=\; \det\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & a \\ 1 & t & \frac{1}{2}t^2 & -\frac{1}{6}t^3 \\ 0 & 1 & t & -\frac{1}{2}t^2 \end{pmatrix} \;=\; a^2 - bc + bt + at^2 + \tfrac{1}{3}ct^3 - \tfrac{1}{12}t^4\,.$$

The system for L to meet $F_2(s)$, $F_2(t)$, and $F_2(u)$,

$$f(a,b,c;s) \;=\; f(a,b,c;t) \;=\; f(a,b,c;u) \;=\; 0\,,$$

is equivalent to the vanishing of the polynomials

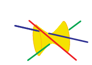
$$12a + 4e_1c - e_1^2 + e_2\,, \qquad 12b - 4e_2c + e_1e_2 - e_3\,, \qquad \text{and}$$

$$(16e_1^2 - 48e_2)c^2 + (-8e_1^3 + 20e_1e_2 + 36e_3)c + e_1^4 - 2e_1^2e_2 + e_2^2 - 12e_1e_3\,,$$

where e_1, e_2, e_3 are the elementary symmetric polynomials in s, t, u. The discriminant of this quadratic in c is

$$-48(s-t)^2(s-u)^2(t-u)^2\,,$$

which is always negative when s, t, u are real and distinct.



We summarize the conclusion of this example.

Proposition 14.3. *When $\kappa = 2$, the unique codimension one Schubert condition for $\mathrm{LG}(3)$, and s, t, u are distinct real numbers, the intersection*

$$L_\kappa F_\bullet(s) \cap L_\kappa F_\bullet(t) \cap L_\kappa F_\bullet(u)$$

is transverse with no points real.

In [**144**], this computation was coupled with a limiting argument using Schubert induction (similar to the proof of Theorem 10.1) show that there are intersections (14.8) in any Lagrangian Grassmannian with no real points. Specifically, let $\kappa := [m] \setminus \{1\}$, the unique codimension one Schubert condition. Then there exist numbers $s_1, s_2, \ldots, s_{\binom{m+1}{2}} \in \mathbb{R}$ such that the intersection

$$X_\kappa F_\bullet(s_1) \cap X_\kappa F_\bullet(s_2) \cap \cdots \cap X_\kappa F_\bullet(s_{\binom{m+1}{2}}) \tag{14.9}$$

is zero-dimensional with no points real. There is no conclusion of transversality.

It is compelling to posit that all Shapiro-type intersections (14.8) in $\mathrm{LG}(m)$ will contain no real points, but this is also false. The same argument as in the proof of Theorem 14.1 gives the following result.

THEOREM 14.4. *Let $\kappa^1, \ldots, \kappa^n$ be a Schubert problem for* $\mathrm{LG}(m)$*. If the corresponding indices $\beta(\kappa^1), \ldots, \beta(\kappa^n)$ also form a Schubert problem for* $\mathrm{Gr}(m, 2m)$*, then for every choice of n distinct points $s_1, \ldots, s_n \in \mathbb{RP}^1$, the intersection*

$$L_{\kappa^1} F_\bullet(s_1) \cap L_{\kappa^2} F_\bullet(s_2) \cap \cdots \cap L_{\kappa^n} F_\bullet(s_n)\,,$$

is transverse with all points real.

These results motivate the following conjecture.

CONJECTURE 14.5 (Shapiro and Discriminant Conjectures for $\mathrm{LG}(m)$). *Let $\kappa^1, \ldots, \kappa^n$ be a Schubert problem for the Lagrangian Grassmannian* $\mathrm{LG}(m)$*. Then for every choice of n distinct points $s_1, , \ldots, s_n \in \mathbb{RP}^1$, the intersection*

$$L_{\kappa^1} F_\bullet(s_1) \cap L_{\kappa^2} F_\bullet(s_2) \cap \cdots \cap L_{\kappa^n} F_\bullet(s_n) \tag{14.10}$$

is transverse. None of the points in the intersection are real, unless $\beta(\kappa^1), \ldots, \beta(\kappa^n)$ form a Schubert problem for $\mathrm{Gr}(m, 2m)$*, in which case all of the points are real.*

Lastly, the discriminant of the intersection (14.10) *is a sum of squares, with each term a monomial in the differences $(s_i - s_j)^2$.*

Tables 14.1 and 14.2 display some of the computational evidence for Conjecture 14.5.

We characterize when $\kappa^1, \ldots, \kappa^n$ and $\beta(\kappa^1), \ldots, \beta(\kappa^n)$ both form Schubert problems. Suppose that $F_\bullet$ is an isotropic flag and let $\kappa \subset [m]$. Then the Schubert variety $L_\kappa F_\bullet$ has dimension $\|\kappa\|$ and codimension $\|\lambda\|$, where $\lambda := [m] \setminus \kappa$. The dimension of $X_{\beta(\kappa)} F_\bullet$ is

$$\begin{aligned} |\beta(\kappa)| &= m{+}1{-}\lambda_1 + \cdots + m{+}1{-}\lambda_l \;+\; m{+}\kappa_k + \cdots + m{+}\kappa_1 \;-1 - \cdots - m \\ &= m^2 + l - \|\lambda\| + \|\kappa\| - \tbinom{m+1}{2} \;=\; m^2 + l - 2\|\lambda\| \;=\; 2\|\kappa\| - k\,, \end{aligned}$$

as the cardinalities l of λ and k of κ sum to m. Thus the codimension of $X_{\beta(\kappa)} F_\bullet$ is $2\|\lambda\| - l$.

