Algebra Autumn 2023

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Twelfth Homework

Write your answers neatly, in complete sentences. I highly recommend recopying your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Hand in for the grader Monday 27 November:

61. Let S be a multiplicative subset of an integral domain R with $0 \notin S$. Show that if R is a principal ideal domain, then so is $R[S^{-1}]$.

    Show that if R is a unique factorization domain, then so is $R[S^{-1}]$.

62. Let R be an integral domain, and for each maximal ideal $\mathfrak{m}$ of R, show that the localization $R_\mathfrak{m}$ is a subring of the quotient field of R.

63. Continuing the previous problem, show that the intersection of the rings $R_\mathfrak{m}$, as $\mathfrak{m}$ ranges over all maximal ideals of R, is R itself.

64. Show that the equation $x^2+1=0$ has infinitely many solutions in Hamilton's Quaternions, $\mathbb{H}$, which is $\mathbb{R} \oplus i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R}$, where $ij=k$, $ji=-k$, etc. These are defined in the Example on page 117 of my copy of Hungerford in Section III.1.

65. Let R be a ring and G be an infinite multiplicative cyclic group with generator ξ. Prove or disprove: The group ring $R[G]$ is isomorphic to the polynomial ring $R[x]$ in one indeterminate x.

66. Show that the polynomial $x+1$ is a unit in the power series ring $\mathbb{Z}[[x]]$, but not in the polynomial ring $\mathbb{Z}[x]$.

    Show that the polynomial x^2+3x+2 is irreducible in $\mathbb{Z}[[x]]$, but not in $\mathbb{Z}[x]$.

67. (a) If D is an integral domain and c is an irreducible element in D, show that $D[x]$ is not a principal ideal domain. (Hint: consider the ideal generated by x and c.)

    (b) Show that $\mathbb{Z}[x]$ is not a principal ideal domain.

    (c) If $\mathbb{F}$ is a field and $n \geq 2$, show that $\mathbb{F}[x_1, \ldots, x_n]$ is not a principal ideal domain. (Hint: show that x_1 is irreducible in $\mathbb{F}[x_1, \ldots, x_{n-1}]$.)