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Answers to Second Test

1. [10] Determine all possible rational canonical forms of a linear transformation on a six-dimensional $\mathbb{Q}$-vector space with characteristic polynomial $x^2(x^2+1)^2$.

   Let T be a linear transformation on $\mathbb{Q}^6$ have characteristic polynomial $\chi(x) := x^2(x^2+1)^2$. The rational canonical form of T will be a block-diagonal matrix where the ith block is the companion matrix of a divisor d_i of χ such that $d_i|d_{i+1}$ and χ is the product of the d_i. These correspond to the decomposition of the $\mathbb{Q}[t]$-module $\mathbb{Q}^6$ with t acting via T.

   There are the four possible factorizations: $d_1 := x^2(x^2+1)^2$, $d_1 := x,\ d_2 := x(x^2+1)^2$, $d_1 := (x^2+1),\ d_2 := x^2(x^2+1)$, and $d_1 := x(x^2+1),\ d_2 := x(x^2+1)$. These give four possible rational canonical forms:

$$\begin{pmatrix}0&0&0&0&0&0\\1&0&0&0&0&0\\0&1&0&0&0&-1\\0&0&1&0&0&0\\0&0&0&1&0&-2\\0&0&0&0&1&0\end{pmatrix}\quad\begin{pmatrix}0&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&0&0&0&0&0\\\mathbf{0}&1&0&0&0&-1\\\mathbf{0}&0&1&0&0&0\\\mathbf{0}&0&0&1&0&-2\\\mathbf{0}&0&0&0&1&0\end{pmatrix}\quad\begin{pmatrix}0&-1&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\1&0&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&0&0&0&0\\\mathbf{0}&\mathbf{0}&1&0&0&0\\\mathbf{0}&\mathbf{0}&0&1&0&-1\\\mathbf{0}&\mathbf{0}&0&0&1&0\end{pmatrix}\quad\begin{pmatrix}0&0&0&\mathbf{0}&\mathbf{0}&\mathbf{0}\\1&0&-1&\mathbf{0}&\mathbf{0}&\mathbf{0}\\0&1&0&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&0&0&0\\\mathbf{0}&\mathbf{0}&\mathbf{0}&1&0&-1\\\mathbf{0}&\mathbf{0}&\mathbf{0}&0&1&0\end{pmatrix}$$

2. [6] Let F/K be a field extension. Please give proper (complete sentences, etc.) definitions of the following concepts.

   $[F\colon K]$.

   The *index*, $[F\colon K]$ of K in F, is the dimension of F as a vector space over K. This is also called the *degree* of the extension.

   F/K is a Galois extension.

   The field extension F/K is a *Galois extension* if the group Gal(F/K) of automorphisms of the field F that fixes elements of K has the property that K is its fixed field: $u \in F \smallsetminus K \Rightarrow \exists\sigma \in \mathrm{Gal}(F/K)$ with $\sigma(u) \neq u$.

3. [6] Suppose that the characteristic of a field K is not 2 and F/K is a field extension of degree two. Show that there is an element $\alpha \in F \smallsetminus K$ with $\alpha^2 \in K$, and that $F = K(\alpha)$.

   First note that if $\alpha \in F \smallsetminus K$, then $F = K + K\alpha \subset K(\alpha) \subset F$, as $[F\colon K] = 2$.

   Let $\beta \in F \smallsetminus K$. As $\beta^2 \in F = K(\beta)$, there exist $c, d \in K$ (not both zero) with $\beta^2 = b\beta + c$. Set $\alpha := \beta - b/2$. Then

$$\alpha \notin K \qquad \text{and} \qquad \alpha^2 \;=\; \beta^2 - b\beta + b^2/4 \;=\; c + b^2/4 \;\in\; K\,.$$

4. [8] Let F/K be a finite Galois extension and E an intermediate field of this extension. A *Galois conjugate* of E is its image $\sigma(E)$ under any element $\sigma \in \mathrm{Gal}(F/K)$. Show that the compositum of all the Galois conjugates of E is Galois over K and is the smallest intermediate field M that contains E and is Galois over K. (This compositum is the subfield generated by all Galois conjugates of E.)

   The compositum M is the smallest intermediate field that contains $\{\sigma E \mid \sigma \in \mathrm{Gal}(F/K)\}$, and is therefore stable. The problem is completed by noting that stable intermediate fields of a finite Galois extension are Galois.