Suppose that $\kappa^1, \ldots, \kappa^n$ form a Schubert problem for $\mathrm{LG}(m)$ and also that $\beta(\kappa^1), \ldots, \beta(\kappa^n)$ form a Schubert problem for $\mathrm{Gr}(m, 2m)$. For each $i = 1, \ldots, n$, set $\lambda^i := [m] \setminus \kappa^i$ and let l_i be its cardinality. Then the sum of the codimensions of the Schubert varieties $L_{\kappa^i} F_\bullet$ equals the dimension of $\mathrm{LG}(m)$,

$$\binom{m+1}{2} \;=\; \|\lambda^1\| \;+\; \|\lambda^2\| \;+\; \cdots \;+\; \|\lambda^n\|\,,$$

and the sum of the codimensions of the Schubert varieties $X_{\beta(\kappa^i)}F_\bullet$ equals the dimension of $\mathrm{Gr}(m, 2m)$,

$$\begin{aligned} m^2 &= 2\|\lambda^1\| - l_1 \ + \ \|\lambda^2\| - l_2 \ + \ \cdots \ + \ \|\lambda^n\| - l_n \\ &= m(m+1) \ - l_1 - l_2 - \cdots - l_n \,. \end{aligned}$$

From this we deduce our characterization.

LEMMA 14.6. *Let $\kappa^1, \ldots, \kappa^n$ be a Schubert problem for* $\mathrm{LG}(m)$ *and define l_i so that $m - l_i$ is the cardinality of κ^i. Then $\beta(\kappa^1), \ldots, \beta(\kappa^n)$ is a Schubert problem for* $\mathrm{Gr}(m, 2m)$ *if and only if $l_1 + \cdots + l_n = m$.*

Since $n \leq m$, these Schubert problems involve few Schubert conditions.

As in Section 13.3, we may consider intersections (14.10) which are real, but where some pairs of the points s_i are complex conjugates. For example, if we let $\square := \{3, 2\}$ be the unique codimension 1 condition in LG(3), then Table 14.1 records how often a given number of real solutions were observed in an experiment

TABLE 14.1. Upper bounds for $\square^6 = 16$ on LG(3).

c	Number of Real Solutions								
	0	2	4	6	8	10	12	14	16
0	25000								
1	25000								
2	172291	23874	194932	67798	135189	4741	175		
3	26229	4717	6312	2902	6889	395	714	122	1720

computing 300,000 instances of the Schubert problem $\square^6 = 16$ on LG(3), for different numbers c of complex conjugate pairs. The first row consists of verifications of the Shapiro Conjecture for this Schubert problem. In contrast to the computations of Section 13.3, there appear to be upper bounds on the possible numbers of real solutions. This computation took 3.6 gigahertz-days.

More intriguing is Table 14.2 which is for 1900 instances of the Schubert problem $\square\square^5 = 24$ on LG(4), where $\square\square = \{4, 3, 1\}$. There appears to be a clear upper bound on the number of real solutions. This computation took 1.25 gigahertz-years.

TABLE 14.2. Computations for $\square\square^5 = 24$ on LG(4).

c	Number of Real Solutions												
	0	2	4	6	8	10	12	14	16	18	20	22	24
0	500												
1	315	441	744	113	253	134							
2	500												

14.3. The Shapiro Conjecture for flag manifolds

The Shapiro Conjecture may be posed for any flag manifold. We explain this in the language of linear algebraic groups. Definitions and background may be found in [**21**].

Let G be a reductive linear algebraic group. A *Borel subgroup* is a maximal connected solvable subgroup B of G, and a *parabolic subgroup* P is any subgroup containing a Borel subgroup. The quotient G/P is a compact space called the *flag manifold*. Its points are identified with the parabolic subgroups conjugate to P.

A Borel subgroup B has finitely many orbits on a flag manifold G/P—these correspond to double cosets BgP of B and P in G. Assuming, as we may, that $B \subset P$, these orbits are naturally indexed by cosets W/W_P of the Weyl group W_P of P in the Weyl group W of G. Each coset has a canonical representative in the Coxeter group W of minimal length. Let $W^P \subset W$ be the set of these minimal representatives, which we will call *Schubert conditions*. For $w \in W^P$, the orbit BwP/P is isomorphic to $\mathbb{C}^{\lg(w)}$, where $\lg(w)$ is the length of w. Its closure is the *Schubert variety X_wB*.

