Math 300/220
Problems from Tests II Fall 2020

1. Prove the following statement: For all real numbers a, b, if $a + b$ is irrational, then either a is irrational or b is irrational.

2. Let $x \in \mathbb{R}$ and assume that for all $\epsilon > 0$, $|x| < \epsilon$. Prove that $x = 0$. (Hint: Prove the implication by contradiction, suppose that $x \neq 0$ and find a specific $\epsilon > 0$ that contradicts the hypothesis.)

3. Let A and B be sets. Give the formal definition of the complement of B in A.

   Suppose that A is the set of positive even integers and B is the set of prime numbers. Describe the complement of B in A.

4. Let A be the set $\Big\{ \emptyset, \{\emptyset\}, \{ \emptyset, \{\{\emptyset\}\} \} \Big\}$.

   (a) What is $|A|$? (b) What is $|\mathcal{P}(A)|$? (c) What is $|\mathcal{P}(A) - A|$?

   (d) Is $\{\emptyset, \{\emptyset\}\} \in A$? (e) Is $\{\emptyset, \{\emptyset\}\} \in \mathcal{P}(A)$?

5. Let A and B be sets.

   (a) Give the formal definition of subset. That is, complete the sentence: "We say that A is a subset of B, written $A \subseteq B$, if..."

   (b) Write the condition $A \subseteq B$ as a statement involving a universal quantifier. You may assume that there is a universal set U containing both A and B.

6. For a set A, let $\mathbf{P}(A)$ denote its power set, the set of all subsets of A. Suppose that $A := \{\square, \bigcirc, \triangle, \spadesuit\}$. Which of the following are true and which are false.

   (a) $\{\triangle\} \subseteq \mathbf{P}(A)$.

   (b) $\emptyset \subseteq \mathbf{P}(A)$.

   (c) $\emptyset \in \mathbf{P}(A)$.

   (d) $\{\bigcirc, \square\} \in \mathbf{P}(A)$.

   (e) $\{\emptyset\} \in \mathbf{P}(A)$.

   (f) $\{\emptyset\} \subseteq \mathbf{P}(A)$.

   (g) $\{\{\square\}, \{\triangle, \spadesuit\}\} \subseteq \mathbf{P}(A)$.

7. Let A and B be sets. Prove, using the definition of subset and union, that $A \subseteq A \cup B$.

8. Let A, B, and C be subsets of a universal set U.
   Give the definitions for (a) $A \cup B$ and for (b) $A \subset C$.
   Prove that for all sets A and B, we have $A \subset A \cup B$.

9. Let A and B be subsets of a universal set U. Prove that $(A \cup B) \cap A^c = B - A$.

10. Let A be a set. What is/are:

    (a) $A \cup \emptyset$.

    (b) $A \cap \emptyset$.

    (c) $A - \emptyset$.

    (d) $A \times \emptyset$.

    (e) $\mathcal{P}(\emptyset)$. ($\mathcal{P}(\ )$ is power set.)

    No proofs are necessary.

11. Suppose that $A := \{2, 3, 5, 7\}$ and $B := \{3, 4, 7\}$. Find the following sets:

    (a) $A \cap B$.

    (b) $A \cup B$.

    (c) $A \times B$.

    (d) $A - B$.

    (e) $B - A$.

    (f) $\mathbf{P}(A \cap B)$, the power set of the intersection of A and B.

12. Let A and B be subsets of a given universal set U. Prove the de Morgan law that

$$(A \cap B)^c \;=\; A^c \cup B^c\,.$$

13. Let A and B be sets. Prove, using the definition of subset and intersection, that $A \cap B \subseteq A$.

14. Let A and B be sets with $A \subset B$. Prove that the power set of A is a subset of the power set of B, that is, prove that $\mathcal{P}(A) \subset \mathcal{P}(B)$.

15. *(True/False/Counterexample.)* For each statement, determine whether it is true or false, and accordingly write "T" or "F" in the blank. If the statement is false, provide a counterexample. (No need to prove true statements.)

_______ For all sets A and B, $A \subseteq B$ if and only if $B \cap A = A$.

_______ For all subsets A and B of a universal set U, $A^c \subseteq B^c$ if and only if $A \subseteq B$.

_______ For all sets A, B, and C, $A \cup (B \cap C) = (A \cup B) \cap C$.

_______ Each set consisting of three elements has exactly eight subsets.

_______ For all integers n, $(-\infty, n] \cup [-n, \infty) = \mathbb{R}$.

16. State the Principal of Mathematical Induction. It should begin: "For each positive integer n, let $P(n)$ be a statement."

17. Prove by mathematical induction that for each positive integer n,

$$1 + 3 + 5 + \cdots + (2n - 1) \;=\; n^2\,.$$

18. Use induction to prove that, for all positive integers n, we have

$$1+2+3+\cdots+n \;=\; \sum_{i=1}^{n} i \;=\; \frac{n(n+1)}{2}\;.$$

19. The *Fibonnaci numbers*, $\{f_n \mid n \geq 1\}$ are defined by $f_1 = f_2 = 1$, and for $n \geq 2$ $f_{n+1} = f_n + f_{n-1}$. Prove that for all $n \geq 1$, $f_1^2 + f_2^2 + f_3^2 + \cdots + f_n^2 = f_n f_{n+1}$.

20. Let $a_1 = 1$, $a_2 = 7$, and $a_{n+1} = 7a_n - 12a_{n-1}$ for all positive integers $n \geq 2$. Prove that for all positive integers n, $a_n = 4^n - 3^n$.

21. Prove the following formula using the Principle of Mathematical Induction.

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} \;=\; \frac{n}{n+1}\;.$$

(The sum is $\frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \cdots + \frac{1}{n(n+1)}$.)

22. Consider the open sentence $P(n)$: $9 + 13 + \cdots + (4n+5) = \frac{4n^2+14n+1}{2}$.

    (a) Verify the implication $\forall k \in \mathbb{N},\; P(k) \Rightarrow P(k+1)$.

    (b) Does it follow that $P(n)$ is true for all $n \in \mathbb{N}$?

23. Let $\mathbf{E}$ and $\mathbf{O}$ denote the sets of even and odd integers, respectively. Prove that $\mathscr{P} := \{\mathbf{E}, \mathbf{O}\}$ is a partition of the integers $\mathbf{Z}$.

24. Let A be a nonempty set. Give the definition of a partition of A. Give a partition of the set $A := \{\square, \bigcirc, \triangle, \Diamond\}$.

25. For each of the following, give its definition as used in our course. Suppose that A and B are sets, that x is a real number, and that a, b, m are integers with $m > 1$.

    (a) $A - B$.

    (b) $|x|$, the absolute value of x.

    (c) a divides b.

    (d) a is congruent to b modulo m.