Algebra Autumn 2023
Frank Sottile
8 November 2023

Eleventh Homework

Write your answers neatly, in complete sentences. I highly recommend recopying your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Hand in to Frank Wednesday November 15: This is part of your test.

52. Let $R \subset \mathbb{C}$ be set whose elements are

$$R \ := \ \{a + b\left(\tfrac{1+\sqrt{-19}}{2}\right) \ \mid \ a, b \in \mathbb{Z}\}\,.$$

Show that R is a subring of $\mathbb{C}$ that is a principal ideal domain but not a Euclidean domain.

Hand in for the grader Friday 17 November:

53. The *center* $Z(R)$ of a ring R is the set of elements that (multiplicatively) commute with every element of R.

    Let G be a finite group. Show that the center of the group algebra $\mathbb{C}[G]$ has dimension as a $\mathbb{C}$-vector space equal to the number of conjugacy classes of G. (The expected answer is a huge hint for a possible basis for $Z(\mathbb{C}[G])$.)

54. Determine all the prime and maximal ideals in the ring $\mathbb{Z}/m\mathbb{Z}$, where $m \in \mathbb{N}$ is an integer greater than 1.

55. Show that a nonzero ideal in a principal ideal domain is maximal if and only if it is a prime ideal.

56. Let $\omega := e^{2\pi\sqrt{-1}/3} = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$. For $a + b\omega \in \mathbb{Z}[\omega]$, set $N(a + b\omega) := a^2 - ab + b^2$. Show that $\mathbb{Z}[\omega]$ is a unique factorization domain.

57. Show that the ring $\mathbb{Z}[\sqrt{10}]$ is not a UFD.

58. If $S = \{2, 4\}$ and $R = \mathbb{Z}/6\mathbb{Z}$, show that $R[S^{-1}] \simeq \mathbb{Z}/3\mathbb{Z}$.

59. Let $p \in \mathbb{Z}$ be a prime number, so that (p) is a prime ideal. What can be said about the relation between the quotient ring $\mathbb{Z}_p$ and the localisation $\mathbb{Z}_{(p)}$?

60. Show that a commutative ring R is local if and only if for all $r, s \in R$, if $r + s = 1$, then either r or s is a unit.