Chapter 1

Varieties

Algebraic geometry uses tools from algebra to study geometric objects called (algebraic) varieties, which are the common zeroes of a collection of polynomials. We develop some basic notions of algebraic geometry, perhaps the most fundamental being the dictionary between algebraic and geometric concepts. The basic objects we introduce and concepts we develop will be used throughout the book. These include affine varieties, important notions from the algebra-geometry dictionary, projective varieties, and maps between varieties. We provide additional algebraic background in the appendices and pointers to other sources of introductions to algebraic geometry in the references provided at the end of the chapter.

1.1 Affine varieties

Let $\mathbb{K}$ be a field, which for us will almost always be either the complex numbers $\mathbb{C}$, the real numbers $\mathbb{R}$, or the rational numbers $\mathbb{Q}$. These different fields have their individual strengths and weaknesses. The complex numbers are *algebraically closed* in that every univariate polynomial has a complex root. Algebraic geometry works best over an algebraically closed field, and many introductory texts restrict themselves to the complex numbers. However, quite often real number answers are needed in applications. Because of this, we will often consider real varieties and work over $\mathbb{R}$. Symbolic computation provides many useful tools for algebraic geometry, but it requires a field such as $\mathbb{Q}$, which can be represented on a computer. Much of what we do remains true for arbitrary fields, such as the Gaussian rationals $\mathbb{Q}[i]$, or $\mathbb{C}(t)$, the field of rational functions in the variable t, or finite fields. We will at times use this added generality.

Algebraic geometry concerns the interplay of algebra and geometry, with its two most basic objects being the ring $\mathbb{K}[x_1,\dots,x_n]$ of polynomials in variables $x_1,\dots,x_n$ with coefficients in $\mathbb{K}$, and the space $\mathbb{K}^n$ of n-tuples $a=(a_1,\dots,a_n)$ of numbers from $\mathbb{K}$, called *affine space*. Evaluating a polynomial $f\in\mathbb{K}[x_1,\dots,x_n]$ at points of $\mathbb{K}^n$ defines a function $f\colon\mathbb{K}^n\to\mathbb{K}$ on affine space. We use these polynomial functions to define our primary object of interest. We will often abbreviate $\mathbb{K}[x_1,\dots,x_n]$ as $\mathbb{K}[x]$, when it is clear from the context that we are working with multivariate polynomials (and not univariate polynomials).

Definition 1.1.1. An *affine variety* is the set of common zeroes of some polynomials. Given a set $S \subset \mathbb{K}[x]$ of polynomials, the affine variety defined by S is the set

$$\mathcal{V}(S) \ := \ \{a \in \mathbb{K}^n \mid f(a) = 0 \quad \text{for } f \in S\}\,.$$

This is a *subvariety* of $\mathbb{K}^n$ or simply a *variety* or (*affine*) *algebraic variety*. When S consists of a single polynomial f, then $\mathcal{V}(S) = \mathcal{V}(f)$ is called a *hypersurface*.

If X and Y are varieties with $Y \subset X$, then Y is a *subvariety* of X. In Exercise 2, you will be asked to show that if $S \subset T$, then $\mathcal{V}(S) \supset \mathcal{V}(T)$.

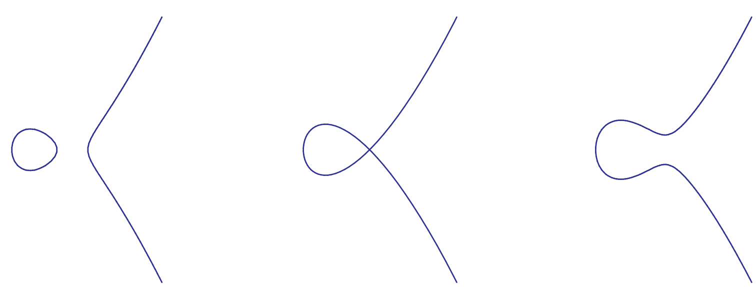
The empty set $\emptyset = \mathcal{V}(1)$ and affine space itself $\mathbb{K}^n = \mathcal{V}(0)$ are varieties. Any linear or affine subspace L of $\mathbb{K}^n$ is a variety. Indeed, an affine subspace L has an equation $Ax = b$, where A is a matrix and b is a vector, and so $L = \mathcal{V}(Ax - b)$ is defined by the linear polynomials which form the entries of the vector $Ax - b$. An important special case is when $L = \{b\}$ is a point of $\mathbb{K}^n$. Writing $b = (b_1, \ldots, b_n)$, then L is defined by the equations $x_i - b_i = 0$ for $i = 1, \ldots, n$.