A *Schubert problem* is a list $w_1, \ldots, w_n$ of Schubert conditions satisfying

$$\sum_{i=1}^{n}(\dim(G/P) - \lg(w_i)) \;=\; \dim(G/P)\,.$$

Given a Schubert problem $w_1, \ldots, w_n$, the Kleiman-Bertini Theorem [**86**] implies that for any Borel subgroup B and general elements $g_1, \ldots, g_n \in G$, the intersection of translates

$$g_1.X_{w_1}B \,\cap\, g_2.X_{w_2}B \,\cap\, \cdots \,\cap\, g_n.X_{w_n}B \tag{14.11}$$

is transverse and zero-dimensional. If $B_i := g_i.B = g_i \cdot B \cdot g_i^{-1}$, then (14.11) becomes

$$X_{w_1}B_1 \,\cap\, X_{w_2}B_2 \,\cap\, \cdots \,\cap\, X_{w_n}B_n\,,$$

which we may compare to (9.6) and (14.6).

The Shapiro Conjecture involves real points of intersections (14.11). For it, we will need to choose an appropriate real form of the group G, as not just any real form will do. For example, the orthogonal group of Section 14.1, which is the subgroup of $GL(2m{+}1,\mathbb{R})$ preserving the form $\langle\cdot,\cdot\rangle$ was not the orthogonal group of rotations in $\mathbb{R}^{2m+1}$. We must use the split real form of the group G. Concretely, this means that every semisimple element of $G(\mathbb{R})$ has only real eigenvalues in any representation of G. That is, their characteristic polynomials split completely into linear factors over $\mathbb{R}$. Equivalently, every maximal torus T of $G(\mathbb{R})$ is isomorphic to $(\mathbb{R}^*)^r$, where r is the rank of G. Then if we choose $T \subset B \subset P$ and $g_1, \ldots, g_n \in G(\mathbb{R})$, we may ask for real points in an intersection (14.11).

The Shapiro Conjecture uses a special choice of elements $g_1, \ldots, g_n$ in (14.11). Let $\mathfrak{g}$ be the Lie algebra of G. The adjoint (conjugation) action of G on $\mathfrak{g}$ preserves the nilpotent elements of $\mathfrak{g}$. A nilpotent element $\eta \in \mathfrak{g}$ is *principal* if its orbit under G is dense in the set of nilpotent elements. Associated to a principal nilpotent element η is a 1-parameter unipotent subgroup of G,

$$t \;\longmapsto\; \Gamma(t) \;:=\; e^{\eta t}\,.$$

When G and $\mathfrak{g}$ are expressed in terms of matrices, then $e^{\eta t} = 1 + \eta \cdot t + \eta^2 \cdot \frac{t^2}{2!} + \cdots$. This group Γ lies in a unique Borel subgroup B_+ of G. Let B be any Borel subgroup having minimal intersection with B_+ so that B_+B is dense in G.

CONJECTURE 14.7 (Shapiro Conjecture for G/P). *For any Schubert problem* $w_1, \ldots, w_n$ *for* G/P *and any distinct real numbers* $s_1, \ldots, s_n$*, the intersection*

$$\Gamma(s_1).X_{w_1}B \cap \Gamma(s_2).X_{w_2}B \cap \cdots \cap \Gamma(s_n).X_{w_n}B$$

is transverse with all points real.

This conjecture, while true for the classical Grassmannians $\mathrm{Gr}(b,m)$ and the orthogonal Grassmannians $\mathrm{OG}(m)$, does not hold for the Lagrangian Grassmannians $\mathrm{LG}(m)$. As we saw in Section 14.2, it can be repaired for $\mathrm{LG}(m)$ and transversality appears to hold. We will see that it fails for the classical flag manifolds when $G = SL(m,\mathbb{R})$ that are not Grassmannians, but may also be repaired. While this conjecture has not been systematically investigated for other flag manifolds, it has been shown to fail for the manifold of isotropic 2-planes in $\mathbb{C}^6$—a flag manifold for the symplectic group $Sp(6,\mathbb{C})$, and there is no clear way to repair it.

The results and conjectures for the Grassmannians $\mathrm{Gr}(b,m)$, $\mathrm{OG}(m)$, and $\mathrm{LG}(m)$ suggest that some variant of Conjecture 14.7 may hold for flag manifolds that are similar to these three families. One attractive class is the class of minuscule and cominuscule [**91**] flag manifolds. Besides these three families, this class only includes the quadrics, a second form of $\mathrm{OG}(m)$ for the even orthogonal groups and two sporadic spaces E_6/D_5 and E_7/E_6.

14.4. The Monotone Conjecture

The most familiar flag manifolds beyond the Grassmannians are the classical flag manifolds for the special linear groups. Let $E_\bullet$ be a (partial) flag in $\mathbb{C}^m$, which is an increasing sequences of subspaces

$$E_\bullet \,:\, E_{a_1} \subset E_{a_2} \subset \cdots \subset E_{a_k} \subset \mathbb{C}^m\,,$$

where $\dim(E_{a_i}) = a_i$. The *type* of $E_\bullet$ is the sequence $\mathbf{a} := (a_1, a_2, \ldots, a_k)$ of dimensions of dimensions, and we write $\mathbb{F}\ell(\mathbf{a};m)$ for the set of all flags of type $\mathbf{a}$. This is a manifold of dimension

$$\dim(\mathbf{a}) \;:=\; \sum_{i=1}^{k}(a_i - a_{i-1})(m - a_i)\,,$$

where we set $a_0 := 0$. For background on these flag manifolds, see [**53**].

