Algebra Autumn 2023
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21 August 2023

First Homework

Write your answers neatly, in complete sentences. I highly recommend recopying your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Hand in to Frank Monday 28 August: (Have this on a separate sheet of paper.)

1. Let α be a rotation about a point p in the plane and ρ be a reflection in a line through p. What is α^{-1}, geometrically? Show that $\rho\alpha\rho = \alpha^{-1}$. Let σ also be a reflection in a line. What is $\rho\sigma$? (There are several cases to consider.)

Hand in for the grader Monday 28 August: (Have this separate from #1.)

A *subgroup* of a group G is a subset H of G which is a group in its own right, under the group operations of G. For example, the set $2\mathbb{Z}$ of even integers is a subgroup of the additive group of integers.

2. Show that a finite group G of even order (G has an even number of elements) has a non-identity element a with $a^2 = e$.
3. Let $GL(2,\mathbb{Z})$ be the collection of 2×2 matrices with integer entries and determinant ± 1. This is a group under multiplication of matrices, with identity $I = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. Let $A := \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ and $B := \left(\begin{smallmatrix}0&1\\-1&1\end{smallmatrix}\right)$. What is the order of A ? Of B ? Of AB?
4. Suppose that G is a group in which the square of every element is the identity. Prove that G is abelian.
5. Let p be a prime number. Consider the following subsets of the rational numbers, $\mathbb{Q}$:

$$\mathbb{Q}[\tfrac{1}{p}] := \{\tfrac{a}{b} \in \mathbb{Q} \mid a, b \in \mathbb{Z}\,,\ \gcd(a,b) = 1\,,\ \text{and } b = p^i \text{ some } i \geq 0\}$$
$$\mathbb{Q}^{(p)} := \{\tfrac{a}{b} \in \mathbb{Q} \mid a, b \in \mathbb{Z}\,,\ \gcd(a,b) = 1\,,\ \text{and } p \nmid b\}$$

Prove that these are subgroups of $\mathbb{Q}$ under addition.

(We intend that $\gcd(0,1) = 1$.)