Math 300

Problems from Tests III Fall 2020

1. Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be defined by

$$f(n) \;=\; \begin{cases} 2n+4 & \text{if } n \text{ is even} \\ 2n-4 & \text{if } n \text{ is odd} \end{cases} .$$

   (a) What is $f[\{1,2,3,4,5,6\}]$?

   (b) What is $f^{-1}[\{1,2,3,4,5,6,7,8,9,10\}]$?

   (c) What is the range of f? Justify your answer.

   (d) Is f injective? Justify your answer.

2. Let A and B be sets and X and Y be subsets of A. Suppose that $f\colon A \to B$ is an injective function. Prove that $f(X) - f(Y) = f(X - Y)$.

3. Let A, B and C be sets and suppose that $f\colon A \to B$ and $g\colon B \to C$ are functions. Prove that if $g \circ f$ is surjective, then g is surjective.

4. Let X, Y, Z be sets, and let $f : X \to Y$ and $g : Y \to Z$ be functions.

   (a) Prove that if $g \circ f$ is one-to-one, then f is one-to-one.

   (b) Give an example of functions f and g for which $g\circ f$ is one-to-one and g is not one-to-one.

5. Let A and B be sets and suppose that $f\colon A \to B$ is a function. Give definitions for the following notions. (I am looking for answers that are complete sentences and mathematically precise.)

   (a) The *image, $f(X)$* of X under f, where X is a subset of A.

   (b) The *inverse image, $f^{-1}(Y)$* of Y under f, where Y is a subset of B.

   (c) The function f is *one to one*.

6. Let $\mathbb{N} := \{0, 1, \dots\}$ be the nonnegative integers. Define a relation R on $\mathbb{N} \times \mathbb{N}$ by

$$(a,b)R(c,d) \;\text{ if and only if }\; a + d \;=\; b + c\,.$$

   Prove or disprove: R is an equivalence relation.

7. Let $a, b \in \mathbb{Z}$ be integers. Prove that $(a,b) = 1 \implies (ab, a+b) = 1$. (Here, (x,y) is the greatest common divisor of the integers x and y.)

8. A congruence class $[a]_9$ is *invertible modulo 9* if there is a $[b]_9$ such that $ab \equiv 1 \mod 9$ (equivalently, if $[a]_9 *_9 [b]_9 = [ab]_9 = [1]_9$.) Which congruence classes modulo 9 are invertible modulo 9? For each, what is its inverse modulo 9?

   (The congruence classes modulo 9 are $\{[0],[1],[2],[3],[4],[5],[6],[7],[8]\}$. Note that I dropped the $_9$ from these, writing $[2]$ for $[2]_9$. It would be fine were you to drop the $[]$ in your answer/calculations, if you wanted.)