Any finite subset $Z \subset \mathbb{K}^1$ is a variety as $Z = \mathcal{V}(f)$, where

$$f \ := \ \prod_{z \in Z} (x - z)$$

is the monic polynomial with simple zeroes at points of Z.

A non-constant polynomial $f(x, y)$ in the variables x and y defines a *plane curve* $\mathcal{V}(f) \subset \mathbb{K}^2$. Here are the real plane cubic curves $\mathcal{V}(f + \frac{1}{20})$, $\mathcal{V}(f)$, and $\mathcal{V}(f - \frac{1}{20})$, where $f(x, y) := y^2 - x^2 - x^3$.



(1.1)

A *quadric* is a variety defined by a single quadratic polynomial. The smooth quadrics in $\mathbb{K}^2$ are the plane conics (circles, ellipses, parabolas, and hyperbolas in $\mathbb{R}^2$) and the smooth quadrics in $\mathbb{R}^3$ are the spheres, ellipsoids, paraboloids, and hyperboloids (a formal definition of smooth variety is given in Section 3.4). Figure 1.1 shows a hyperbolic paraboloid $\mathcal{V}(xy + z)$ and a hyperboloid of one sheet $\mathcal{V}(x^2 - x + y^2 + yz)$.

These examples, finite subsets of $\mathbb{K}^1$, plane curves, and quadrics, are varieties defined by a single polynomial and are called *hypersurfaces*. Any variety is an intersection

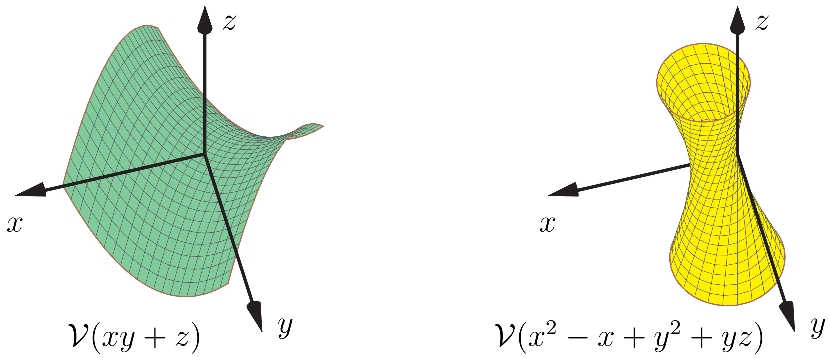
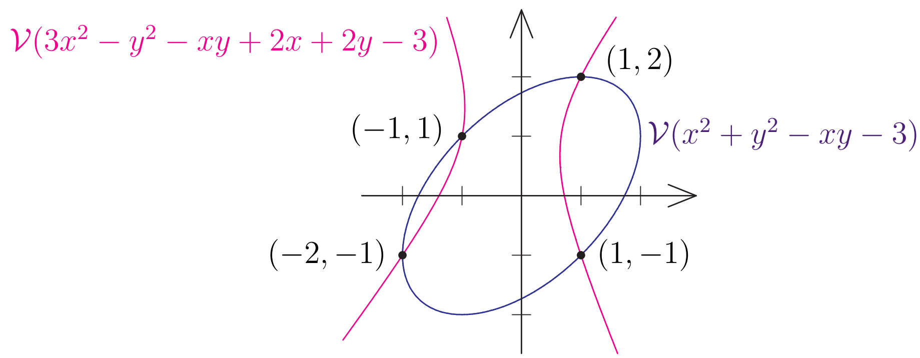


Figure 1.1: Two hyperboloids.

of hypersurfaces, one for each polynomial defining the variety. The set of four points $\{(-2,-1),(-1,1),(1,-1),(1,2)\}$ in $\mathbb{K}^2$ is a variety. It is the intersection of an ellipse $\mathcal{V}(x^2+y^2-xy-3)$ and a hyperbola $\mathcal{V}(3x^2-y^2-xy+2x+2y-3)$.



The quadrics of Figure 1.1 meet in the variety $\mathcal{V}(xy+z,\ x^2-x+y^2+yz)$, which is shown on the right in Figure 1.2. This intersection is the union of two space curves. One is the line $x=1, y+z=0$, while the other is the cubic space curve which has parametrization $t \mapsto (t^2, t, -t^3)$. Observe that the sum of the degrees of these curves, 1 (for the line) and 3 (for the space cubic) is equal to the product $2 \cdot 2$ of the degrees of the quadrics defining the intersection. We will have more to say on this in Section 3.6.

The intersection of the hyperboloid $x^2+(y-\frac{3}{2})^2-z^2=\frac{1}{4}$ with the sphere $x^2+y^2+z^2=4$ centered at the origin with radius 2 is a singular space curve (the figure ∞ on the left sphere in Figure 1.3). If we instead intersect the hyperboloid with the sphere centered at the origin having radius 1.9, then we obtain the smooth quartic space curve drawn on the right sphere in Figure 1.3.