A Schubert variety of $\mathbb{F}\ell(\mathbf{a};m)$ is the set of flags $E_\bullet$ having specified position with respect to a fixed flag $F_\bullet$. The Shapiro Conjecture for the flag manifold $\mathbb{F}\ell(\mathbf{a};m)$ considers intersections of Schubert varieties when the fixed flags osculate a real rational normal curve. It turns out that such intersections need not have any real points and they need not be transverse, so the Shapiro Conjecture is false for the flag manifold $\mathbb{F}\ell(\mathbf{a};m)$. There is however a way to repair it, which we call the Monotone Conjecture. We give a detailed example of the failure of the Shapiro Conjecture, formulate the Monotone Conjecture, and discuss evidence for it.

EXAMPLE 14.8. The flag manifold $\mathbb{F}\ell(2,3;4)$ may be realized as the set of flags consisting of a line h lying on a plane H in $\mathbb{P}^3$. The natural Schubert conditions for these flags $h \subset H$ are either that h meets a given line ℓ, or that H contains a given point p. Consider the Schubert problem of flags $h \subset H$ where h meets three given lines ℓ_1, ℓ_2, and ℓ_3 and H contains two given points p and q. We solve this by reducing it to the problem of four lines. The plane H contains the line $\overline{p,q}$ spanned by the points p and q. Then h is constrained to meet a fourth line, $\overline{p,q}$ as $h \subset H$. Thus there are two lines h_1 and h_2 meeting the lines ℓ_1, ℓ_2, ℓ_3, and $\overline{p,q}$. The planes H_i are determined by the lines, $H_i := \mathrm{span}\{h_i, p\} = \mathrm{span}\{h_i, q\}$.

For this problem, the Shapiro Conjecture asks for the flags $h \subset H$ when the lines ℓ_i osculate a rational normal curve γ and p, q are points of γ. Suppose that γ is the rational curve of (1.6) from Example 1.10,

$$\gamma \ : \ t \ \longmapsto \ \left(6t^2 - 1,\ \tfrac{7}{2}t^3 + \tfrac{3}{2}t,\ \tfrac{3}{2}t - \tfrac{1}{2}t^3\right),$$

the three tangent lines are $\ell(-1)$, $\ell(0)$, and $\ell(1)$, and the two points are $\gamma(v)$ and $\gamma(w)$, where $v \neq w$. The quadric Q containing the three tangent lines is shown in Figure 1.1. As we have seen, the lines h meeting the four given lines, and hence the solutions $h \subset H$ to our Schubert problem will be real if and only if the secant line $\ell(v,w) := \overline{\gamma(v), \gamma(w)}$ meets the quadric in real points.

This reality depends upon the configuration of the points v, w. Suppose first that the secant line $\ell(v,w)$ is close to a tangent line. Figure 14.1 shows such a secant line $\ell(v,w)$ which meets the quadric in two real points, and therefore these

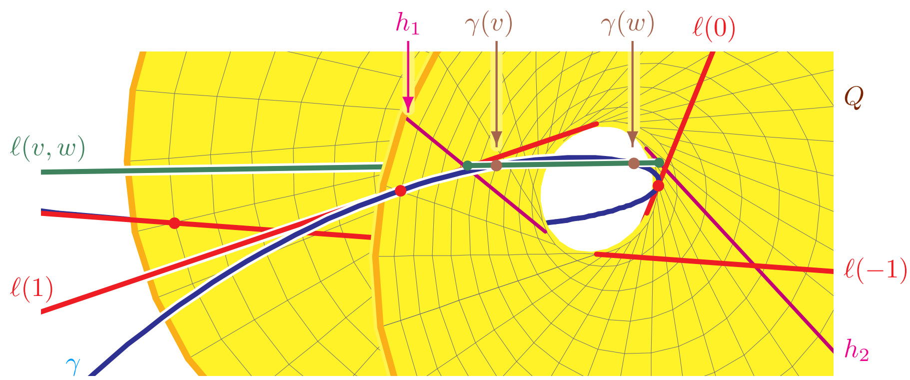
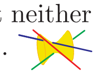


FIGURE 14.1. Secant line meeting Q, giving two real solutions.

choices for v and w give two real solutions to our Schubert problem. There is also a secant line meeting the quadric Q in two complex conjugate points, so that neither flag solving our problem is real. We show this configuration in Figure 14.2.



