Hyperbolic Angle Sum Formula

Find $\sinh(x+y)$ and $\cosh(x+y)$ in terms of $\sinh x$, $\cosh x$, $\sinh y$ and $\cosh y$.

Solution

$\sinh(x+y)$

Recall that:

$$\sinh(u) = \frac{e^u - e^{-u}}{2} \quad \text{and} \quad \cosh(u) = \frac{e^u + e^{-u}}{2}.$$

The easiest way to approach this problem might be to guess that the hyperbolic trig. angle sum formulas will be similar to those from regular trigonometry, then adjust those formulas to fit.

To proceed without consulting the angle sum formulas, we start by rewriting $\sinh(x+y)$ in terms of e^x and e^y and then attempt to separate the terms. A first attempt might look like:

$$\begin{aligned}
\sinh(x+y) &= \frac{e^{x+y} - e^{-x-y}}{2} \\
&= \frac{1}{2}(e^{x+y} - e^{-x-y}) \\
&= \frac{1}{2}(e^{x+y} - e^{-x-y} + e^{y-x} - e^{x-y} - e^{y-x} + e^{x-y})
\end{aligned}$$

Using the fact that:

$$(e^a + e^{-a})(e^b - e^{-b}) = e^{a+b} + e^{b-a} - e^{a-b} + e^{-a-b}$$

we could write $\sinh(x+y)$ as $2\cosh x \sinh y$ plus some left over stuff. However, we would then have to do something with the left over stuff, and the coefficient of 2 on $2\cosh x \sinh y$ seems out of place. Let's try again:

$$\begin{aligned}
\sinh(x+y) &= \frac{e^{x+y} - e^{-x-y}}{2} \\
&= \frac{1}{4}(2e^{x+y} - 2e^{-x-y}) \\
&= \frac{1}{4}(e^{x+y} - e^{-x-y} + e^{y-x} - e^{x-y} + e^{x+y} - e^{-x-y} - e^{y-x} + e^{x-y}) \\
&= \frac{1}{4}(e^{x+y} - e^{-x-y} + e^{y-x} - e^{x-y}) + \frac{1}{4}(e^{x+y} - e^{-x-y} - e^{y-x} + e^{x-y}) \\
&= \frac{1}{4}(e^x + e^{-x})(e^y - e^{-y}) + \frac{1}{4}(e^x - e^{-x})(e^y + e^{-y}) \\
&= \frac{e^x + e^{-x}}{2}\frac{e^y - e^{-y}}{2} + \frac{e^x - e^{-x}}{2}\frac{e^y + e^{-y}}{2} \\
\sinh(x+y) &= \cosh x \sinh y + \sinh x \cosh y
\end{aligned}$$

We know that $\sin(x+y) = \sin x \cos y + \cos x \sin y$, so this seems like a reasonable answer. It's a straightforward exercise to check this result using our formulas for $\sinh u$ and $\cosh u$.

$\cosh(x+y)$

We proceed as before, without the false start.

$$\begin{aligned}
\cosh(x+y) &= \frac{e^{x+y}+e^{-x-y}}{2} \\
&= \frac{1}{2}(e^{x+y}+e^{-x-y}) \\
&= \frac{1}{4}(e^{x+y}+e^{-x-y}+e^{y-x}+e^{x-y}+e^{x+y}+e^{-x-y}-e^{y-x}-e^{x-y}) \\
&= \frac{1}{4}(e^{x+y}+e^{-x-y}+e^{y-x}+e^{x-y})+\frac{1}{4}(e^{x+y}+e^{-x-y}-e^{y-x}-e^{x-y}) \\
&= \frac{1}{4}(e^{x}+e^{-x})(e^{y}+e^{-y})+\frac{1}{4}(e^{x}-e^{-x})(e^{y}-e^{-y}) \\
\cosh(x+y) &= \cosh x\cosh y+\sinh x\sinh y
\end{aligned}$$

We know that $\cos(x+y) = \cos x \cos y - \sin x \sin y$, so again this is a plausible result.