Algebra Autumn 2023
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11 September 2023 Fourth Homework

Write your answers neatly, in complete sentences. I highly recommend recopying your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Hand in for the grader Monday 18 September:

16. Show that A_4 has no subgroup of order 6. Does the converse to Lagrage's Theorem hold?
17. Let D_{2n} be the dihedral group of order $2n$ (Hungerford writes this as D_n). Determine all of its normal subgroups.
18. Let G be a group and let $C(G) := \{g \in G \mid gh = hg \text{ for all } h \in G\}$ be its *center*.
    (a) Show that $C(G)$ is a normal subgroup of G.
    (b) Prove that if $G/C(G)$ is cyclic, then G is abelian.
    (c) Let p be a prime number. Prove that any group of order p^2 is abelian.
19. Let G be a finite group of order n and let $\varphi\colon G \hookrightarrow S_n$ be the right regular representation of G on itself (the Cayley embedding). Find necessary and sufficient conditions on G so that its image under φ is a subgroup of the alternating group, A_n.
20. Suppose that G and K are finite groups with respective normal subgroups $H \triangleleft G$ and $L \triangleleft K$. Give examples showing that each of the following statements do not hold for all groups.
    (a) $G \simeq K$ and $H \simeq L$ implies that $G/H \simeq K/L$.
    (b) $G \simeq K$ and $G/H \simeq K/L$ implies that $H \simeq L$.
    (c) $G/H \simeq K/L$ and $H \simeq L$ implies that $G \simeq K$.