To investigate this failure of the Shapiro Conjecture, note that as a curve in $\mathbb{P}^3$, γ has the parametrization

$$\gamma \ : \ t \ \longmapsto \ [2 : 12t^2{-}2 : 7t^3{+}3t : 3t{-}t^3]\,.$$

Then the lines tangent to γ at $\gamma(-1)$, $\gamma(0)$, and $\gamma(1)$ lie on the quadric

$$Q \ : \ x_0^2 - x_1^2 + x_2^2 - x_3^2 \ = \ 0\,.$$

If we parameterize the secant line $\ell(v,w)$ as $(\frac{1}{2}+t)\gamma(v) + (\frac{1}{2}-t)\gamma(w)$ and then substitute this into the equation for Q, we obtain a quadratic polynomial in t. Its discriminant is

$$16(v-w)^2\,(2vw+v+w)(3vw+1)(1-vw)(v+w-2vw)\,. \qquad (14.12)$$

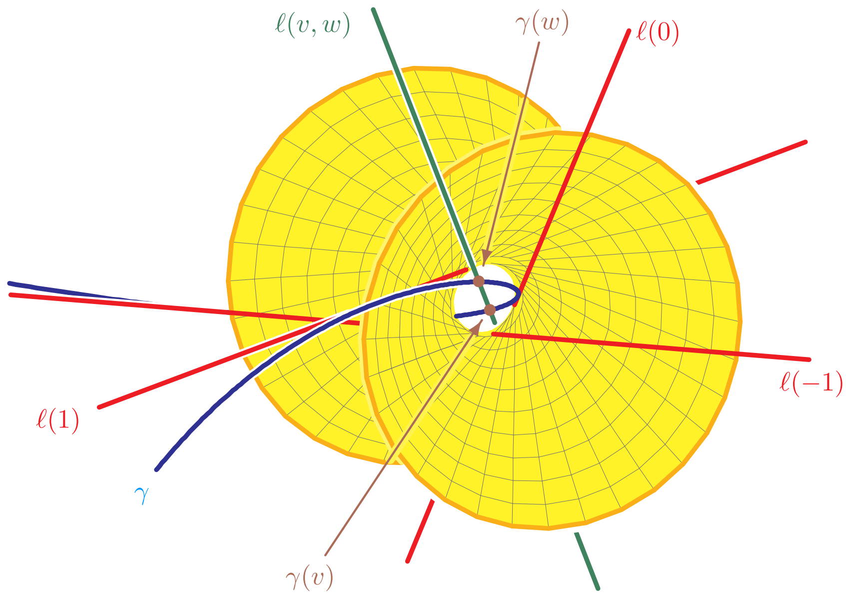


Figure 14.2. Secant line with no real solutions.

Figure 14.3 shows this discriminant in the square $v, w \in [-2, 2] \subset \mathbb{RP}^1 \times \mathbb{RP}^1$, shading the regions where it is negative. The broken lines are $v, w = \pm 1$, the diagonal line is $v = w$, the cross is the value of (v, w) in Figure 14.1, and the dot is the value in Figure 14.2. The discriminant is nonnegative if (v, w) lies in one of the

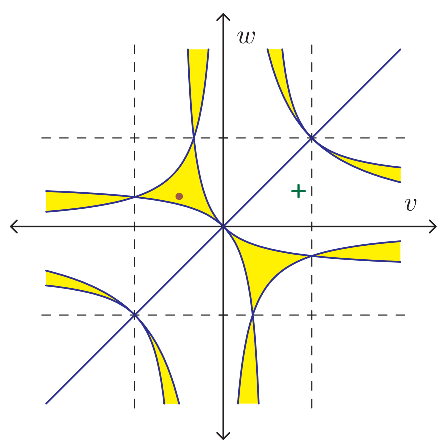


Figure 14.3. Discriminant of the Schubert problem of Example 14.8.

squares $(-1, 0)^2$, $(0, 1)^2$, or if $(\frac{1}{v}, \frac{1}{w}) \in (-1, 1)^2$ and it is positive on the triangles into which the line $v = w$ subdivides these squares. These squares are when v and w both lie within one of the three intervals of $\mathbb{RP}^1$ determined by $-1, 0, 1$. Write $Y(s)$ for those flags $h \subset H$ where h meets $\ell(s)$ and $Z(s)$ for those flags $h \subset H$ where H contains $\gamma(s)$. Then our Schubert problem is

$$Y(s) \cap Y(t) \cap Y(u) \ \cap \ Z(v) \cap Z(w)\,, \tag{14.13}$$

for general $s, t, u, v, w \in \mathbb{RP}^1$. Möbius transformations of $\mathbb{RP}^1$, together with our analysis of this discriminant imply following proposition.

PROPOSITION 14.9. *If there are disjoint intervals I_1 and I_2 of $\mathbb{RP}^1$ so that $s, t, u \in I_1$ and $v, w \in I_2$, then the intersection* (14.13) *is transverse with both points real.*

Figure 14.4 displays cartoons of the configurations of secant and tangent lines to the rational normal curve γ (represented as a circle) of Figures 14.1 and 14.2. In

Fig. 14.1

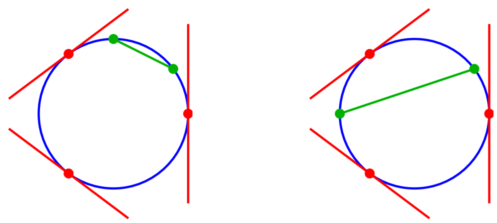


Fig. 14.2

FIGURE 14.4. Configuration of secants and tangents.

the configuration for Figure 14.1, one may travel along the circle, first encountering the three points of the tangent lines and then the two secant points. Recording the dimension of the piece of the flag $h \subset H$ affected by the tangent line or point gives the weakly increasing sequence $(1,1,1,2,2)$. The configuration of Figure 14.2 gives either $(1,1,2,1,2)$, or $(2,1,2,1,1)$, or $(1,2,1,2,1)$, none of which are weakly increasing. We believe that this is the key to understanding reality.

