Algebra Autumn 2023
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25 September 2023 Sixth Homework

Write your answers neatly, in complete sentences. I highly recommend recopying your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Hand in for the grader Monday 2 October:

22. Recall the definition of a (finitely generated) reflection or *Coxeter* group G of rank $n \in \mathbb{N}$. Let $S = \{s_i \mid i \in [n]\}$ be an n element set and $M = (m_{i,j})$ a symmetric matrix of size $n \times n$ whose entries are positive integers and whose diagonal entries are all 1. Let G be the group generated by S with relations $\{(s_i s_j)^{m_{i,j}} = e \mid i, j \in [n]\}$.

    Prove that the map $s_i \mapsto -1$ induces a group homomorphism $\text{sgn}\colon G \to \{\pm 1\}$, and conclude that G has a subgroup of index 2. Is this subgroup normal?

23. Prove that the operation $*$ of free product is commutative and associative; If G, H, and K are groups, then $G * H \simeq H * G$, and $(G * H) * K \simeq G * (H * K)$.

24. A subset X of an abelian group F is *linearly independent* if $n_1x_1 + n_2x_2 + \cdots + n_kx_k = 0$ implies that $n_i = 0$ for all i, where $n_i \in \mathbb{Z}$ and $x_1, \dots, x_k$ are distinct elements of X.

    (a) Show that X is linearly independent if and only if every nonzero element of the subgroup $\langle X \rangle$ it generates may be written uniquely in the form $n_1x_1 + \cdots + n_kx_k$, where $n_i \in \mathbb{Z}$ and $x_1, \dots, x_k$ are distinct elements of X.

    (b) Prove or give a counterexample to the following statement:

        If F is free abelian of finite rank n, then every linearly independent subset of n elements is a basis.

    (c) Prove or give a counterexample to the following statement:

        If F is free abelian, then every linearly independent subset of F may be extended to a basis of F.

    (d) Prove or give a counterexample to the following statement:

        If F is free abelian, then every generating set of F contains a basis of F.

25. Prove that the additive group of the rational numbers $\mathbb{Q}$ is not a free abelian group.

26. Prove that the multiplicative group $\mathbb{Q}^\times$ of the nonzero rational numbers is a free abelian group.