The product $X \times Y$ of two varieties X and Y is again a variety. Indeed, suppose that $X \subset \mathbb{K}^n$ is defined by the polynomials $f_1,\ldots,f_s \in \mathbb{K}[x_1,\ldots,x_n]$ and that $Y \subset \mathbb{K}^m$ is defined by the polynomials $g_1,\ldots,g_t \in \mathbb{K}[y_1,\ldots,y_m]$. Then $X \times Y \subset \mathbb{K}^n \times \mathbb{K}^m = \mathbb{K}^{n+m}$ is

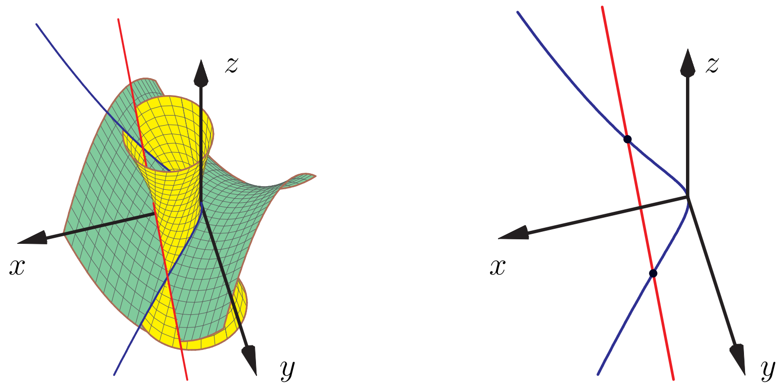


Figure 1.2: Intersection of two quadrics.

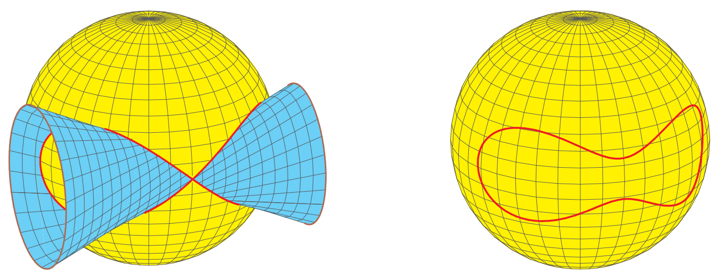


Figure 1.3: Quartics on spheres.

defined by the polynomials $f_1, \ldots, f_s, g_1, \ldots, g_t \in \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Given a point $x \in X$, the product $\{x\} \times Y$ is a subvariety of $X \times Y$ which may be identified with Y simply by forgetting the coordinate x.

The set $\mathrm{Mat}_{m\times n}$ or $\mathrm{Mat}_{m\times n}(\mathbb{K})$ of $m\times n$ matrices with entries in $\mathbb{K}$ is identified with the affine space $\mathbb{K}^{mn}$, which may be written $\mathbb{K}^{m\times n}$. An interesting class of varieties are linear algebraic groups, which are algebraic subvarieties of the space $\mathrm{Mat}_{n\times n}$ square matrices that are closed under multiplication and taking inverses. The *special linear group* is the set of matrices with determinant 1,

$$SL_n \ := \ \{M \in \mathrm{Mat}_{n\times n} \mid \det M = 1\}\,,$$

which is a linear algebraic group. Since the determinant of a matrix in $\mathrm{Mat}_{n\times n}$ is a polynomial in its entries, SL_n is the variety $\mathcal{V}(\det -1)$. We will later show that SL_n is smooth, irreducible, and has dimension $n^2 - 1$. (We must first, of course, define these notions.)

The general linear group $GL_n := \{M \in \mathrm{Mat}_{n\times n} \mid \det M \neq 0\}$ at first does not appear to be a variety as it is defined by an inequality. You will show in Exercise 7 that it may be identified with the set $\{(t, M) \in \mathbb{K} \times \mathrm{Mat}_{n\times n} \mid t\det M = 1\}$, which is a variety. When $n = 1$, $GL_1 = \{a \in \mathbb{K} \mid a \neq 0\}$ is the group of units (invertible elements) in $\mathbb{K}$, written $\mathbb{K}^\times$.

There is a general construction of other linear algebraic groups. Let g^T be the transpose of a matrix $g \in \mathrm{Mat}_{n\times n}$. For a fixed matrix $M \in \mathrm{Mat}_{n\times n}$, set

$$G_M \ := \ \{g \in SL_n \mid gMg^T = M\}\,.$$

This a linear algebraic group, as the condition $gMg^T = M$ is n^2 polynomial equations in the entries of g, and G_M is closed under matrix multiplication and matrix inversion.