Schubert varieties in $\mathbb{F}\ell(\mathbf{a}; m)$ are indexed by permutations w in the symmetric group on m letters whose descents $\{i \mid w(i) > w(i{+}1)\}$ lie in $\mathbf{a}$. These descents have a geometric interpretation. For each $b \in \mathbf{a}$, there is a natural projection

$$\pi_b \ :\ \mathbb{F}\ell(\mathbf{a}; m) \ \longrightarrow\ \mathrm{Gr}(b, m) \tag{14.14}$$

obtained by forgetting all subspaces of the flag $E_\bullet$ except for E_b. The image of a Schubert variety $X_w F_\bullet$ of $\mathbb{F}\ell(\mathbf{a}; m)$ under the projection π_b is the Schubert variety $X_{\pi_b(w)} F_\bullet$ of $\mathrm{Gr}(b; n)$ where $\pi_b(w)$ is the increasing rearrangement of the sequence

$$(n{+}1{-}w(1)\,,\ n{+}1{-}w(2)\,,\ \cdots\,,\ n{+}1{-}w(b))\,.$$

These projections define the Schubert variety $X_w F_\bullet$,

$$X_w F_\bullet \ =\ \bigcap_{b \in \mathbf{a}} \pi_b^{-1}\bigl(X_{\pi_b(w)} F_\bullet\bigr)\,. \tag{14.15}$$

This explains the significance of the descent set of w.

LEMMA 14.10. *The intersection* (14.15) *may be taken over the descents of w,*

$$X_w F_\bullet \ =\ \bigcap_{b\,:\ w(b) > w(b+1)} \pi_b^{-1}\bigl(X_{\pi_b(w)} F_\bullet\bigr)\,.$$

In particular, if w has only a single descent at b, then $X_w F_\bullet = \pi_b^{-1}\bigl(X_{\pi_b(w)} F_\bullet\bigr)$.

A *Grassmannian Schubert condition* w for $\mathbb{F}\ell(\mathbf{a}; m)$ is a permutation which has only a single descent, necessarily at some position in $\mathbf{a}$. Write $\mathrm{des}(w)$ for the position of this descent. By Lemma 14.10, a Schubert variety given by a Grassmannian condition w with descent b is the pullback of a Schubert variety in the Grassmannian

$\mathrm{Gr}(b,m)$ along the projection $\pi_b\colon \mathbb{F}\ell(\mathbf{a};m) \to \mathrm{Gr}(b,m)$. In Example 14.8, both conditions were Grassmannian—one had descent 2 (for the requirement that h meet a tangent line) and the other had descent 3 (for the requirement that H meet a point on the rational normal curve.) These differed from the dimension of the linear spaces h and H, as we were working in $\mathbb{P}^3$, rather than in $\mathbb{C}^4$.

A set of points $\{s_1 \ldots, s_n\} \subset \mathbb{RP}^1$ may be *cyclically ordered* by the order in which they appear along $\mathbb{RP}^1$ starting from some point and moving in some direction. There are $2n$ such cyclic orderings of a given set of n points. A collection of points $\{s_1,\ldots,s_n\} \subset \mathbb{RP}^1$ is *monotone* with respect to a Grassmannian Schubert problem $w_1,\ldots,w_n$ if there is a cyclic order $\prec$ of $\{s_1,\ldots,s_n\}$ such that for all $i \neq j$,

$$\mathrm{des}(w_i) < \mathrm{des}(w_j) \implies s_i \prec s_j\,. \tag{14.16}$$

Fix a real rational normal curve γ and let $F_\bullet(s)$ be the flag of subspaces osculating the curve γ at the point $\gamma(s)$.

Conjecture 14.11 (Monotone Conjecture). *Let $w_1,\ldots,w_n$ be a Grassmannian Schubert problem on $\mathbb{F}\ell(\mathbf{a};m)$. Then for any collection $\{s_1,\ldots,s_n\}$ of n points in $\mathbb{RP}^1$ that is monotone with respect to $w_1,\ldots,w_n$,*

$$X_{w_1}F_\bullet(s_1) \cap X_{w_2}F_\bullet(s_2) \cap \cdots \cap X_{w_n}F_\bullet(s_n) \tag{14.17}$$

is transverse with all points real.

