Math 300/220
Problems from Tests

1. Give the formal definition of a contradiction. Give a simple contradiction involving statement forms.

2. Give the formal definition of tautology. Give a simple tautology involving statement forms.

3. Consider the following statement:
"If it rains on Saturday, I will eat ice cream or read *Moby Dick*."
Please give (a) its converse, (b) its contrapositive, and (c) a useful negation.

4. Let $P(x)$ and $Q(x)$ be predicates, with x taking values in some universe U. Are the two statements
$$(\exists x \in U)\,(P(x) \wedge Q(x)) \qquad \text{and} \qquad ((\exists y \in U)P(y)) \wedge ((\exists z \in U)Q(z))$$
logically equivalent? If so, provide a proof. If not, then given an example of predicates $P(x)$ and $Q(x)$ for which they are not logically equivalent, together with a convincing demonstration.

5. Your roommate, who is taking Calculus I with an old-school teacher, has been asked to negate the following mathematical sentence in a useful manner:
$$\forall \epsilon > 0 \left(\exists N \in \mathbb{R} \left(\forall x \in \mathbb{R}, \text{ if } x > N, \text{ then } |f(x) - \ell| < \epsilon \right) \right).$$
This is about a function f and a real number ℓ, and the other symbols are also real numbers. Please help him do his homework.
Bonus 5 points: What is this statement, besides logical gobbledygook?

6. Consider the statement P: "The sum of two even integers is divisible by 4".
   (a) Write P as a statement of the form: "some quantifier. . . , if . . . , then . . . ."
   (b) Write $\neg P$ in this form.
   (c) Prove whichever of P or $\neg P$ is true.

7. Let P, Q, and R be mathematical statements/propositions. Determine whether or not the two expressions in each pair are logically equivalent. In each case, demonstrate that your answer is correct, either using a truth table to prove equivalence or finding an assignment of truth values to P, Q, and R which shows they are not equivalent.
   (a) $(P \vee Q) \wedge R$ $\quad$ $(P \wedge R) \vee (Q \wedge R)$
   (b) $(P \Rightarrow Q) \Rightarrow R$ $\quad$ $P \Rightarrow (Q \Rightarrow R)$
   (c) $P \Rightarrow (Q \vee R)$ $\quad$ $(P \wedge \neg Q) \Rightarrow R$

8. Let P, Q, and R be mathematical statements/propositions. Prove that the statement $P \Rightarrow (Q \vee R)$ is logically equivalent to the statement $(P \wedge \neg Q) \Rightarrow R$. You may use a truth table.

9. Consider the statement: For all integers m and n, if m and n are odd, then mn is odd.
   (a) Give the converse of this statement.
   (b) Give the contrapositive of this statement.
   (c) Give the negation of this statement.

10. Let m and n be integers.
    (a) Prove the statement "If n and m are even, then $n+m$ is even."
    (b) State the contrapositive of this statement.
    (c) Prove or disprove the converse to this statement.

11. Prove that for all integers n, $n^2 + 3n$ is even.

12. Prove that there do not exist integers m and n for which $6m - 14n = 7$.

13. Prove the following statement, using definitions from the course.
    For all integers m and n, if the product mn is even, then m is even or n is even.

14. Let $x \in \mathbb{R}$ and assume that $x > 0$. Determine whether or not one of the expressions $\frac{x+1}{x+2}$ or $\frac{x}{x+1}$ is always larger. Prove your assertion.

15. Consider the statement: For all real numbers x and y, if x and y are irrational, then xy is irrational.
    (a) Write the converse of this statement.
    (b) Write the contrapositive of this statement.
    (c) Write the negation of this statement.
    (d) Which of the above four statements *(the proposition, its converse (a), its contrapositive (b), its negation (c))* are true? (You need not justify your answer.)

16. Recall that a real number x is *rational* if it is a quotient of integers, $x = m/n$, where $m, n \in \mathbb{Z}$, and otherwise it is *irrational*.
    (a) Prove the following statement: For all real numbers a, b, if $a + b$ is irrational, then either a is irrational or b is irrational.
    (b) Prove that the sum of a rational number and an irrational number is irrational.
    (c) In the previous question, if the universe for the universal quantifier on a, b is restricted to the rational numbers (instead of the real numbers) is the statement true or false? Give a valid reason for your answer.