When M is skew-symmetric and invertible, G_M is a *symplectic group*. In this case, n is necessarily even. If we let J_n denote the $n \times n$ matrix with 1s on its anti-diagonal, then the matrix

$$\begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}$$

is conjugate to every other invertible skew-symmetric matrix in $\mathrm{Mat}_{2n\times 2n}$. We assume M is this matrix and write Sp_{2n} for the symplectic group.

When M is symmetric and invertible, G_M is a *special orthogonal group*. When $\mathbb{K}$ is algebraically closed, all invertible symmetric matrices are conjugate, and we may assume $M = J_n$. For general fields, there may be many different forms of the special orthogonal group. For instance, when $\mathbb{K} = \mathbb{R}$, let k and l be, respectively, the number of positive and negative eigenvalues of M (these are conjugation invariants of M). Then we obtain the group $SO_{k,l}\mathbb{R}$. We have $SO_{k,l}\mathbb{R} \simeq SO_{l,k}\mathbb{R}$.

Consider the two extreme cases. When $l = 0$, we may take $M = I_n$, and so we obtain the special orthogonal group $SO_{n,0} = SO_n(\mathbb{R})$ of rotation matrices in $\mathbb{R}^n$, which is compact in the usual topology. The other extreme case is when $|k - l| \leq 1$, and we may take $M = J_n$. This is known as the split form of the special orthogonal group which is not compact.

When $n = 2$, consider the two different real groups:

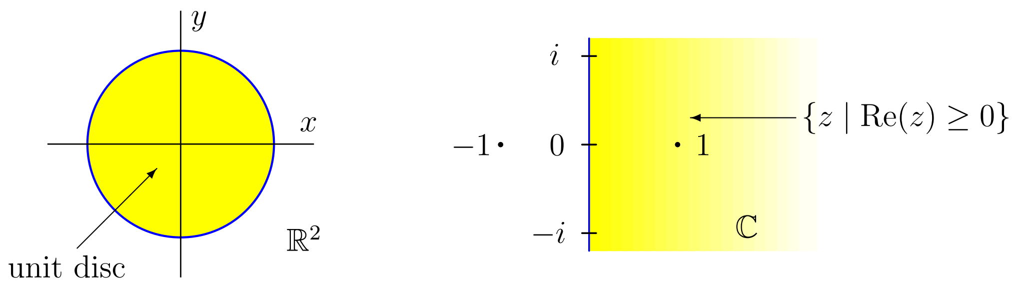
$$\begin{aligned} SO_{2,0}\mathbb{R} \ &:= \ \left\{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mid \theta \in [0, 2\pi)\right\} \\ SO_{1,1}\mathbb{R} \ &:= \ \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{R}^\times\right\} \end{aligned}$$

Note that in the Euclidean topology (see Appendix A.2) $SO_{2,0}(\mathbb{R})$ is compact, while $SO_{1,1}(\mathbb{R})$ is not. The complex group $SO_2(\mathbb{C})$ is also not compact in the Euclidean topology.

We point out some subsets of $\mathbb{K}^n$ which are not varieties. The set $\mathbb{Z}$ of integers is not a variety. The same is true for any other infinite proper subset of $\mathbb{K}$, for example, the infinite sequence $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ is not a subvariety of $\mathbb{R}$ or of $\mathbb{C}$.

Other subsets which are not varieties (for the same reasons) include the unit disc in

$\mathbb{R}^2$, $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$ or the complex numbers with positive real part.



Sets like these last two which are defined by inequalities involving real polynomials are called *semi-algebraic*. We will study them in Chapter 5.

Exercises

1. Show that no proper nonempty open subset S of $\mathbb{R}^n$ or $\mathbb{C}^n$ is a variety. Here, we mean open in the usual (Euclidean) topology on $\mathbb{R}^n$ and $\mathbb{C}^n$. (Hint: Consider the Taylor expansion of any polynomial that vanishes identically on S.)

2. Let $S \subset T$ be sets of multivariate polynomials in $\mathbb{K}[x]$. Show that $\mathcal{V}(S) \supset \mathcal{V}(T)$.

3. Show that any finite subset Z of $\mathbb{K}^n$ is a variety. (Hint: for a linear form $\Lambda : \mathbb{K}^n \to \mathbb{K}$, the polynomial
$$\Lambda_Z \ := \ \prod_{z\in Z} (\Lambda(x) - \Lambda(z))$$
vanishes on Z. Show that there is a set L of linear forms such that the polynomials Λ_Z for $\Lambda \in L$ define Z.) This may be too complicated

4. Prove that in $\mathbb{K}^2$ we have $\mathcal{V}(y-x^2) = \mathcal{V}(y^3-y^2x^2,\ x^2y-x^4)$.

5. Show that the following sets are not algebraic varieties.

   (i) $\mathbb{Z} \subset \mathbb{C}$ and $\mathbb{Z} \subset \mathbb{R}$.