If we have ordered the Schubert conditions so that $\mathrm{des}(w_1) \leq \mathrm{des}(w_2) \leq \cdots \leq \mathrm{des}(w_n)$, then we may reparameterize $\mathbb{RP}^1$ so that the points s_i all lie in $\mathbb{R}$ and satisfy $s_1 < s_2 < \cdots < s_n$.

The Monotone Conjecture is an extension of the Shapiro Conjecture for Grassmannians. Indeed, the monotone condition (14.16) is vacuous when all permutations have the same descent, which is the case when the flag manifold $\mathbb{F}\ell(\mathbf{a};m)$ is a Grassmannian. Besides this reduction to the Shapiro Conjecture, other evidence for the Monotone Conjecture was provided by Eremenko, Gabrielov, Shapiro, and Vainshtein, whose main theorem in [**47**] implied it for the flag manifolds $\mathbb{F}\ell(m{-}2,m{-}1;m)$ and $\mathbb{F}\ell(1,2;m)$.

As with the Shapiro Conjecture for $\mathrm{Gr}(b,m)$, $\mathrm{OG}(m)$, and $\mathrm{LG}(m)$, there is also a related Discriminant Conjecture. The *discriminant* of a Grassmannian Schubert problem $w_1,\ldots,w_n$ is the polynomial defining the hypersurface in $\mathbb{C}^n$ consisting of points $(s_1,\ldots,s_n)$ where the intersection (14.17) fails to be transverse.

The *preprime* generated by polynomials $g_1, g_2, \ldots, g_n$ is the collection of polynomials f of the form

$$f \;=\; S_0 \;+\; S_1g_1 \;+\; S_2g_2 \;+\; \cdots \;+\; S_ng_n\,, \tag{14.18}$$

where the polynomials S_i are sums of squares. A polynomial f in this preprime is obviously positive on the set

$$K \;:=\; \{x \mid g_i(x) > 0\}$$

and a representation (14.18) of f as an element of this preprime is a certificate for its positivity on K.

Conjecture 14.12 (Discriminant Conjecture). *The discriminant of a Grassmannian Schubert problem $w_1,\ldots,w_n$ lies in the preprime generated by $(s_i - s_j)$ for $\mathrm{des}(w_i) > \mathrm{des}(w_j)$.*

The point of this conjecture is that not only is the discriminant positive on the set $\{(s_1, s_2, \ldots, s_n) \mid \operatorname{des}(w_i) > \operatorname{des}(w_j) \Rightarrow s_i > s_j\}$ of monotone parameters, but that it has a special form for which this positivity is transparent. Not all polynomials that are positive on a set of this form can lie in the preprime generated by the differences $s_i - s_j$ for $\operatorname{des}(w_i) > \operatorname{des}(w_j)$ [**126**, §6.7].

The Monotone Conjecture was formulated in [**123**]. That paper also reported on a computer experiment studying the Monotone Conjecture for 1126 different Schubert problems on 29 different flag manifolds and using 15.76 gigahertz-years of computing. The Monotone Conjecture was verified in each of the $165,666,089$ instances checked. That experiment tested much more, also computing intersections (14.17) where monotonicity did not hold, recording the numbers of real and complex solutions in each instance. We describe a small part of these data.

We may decorate the points $s_1, \ldots, s_n \in \mathbb{RP}^1$ of an intersection (14.17) with the corresponding Schubert conditions $w_1, \ldots, w_n$. This configuration of labeled points, considered up to isotopy and reversal, is called a *necklace*. Write necklaces as a word using one of their cyclic orderings, so that

$$YYYZZ\,,\ YYZZY\,,\ YZZYY\,,\ ZZYYY\,,\ \text{ and }\ ZYYYZ\,,$$

all represent the same necklace. For example, the configurations of Figure (14.4) have necklaces $YYYZZ$ and $YYZYZ$ (equivalently, $(1,1,1,2,2)$ and $(1,1,2,1,2)$). In this experiment the number of real solutions was recorded as a function of the necklace. Table 14.3 shows the result of testing $400,000$ instances for each of eight necklaces for the Schubert problem

$$X(s_1) \cap X(s_2) \cap X(s_3) \cap X(s_4)\ \cap\ Y(s_5) \cap Y(s_6) \cap Y(s_7) \cap Y(s_8)$$

on $\mathbb{F}\ell(2,3;5)$, where $X(s)$ is the Schubert variety of flags $E_2 \subset E_3$ where E_2 meets the osculating 3-plane to γ at $\gamma(s)$ and $Y(s)$ is the Schubert variety where E_3 meets the osculating 2-plane. This computation took 213 gigahertz-days. The first row represents computations testing the Monotone Conjecture. The other rows show that even when there are nonmonotone evaluations, some reality survives, and only for the most interlacing necklace of the last row is it possible to get no real solutions.

Table 14.3. The Schubert problem X^4Y^4 on $\mathbb{F}\ell(2,3;5)$.

