Toric Varieties   Fall 2024
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Some exercises on toric varieties

1. Let $\mathcal{A}=\{\binom{0}{0},\binom{1}{0},\binom{2}{0},\binom{0}{1},\binom{1}{1}\}$. Give a generating set for the toric ideal $I_{\mathcal{A}}$.

2. Set $n:=4$ and $k:=2$. Let $\mathbf{e}_1,\ldots,\mathbf{e}_n$ be a the standard basis for $\mathbb{Z}^n$, and set
$$\mathcal{A} \;:=\; \{\mathbf{e}_{i_1}+\mathbf{e}_{i_2}+\cdots+\mathbf{e}_{i_k} \mid 1\leq i_1<i_2<\cdots<i_k\leq n\}\,.$$
   - Determine $I_{\mathcal{A}}$.
   - Describe the lattices N and M associated to the toric variety $X_{\mathcal{A}}$.
   - What is $\dim X_{\mathcal{A}}$?

3. For each of the following monoids $S\subset\mathbb{Z}^2$, describe spec$(\mathbb{K}[S])$.
   - $S=\mathbb{N}^2$.
   - $S=\mathbb{N}\times\mathbb{Z} \;=\; \{(a,b)\mid a\geq 0\}$.
   - $S:=\;\{(a,b)\mid a\geq |b|\}$.

4. Let X be an affine variety equipped with the action of a torus $\mathbb{T}_N$, and let
$$\mathbb{K}[X] =: A \;=\; \bigoplus_{u\in S} A_u$$
be the decomposition of its coordinate ring into isotypical pieces. Here $S\subset M$ is a submonoid and A_u is that piece of A on which $\mathbb{T}_N$ acts via the character $u\in M$.

   Suppose that in addition we have that

   (a) X is normal,

   (b) $\mathbb{T}_N$ acts faithfully on X (generically there are no stabilizers),

   (c) and $\mathbb{T}_N$ as a dense orbit in X.

   Then the grading monoid S is saturated (it has the form $\tau^\vee$), and the dimension of each graded piece A_u of A for $u\in S$ equals 1. (Thus $X=U_\tau=\text{spec}\,\mathbb{K}[S]$.)

5. (a) Using the definition $\mathbb{T}_N(K) := \text{Hom}_{\text{monoid}}(M, \mathbb{K}^\times)$, show that the algebra maps

$$\begin{array}{rccl} \Delta : & \mathbb{K}[M] \to \mathbb{K}[M] \otimes \mathbb{K}[M] & \text{where} & \chi^\alpha \mapsto \chi^\alpha \otimes \chi^\alpha \\ \epsilon : & \mathbb{K}[M] \to \mathbb{K} & & \chi^\alpha \mapsto 1 \\ \Delta : & \mathbb{K}[M] \to \mathbb{K}[M] & & \chi^\alpha \mapsto \chi^{-\alpha} \end{array}$$

dualize (under spec) to induce the usual group structure on $\mathbb{T}_N(\mathbb{K})$ with Δ giving the multiplication, ϵ the unit element, and S the inverse. (Together with the usual multiplication and identity of the algebra $\mathbb{K}[M]$ these maps give it the structure of a finitely generated (over $\mathbb{K}$) commutative Hopf algebra.)

(b) Do the same for the action of $\mathbb{T}_N(\mathbb{K})$ on the toric variety

$$X \;:=\; \text{spec}\mathbb{K}[S] \;=\; \text{Hom}_{\text{monoid}}(S, \mathbb{K})\,.$$

Here, the relevant map is $\rho\colon \mathbb{K}[S] \mapsto \mathbb{K}[M] \otimes \mathbb{K}[S]$ where $\rho(\chi^\alpha) = \chi^\alpha \otimes \chi^\alpha$.

6. Consider the cone $\tau := \text{cone}(\mathbf{e}_i + \mathbf{e}_j \mid 1 \le i < j \le 4)$ of Exercise 3.

- Describe the faces σ of τ.
- For each face σ of τ, determine $\sigma^\perp$, $\sigma^\vee$, and σ^*.

The symmetry of this problem implies that this is less work than it may seem at first.

7. Try your hand at constructing toric varieties associated to the following fans in $\mathbb{Z}^2$.