   (ii) $\{(x,y) \in \mathbb{R}^2 \mid y = \sin x\}$.

   (iii) $\{(\cos t, \sin t, t) \in \mathbb{R}^3 \mid t \in \mathbb{R}\}$.

   (iv) $\{(x, e^x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$.

6. Express the cubic space curve C with parametrization (t, t^2, t^3) for $t \in \mathbb{K}$ as a variety in each of the following ways.

   (a) The intersection of a quadric hypersurface and a cubic hypersurface.

   (b) The intersection of two quadrics.

   (c) The intersection of three quadrics.

7. Let $\mathbb{K}^{n\times n}$ be the set of $n \times n$ matrices over $\mathbb{K}$.

   (a) Show that the set $SL_n(\mathbb{K}) \subset \mathbb{K}^{n\times n}$ of matrices with determinant 1 is an algebraic variety.

   (b) Show that the set of singular matrices in $\mathbb{K}^{n\times n}$ is an algebraic variety.

   (c) Show that the set $GL_n(\mathbb{K})$ of invertible matrices is not an algebraic variety in $\mathbb{K}^{n\times n}$. Show that $GL_n(\mathbb{K})$ can be identified with an algebraic subset of $\mathbb{K}^{n^2+1} = \mathbb{K}^{n\times n} \times \mathbb{K}^1$ via a map $GL_n(\mathbb{K}) \to \mathbb{K}^{n^2+1}$.

8. An $n\times n$ matrix with complex entries is *unitary* if its columns are orthonormal under the complex inner product $\langle z, w\rangle = z \cdot \overline{w}^t = \sum_{i=1}^n z_i\overline{w_i}$. Show that the set $\mathbf{U}(n)$ of unitary matrices is not a complex algebraic variety. Show that it can be described as the zero locus of a collection of polynomials with real coefficients in $\mathbb{R}^{2n^2}$, and so it is a real algebraic variety.

9. Let $\mathbb{K}^{m\times n}$ be the set of $m \times n$ matrices over $\mathbb{K}$.

   (a) Show that the set of matrices of rank at most r is an algebraic variety.

   (b) Show that the set of matrices of rank exactly r is not an algebraic variety when $r > 0$.

1.2 Varieties and Ideals

The strength and richness of algebraic geometry as a subject and source of tools for applications comes from its dual, simultaneously algebraic and geometric, nature. Intuitive geometric concepts are tamed via the precision of algebra while basic algebraic notions are enlivened by their geometric counterparts. The source of this dual nature is a correspondence—in fact an equivalence—between algebraic concepts and geometric concepts that we refer to as the algebra-geometry dictionary.

We defined varieties $\mathcal{V}(S)$ associated to sets $S \subset \mathbb{K}[x]$ of multivariate polynomials,

$$\mathcal{V}(S) \;=\; \{x \in \mathbb{K}^n \mid f(x) = 0 \text{ for all } f \in S\}\,.$$

We would like to invert this association. Given a subset Z of $\mathbb{K}^n$, consider the collection of polynomials that vanish on Z,

$$\mathcal{I}(Z) \;:=\; \{f \in \mathbb{K}[x] \mid f(z) = 0 \text{ for all } z \in Z\}\,.$$

The map $\mathcal{I}$ reverses inclusions so that $Z \subset Y$ implies $\mathcal{I}(Z) \supset \mathcal{I}(Y)$.

These two inclusion-reversing maps

$$\{\text{Subsets } S \text{ of } \mathbb{K}[x]\} \quad \underset{\mathcal{I}}{\overset{\mathcal{V}}{\rightleftarrows}} \quad \{\text{Subsets } Z \text{ of } \mathbb{K}^n\} \tag{1.2}$$

form the basis of the algebra-geometry dictionary of affine algebraic geometry. We will refine this correspondence to make it more precise.

