Honors Multivariate Calculus — Math 221H Section 201

Thirteenth Homework: Due in recitation: Thursday 30 November 2023

1. Use Green's Theorem to evaluate $\int_C 2xy\,dx + x^2\,dy$, where C is the cardioid curve defined by $r = 1 - \sin(\theta)$.

2. Evaluate the integral $\displaystyle\iint_D (3xy - 4x^2y)\,dA$ where D is the unit disc directly and using Green's theorem.

3. Compute the curl $\nabla\times$ and divergence $\nabla\cdot$ of the following vector fields.
   (a) $\mathbf{F}(x,y,z) = \sin x\mathbf{i} + \cos x\mathbf{j} + z^2\mathbf{k}$.
   (b) $\mathbf{F}(x,y,z) = e^{xyz}\mathbf{i} + \sin(x-y)\mathbf{j} - \frac{xy}{2}\mathbf{k}$.

4. Which of the following vector fields on $\mathbb{R}^3$ are conservative.
   (a) $\mathbf{F}(x,y,z) = z\mathbf{i} + 2yz\mathbf{j} + (x^2+y^2)\mathbf{k}$. (b) $\mathbf{F}(x,y,z) = x\mathbf{i} + e^y\sin z\mathbf{j} + e^y\cos z\mathbf{k}$.

5. Suppose that $\mathbf{F}$ and $\mathbf{G}$ are vector fields whose components have continuous second partial derivatives. Prove the identities.
   (a) $\nabla\cdot(\mathbf{F}+\mathbf{G}) = \nabla\cdot\mathbf{F} + \nabla\cdot\mathbf{G}$.
   (b) $\nabla(\mathbf{F}\cdot\mathbf{G}) = (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} + \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F})$.

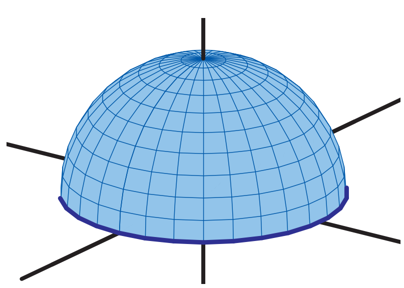
   The operator $\nabla\cdot\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the *Laplacian* and often written Δ.

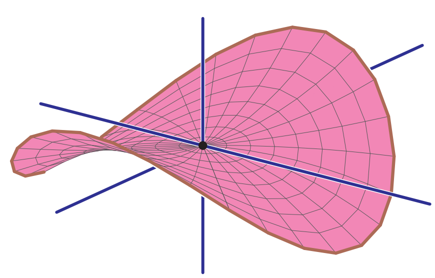
6. Use the normal form of Green's Theorem to deduce *Green's First identity*:

$$\iint_D f\,\nabla^2 g\,dA \;=\; \oint_C f(\nabla g)\cdot\mathbf{n}\,ds \;-\; \iint_D \nabla f\cdot\nabla g\,dA\,.$$

   Here, $D\subset\mathbb{R}^2$ is a domain with piecewise smooth positively oriented boundary $C=\partial D$ (C,D satisfy the hypotheses of Green's Theorem).

7. Evaluate the surface integrals
   (a) $\displaystyle\iint_S xz\,dS$, where S is the triangle with vertices $(1,0,0)$, $(0,2,0)$, $(0,0,3)$.
   (b) $\displaystyle\iint_S (x^2z + y^2z)\,dS$, where S is the hemisphere $x^2+y^2+z^2=3$ and $z\geq 0$.


(b) Hemisphere


(c) Surface

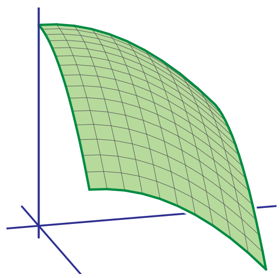
   (c) $\displaystyle\iint_S xy\,dS$, where S is the surface with parametrization $x=u+v$, $y=u-v$, $z=uv$, and $u^2+v^2\leq 1$. What surface is this?

8. Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, for the given vector field $\mathbf{F}$ and surface S.

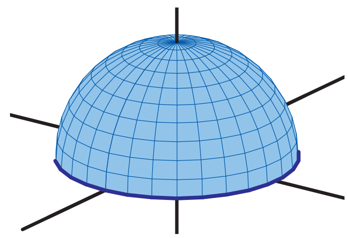
(a) $\mathbf{F}(x, y, z) = xy\mathbf{i} - 2x^2y^2\mathbf{j} + yz\mathbf{k}$, where S is that part of the paraboloid $z = 16 - x^2 - 2y^2$ lying above the rectangle $0 \leq x \leq 3$ and $0 \leq y \leq 2$, oriented upward.

(b) $\mathbf{F}(x, y, z) = -y\mathbf{i} + 2x\mathbf{j} + 3z\mathbf{k}$, where S is the upper hemisphere of the sphere of radius 4, oriented upward.

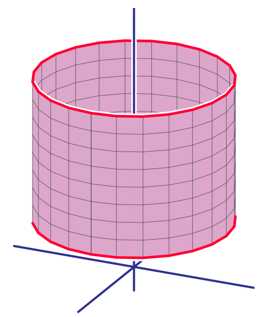
(c) $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j} - e^{xz}\mathbf{k}$, where S is that part of the cylinder $x^2 + y^2 = 4$ where $1 \leq z \leq 4$, and $\mathbf{n}$ is pointing outwards.



(a) Part of Paraboloid



(b) Hemisphere



(c) Cylinder