Necklace	Number of Real Solutions						
	0	2	4	6	8	10	12
XXXXYYYY							400000
XXYXXYYY			118	65425	132241	117504	84712
XXXYYXYY			104	65461	134417	117535	82483
XXYYXXYY			1618	57236	188393	92580	60173
XXYXYYXY			25398	90784	143394	107108	33316
XXYYXYXY		2085	79317	111448	121589	60333	25228
XXXYXYYY		7818	34389	58098	101334	81724	116637
XYXYXYXY	15923	41929	131054	86894	81823	30578	11799

For example, there are no real points in the intersection

$$X(-8) \cap Y(-4) \cap X(-2) \cap Y(-1) \cap X(1) \cap Y(2) \cap X(4) \cap Y(8)\,.$$

Table 14.4 shows calculations for a Schubert problem on $\mathbb{F}\ell(2,3;6)$ with 21 solutions. Here, W represents the Grassmannian Schubert condition that is $\mathbb{T}$ pulled back from $\mathrm{Gr}(2,6)$ and X is the codimension one Grassmannian Schubert condition pulled back from $\mathrm{Gr}(3,6)$. This took 191 gigahertz-days of computing. The first row represents instances of the Monotone Conjecture. Unlike the previous

TABLE 14.4. Enumerative Problem $W^3X^5 = 21$ on $\mathbb{F}\ell(2,3;6)$.

Necklace	Number of Real Solutions										
	1	3	5	7	9	11	13	15	17	19	21
WWWXXXXX											200000
WWXWXXXX						82	4173	36937	46363	25298	87147
WWXXWXXX						1471	11933	49180	82745	36295	18376
WXWXWXXX						2570	27139	61578	55244	23863	29606
WXWXXWXX						4544	31410	79160	55345	22079	7462
Total						8667	74655	226855	239697	107535	342591

Schubert problem, there is a clear and near-uniform lower bound on the number of real solutions to this problem. There is currently no explanation for this and many other lower bounds and gaps observed in the data.

14.5. The Monotone Secant Conjecture

The Monotone Secant Conjecture is the common generalization of the Secant Conjecture for the Grassmannians and the Monotone Conjecture for flag manifolds. It involves flags that are secant to a rational normal curve along disjoint intervals, where the intervals are monotone with respect to the Schubert problem in the same way that the osculating points were monotone in the Monotone Conjecture.

As with points, a collection $I^1, \ldots, I^n$ of pairwise disjoint intervals of a rational normal curve γ has $2n$ cyclic orderings as do flags $F_\bullet^1, \ldots, F_\bullet^n$ which are secant along disjoint intervals of γ. Such a set of flags is *monotone* with respect to a Grassmannian Schubert problem $w_1, \ldots, w_n$ if there is a cyclic ordering $\prec$ of the flags such that for all i, j,

$$\mathrm{des}(w_i) < \mathrm{des}(w_j) \implies F^i \prec F^j\,.$$

CONJECTURE 14.13 (Monotone Secant Conjecture). *Let $w_1, \ldots, w_n$ be a Grassmannian Schubert problem on $\mathbb{F}\ell(\mathbf{a}; m)$ and γ a rational normal curve. Then for any flags $F_\bullet^1, \ldots, F_\bullet^n$ secant to γ along disjoint intervals that are monotone with respect to $w_1, \ldots, w_n$, the intersection*

$$X_{w_1}F_\bullet^1 \cap X_{w_2}F_\bullet^2 \cap \cdots \cap X_{w_n}F_\bullet^n \tag{14.19}$$

is transverse with all points real.

The Monotone Secant Conjecture reduces to the Secant Conjecture when the flag manifold is a Grassmannian. It also reduces to the Monotone Conjecture in the limit as the intervals of secancy shrink to points, so that the secant flags become osculating flags, as in Lemma 13.12. While these reductions provide some justification for the Monotone Secant Conjecture, there is also considerable experimental evidence supporting it. As of February 2011, an experiment at Texas A&M University studying the Monotone Secant Conjecture in about 720 Schubert problems on 17 flag manifolds had used over 600 gigahertz-years of computing. This tested not only instances of the Monotone Secant Conjecture, but also instances of nonmonotone secant flags, and related instances of the Monotone Conjecture, in order to compare the two conjectures. The results are tabulated on-line and the similarity of the tables for the two conjectures is striking.

For example, Table 14.5 displays computations for the same Schubert problem as Table 14.4. This computation required 7.670 gigahertz-years of computing. In

TABLE 14.5. Enumerative Problem $W^3X^5 = 21$ on $\mathbb{F}\ell(2,3;6)$.

Necklace	Number of Real Solutions										
	1	3	5	7	9	11	13	15	17	19	21
WWWXXXXX											80000
WWXWXXXX							921	16549	26267	14475	21788
WWXXWXXX						39	1208	24559	39013	13947	1234
WXWXWXXX						3244	19887	31931	13688	3632	7618
WXWXXWXX						612	9544	43256	23583	2927	78
Total						3895	31560	116295	102551	34981	110718

both, there is an observed lower bound on the number of real solutions, and the conjectures are verified in all cases tested (corresponding to the first rows in each table). However, there is one cell that is empty in Table 14.5 but which is occupied in Table 14.4 (it is shaded). There is currently no theory or conjectures for these observed phenomena that are beyond the Monotone Secant Conjecture.

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