An *ideal* is a subset $I \subset \mathbb{K}[x]$ which is closed under addition and under multiplication by polynomials in $\mathbb{K}[x]$. If $f, g \in I$ then $f + g \in I$ and if we also have $h \in \mathbb{K}[x]$, then $hf \in I$. The *ideal* $\langle S\rangle$ generated by a subset S of $\mathbb{K}[x]$ is the smallest ideal containing S. It is the set of all expressions of the form

$$h_1f_1 + \cdots + h_mf_m$$

where $f_1, \ldots, f_m \in S$ and $h_1, \ldots, h_m \in \mathbb{K}[x]$. We work with ideals because if f, g, and h are polynomials and $x \in \mathbb{K}^n$ with $f(x) = g(x) = 0$, then $(f + g)(x) = 0$ and $(hf)(x) = 0$. Thus $\mathcal{V}(S) = \mathcal{V}(\langle S\rangle)$, and so we may restrict the map $\mathcal{V}$ of the correspondence (1.2) to the ideals of $\mathbb{K}[x]$. In fact, we lose nothing if we restrict the left-hand-side of the correspondence (1.2) to the ideals of $\mathbb{K}[x]$.

Lemma 1.2.1. *For any subset S of $\mathbb{K}^n$, $\mathcal{I}(S)$ is an ideal of $\mathbb{K}[x]$.*

Proof. Let $f, g \in \mathcal{I}(S)$ be polynomials which vanish at all points of S. Then $f+g$ vanishes on S, as does hf, where h is any polynomial in $\mathbb{K}[x]$. This shows that $\mathcal{I}(S)$ is an ideal of $\mathbb{K}[x]$. □

When S is infinite, the variety $\mathcal{V}(S)$ is defined by infinitely many polynomials. Hilbert's Basis Theorem tells us that only finitely many of these polynomials are needed.

Hilbert's Basis Theorem. *Every ideal I of $\mathbb{K}[x]$ is finitely generated.*

We will prove a stronger form of this (Theorem 2.2.10) in Chapter 2, but use it here. Hilbert's Basis Theorem implies important finiteness properties of algebraic varieties.

Corollary 1.2.2. *Any variety $Z \subset \mathbb{K}^n$ is the intersection of finitely many hypersurfaces.*

Proof. Let $Z = \mathcal{V}(I)$ be defined by the ideal I. By Hilbert's Basis Theorem, I is finitely generated, say by $f_1, \ldots, f_s$, and so $Z = \mathcal{V}(f_1, \ldots, f_s) = \mathcal{V}(f_1) \cap \cdots \cap \mathcal{V}(f_s)$. □

Example 1.2.3. The ideal of the cubic space curve C of Figure 1.2 with parametrization $(t^2, t, -t^3)$ not only contains the polynomials $xy+z$ and $x^2-x + y^2+yz$, but also y^2-x, x^2+yz, and y^3+z. Not all of these polynomials are needed to define C as $x^2-x+y^2+yz = (y^2 - x) + (x^2 + yz)$ and $y^3 + z = y(y^2 - x) + (xy + z)$. In fact three of the quadrics suffice,

$$\mathcal{I}(C) \;=\; \langle xy+z,\; y^2-x,\; x^2+yz\rangle\,. \qquad \diamond$$

Lemma 1.2.4. *For any subset Z of $\mathbb{K}^n$, if $X = \mathcal{V}(\mathcal{I}(Z))$ is the variety defined by the ideal $\mathcal{I}(Z)$, then $\mathcal{I}(X) = \mathcal{I}(Z)$ and X is the smallest variety containing Z.*

Proof. Set $X := \mathcal{V}(\mathcal{I}(Z))$. Then $\mathcal{I}(Z) \subset \mathcal{I}(X)$, since if f vanishes on Z, it will vanish on X. However, $Z \subset X$, and so $\mathcal{I}(Z) \supset \mathcal{I}(X)$, and thus $\mathcal{I}(Z) = \mathcal{I}(X)$.

If Y was a variety with $Z \subset Y \subset X$, then $\mathcal{I}(X) \subset \mathcal{I}(Y) \subset \mathcal{I}(Z) = \mathcal{I}(X)$, and so $\mathcal{I}(Y) = \mathcal{I}(X)$. But then we must have $Y = X$ for otherwise $\mathcal{I}(X) \subsetneq \mathcal{I}(Y)$, as is shown in Exercise 4. □

Thus we also lose nothing if we restrict the right-hand-side of the correspondence (1.2) to the subvarieties of $\mathbb{K}^n$. Our correspondence now becomes

$$\{\text{Ideals } I \text{ of } \mathbb{K}[x]\} \quad \underset{\mathcal{I}}{\overset{\mathcal{V}}{\rightleftarrows}} \quad \{\text{Subvarieties } X \text{ of } \mathbb{K}^n\} \,. \tag{1.3}$$

This association is not a bijection. In particular, the map $\mathcal{V}$ is not one-to-one and the map $\mathcal{I}$ is not onto. There are several reasons for this.

