Honors Multivariate Calculus   Math 221H Section 201

Sixth Homework: Due in recitation: Thursday 28 September 2023

Homework about the chain rule and directional derivatives

1. The length ℓ, width w, and height h of a box change with time. At some time, we have $\ell = 1$ and $w = h = 2$ (all in metres), and ℓ and w are decreasing at a rate of 1 m/s, while h is increasing at a rate of 3 m/s. At this point in time, find the rates at which the following quantities are changing.

   (a) The volume. (b) The surface area. (c) The main diagonal.

2. Write out the chain rule for the partial derivatives of u with respect to x, y, z, and w, if

$$u = f(s,t)\,, \qquad s = s(w,x,y,z)\,, \quad \text{and} \quad t = t(w,x,y,z)\,.$$

3. If $z = f(x, y)$, where $x = s + t$ and $y = s - t$, show that $\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \frac{\partial z}{\partial s}\frac{\partial z}{\partial t}$.

4. Suppose that $f(x, y) = \sin(x + 2y)$. Find the directional derivative of f at the point $(x, y) = (4, -2)$ in the direction of the angle $\theta = -2\pi/3$.

5. Suppose that $f(x, y, z) = x^3y^2z$. Find the gradient of f, evaluate it at the point $(1, -2, 1)$, and find the rate of change of f at this point in the direction of $\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$.

6. Suppose that $f(x, y, z) = x^3 - y^2z$. Find the directional derivative $D_u f(2, 1, 6)$ where u has the same direction as $\langle 12, 3, 4\rangle$.

7. Find the maximum rate of change of the function at the given point, and the direction at which it occurs.

$$f(x,y) = \ln(x^2 + 2y^2) \text{ at } (x,y) = (3,4) \qquad f(x,y,z) = \tfrac{x}{y} + \tfrac{y}{z} \text{ at } (x,y,z) = (9,3,1)\,.$$

8. Suppose that the electric potential V in a region of space has the formula $xyz + 5y^2 - 3xy$. What is the rate of change of the potential at the point $(x, y, z) = (4, 3, 5)$ in the direction of $\mathbf{i} + \mathbf{j} - \mathbf{k}$?

   In what direction does V change most rapidly at this point?

   What is the maximum rate of change at this point?

9. Find equations of the tangent plane and the normal line to the implicit surface(s) at the given point(s).

   (a) $x^2 - 2y^2 + z^2 = 3$ at $(x, y, z) = (-1, 1, 2)$ (b) $xe^{yz} = 1$ at $(x, y, z) = (1, 0, 2)\,.$

10. If $f(x, y) = 2x^3 - 3xy + y^2$ find the gradient vector $\nabla f(1, 3)$ and use it to give an equation for the tangent line to the curve $f(x, y) = 2$ at the point $(x, y) = (1, 3)$.

11. Show that the tangent plane to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ at the point (x_0, y_0, z_0) is given by the equation $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1$.

12. Show that the sum of the x-, y-, and z-intercepts of any tangent plane to the surface $\sqrt{x}+\sqrt{y}+\sqrt{z} = a$ (a is a fixed positive real number) is a constant.

13. Find the points on the ellipsoid $3x^2 + 2y^2 + z^2 = 1$ where the tangent plane is parallel to the plane $3x + y - 3z = 9$.

14. Find the points on the graph of the function $z = f(x, y) = \frac{(x+y+1)^2}{x^2+y^2+1}$ where the tangent plane is horizontal.