For example, when $\mathbb{K}=\mathbb{Q}$ and $n=1$, we have $\emptyset=\mathcal{V}(1)=\mathcal{V}(x^2-2)$. The problem here is that the rational numbers are not algebraically closed and we need to work with a larger field (for example $\mathbb{Q}(\sqrt{2})$) to study $\mathcal{V}(x^2-2)$. When $\mathbb{K}=\mathbb{R}$ and $n=1$, $\emptyset\neq\mathcal{V}(x^2-2)$, but we have $\emptyset=\mathcal{V}(1)=\mathcal{V}(1+x^2)=\mathcal{V}(1+x^4)$. While the problem here is again that the real numbers are not algebraically closed, we view this as a manifestation of positivity. The two polynomials $1+x^2$ and $1+x^4$ only take positive values. When working over $\mathbb{R}$ (as our interest in applications leads us to do so) positivity of polynomials plays an important role, as we will see in Chapters 6 and 7.

The problem with the map $\mathcal{V}$ is more fundamental than these examples reveal and occurs even when $\mathbb{K}=\mathbb{C}$. When $n=1$ we have $\{0\}=\mathcal{V}(x)=\mathcal{V}(x^2)$, and when $n=2$, we invite the reader to check that $\mathcal{V}(y-x^2)=\mathcal{V}(y^2-yx^2,xy-x^3)$. Note that while $x\notin\langle x^2\rangle$, we have $x^2\in\langle x^2\rangle$. Similarly, $y-x^2\notin\mathcal{V}(y^2-yx^2,xy-x^3)$, but

$$(y-x^2)^2 \;=\; y^2-yx^2 \;-\; x(xy-x^3) \;\in\; \langle y^2-yx^2, xy-x^3\rangle\,. \tag{1.4}$$

These two cases reveal a source for lack of injectivity of the map $\mathcal{V}$—the polynomials f and f^N have the same set of zeroes, for any positive integer N. For example, if $f_1,\dots,f_s$ are polynomials, then the two ideals

$$\langle f_1,\, f_2,\, \dots,\, f_s\rangle \qquad \text{and} \qquad \langle f_1,\, f_2^2,\, f_3^3,\, \dots,\, f_s^s\rangle$$

both define the same variety, and for any $Z\subset\mathbb{K}^n$, if $f^N\in\mathcal{I}(Z)$, then $f\in\mathcal{I}(Z)$.

We clarify this point with a definition. An ideal $I\subset\mathbb{K}[x]$ is *radical* if whenever $f^N\in I$ for some positive integer N, then $f\in I$. The radical $\sqrt{I}$ of an ideal I of $\mathbb{K}[x]$ is

$$\sqrt{I} \;:=\; \{f\in\mathbb{K}[x] \mid f^N\in I\,,\ \text{for some } N\geq 1\}\,.$$

You will show in Exercise 3 that $\sqrt{I}$ is the smallest radical ideal containing I. For example (1.4) shows that

$$\sqrt{\langle y^2-yx^2, xy-x^3\rangle} \;=\; \langle y-x^2\rangle\,.$$

The reason for this definition is twofold: first, $\mathcal{I}(Z)$ is radical, and second, an ideal I and its radical$\sqrt{I}$ both define the same variety. We record these facts.

Lemma 1.2.5. *For $Z\subset\mathbb{K}^n$, $\mathcal{I}(Z)$ is a radical ideal. If $I\subset\mathbb{K}[x]$ is an ideal, then $\mathcal{V}(I)=\mathcal{V}(\sqrt{I})$.*

When $\mathbb{K}$ is algebraically closed, the precise nature of the correspondence (1.3) follows from Hilbert's Nullstellensatz (null=zeroes, stelle=places, satz=theorem), another of Hilbert's foundational results in the 1890's that helped to lay the foundations of algebraic geometry and usher in twentieth century mathematics. We first state an apparently weak form of the Nullstellensatz, which describes the ideals defining the empty set.

Theorem 1.2.6 (Weak Nullstellensatz)**.** *Suppose that $\mathbb{K}$ is algebraically closed. If I is an ideal of $\mathbb{K}[x]$ with $\mathcal{V}(I) = \emptyset$, then $I = \mathbb{K}[x]$.*

Let $b = (b_1, \dots, b_n) \in \mathbb{K}^n$. We observed that the point $\{b\}$ is defined by the linear polynomials $x_i - b_i$ for $i = 1, \dots, n$. A polynomial $f \in \mathbb{K}[x]$ is equal to the constant $f(b)$ modulo the ideal $\mathfrak{m}_b := \langle x_1 - b_1, \dots, x_n - b_n\rangle$, thus the quotient ring $\mathbb{K}[x_1, \dots, x_n]/\mathfrak{m}_b$ is isomorphic to the field $\mathbb{K}$ and so $\mathfrak{m}_b$ is a maximal ideal. In fact when $\mathbb{K}$ is algebraically closed, these are the only maximal ideals of $\mathbb{K}[x]$.

Theorem 1.2.7. *Suppose that $\mathbb{K}$ is algebraically closed. Then every maximal ideal $\mathfrak{m}$ of $\mathbb{K}[x_1, \dots, x_n]$ has the form $\mathfrak{m}_b$ for some $b \in \mathbb{K}^n$.*

Proof. We prove this when $\mathbb{K}$ is an uncountable field, e.g. $\mathbb{K} = \mathbb{C}$. As $\mathfrak{m}$ is a maximal ideal, $\mathbb{K}[x]/\mathfrak{m}$ is a field, L, that contains $\mathbb{K}$ whose dimension as a $\mathbb{K}$-vector space is at most countable (L is spanned by the images of the countably many monomials). Since $\mathbb{K}$ is algebraically closed, we have $L \neq \mathbb{K}$ only if L contains an element ξ that does not satisfy any algebraic equations with coefficients in $\mathbb{K}$ (ξ is transcendental over $\mathbb{K}$). But then the subfield of L generated by $\mathbb{K}$ and ξ is isomorphic to the field $\mathbb{K}(t)$ of rational functions (quotients of polynomials) in the indeterminate t, under the map $t \mapsto \xi$. Consider the uncountable subset of $\mathbb{K}(t)$,

$$\left\{\frac{1}{t-a} \mid a \in \mathbb{K}\right\}.$$

We claim that this set is linearly independent over $\mathbb{K}$. Indeed, suppose that there is a linear dependency among elements of this set,

$$0 \;=\; \sum_{i=1}^{m} \lambda_i \frac{1}{t-a_i}\,.$$

For any $i = 1, \dots, m$, if we multiply this by $(t - a_i)$ and simplify, and then substitute $t = a_i$, we obtain the equation $\lambda_i = 0$. This shows that the elements $\frac{1}{t-a}$ for $a \in \mathbb{K}$ are linearly independent over $\mathbb{K}$. Thus $\mathbb{K}(t)$ has uncountable dimension over $\mathbb{K}$ and so L cannot contain a subfield isomorphic to $\mathbb{K}(t)$.

We conclude that $L = \mathbb{K}$. If $b_i \in \mathbb{K}$ is the image of the variable x_i, then we see that $\mathfrak{m} \supset \mathfrak{m}_b$. As both are maximal ideals, they are equal. □

Proof of the weak Nullstellensatz. We prove the contrapositive, if $I \subsetneq \mathbb{K}[x]$ is a proper ideal, then $\mathcal{V}(I) \neq \emptyset$. There is a maximal ideal $\mathfrak{m}_b$ with $b \in \mathbb{K}^n$ of $\mathbb{K}[x]$ which contains I. But then

$$\{b\} \;=\; \mathcal{V}(\mathfrak{m}_b) \;\subset\; \mathcal{V}(I)\,,$$

and so $\mathcal{V}(I) \neq \emptyset$. Thus if $\mathcal{V}(I) = \emptyset$, we must have $I = \mathbb{K}[x]$, which proves the weak Nullstellensatz. $\square$

A consequence of this proof is that there is a 1-1 correspondence

$$\{\text{Points } b \in \mathcal{V}(I)\} \longleftrightarrow \{\text{Maximal ideals } \mathfrak{m}_b \supset I\}\,.$$

The Fundamental Theorem of Algebra states that any nonconstant univariate polynomial $f \in \mathbb{C}[x]$ has a root (a solution to $f(x) = 0$). We recast the weak Nullstellensatz as the multivariate fundamental theorem of algebra.

Theorem 1.2.8 (Multivariate Fundamental Theorem of Algebra)**.** *Let $\mathbb{K}$ be an algebraically closed field. If the polynomials $f_1, \ldots, f_m \in \mathbb{K}[x_1, \ldots, x_n]$ generate a proper ideal, then the system of polynomial equations*

$$f_1(x) \;=\; f_2(x) \;=\; \cdots \;=\; f_m(x) \;=\; 0$$

has a solution in $\mathbb{K}^n$.

We now deduce the strong Nullstellensatz, which we will use to complete the characterization (1.3). For this, we assume that $\mathbb{K}$ is algebraically closed.

Theorem 1.2.9 (Nullstellensatz)**.** *Let $\mathbb{K}$ be an algebraically closed field. If $I \subset \mathbb{K}[x]$ is an ideal, then $\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}$.*

Proof. Since $\mathcal{V}(I) = \mathcal{V}(\sqrt{I})$, we have $\sqrt{I} \subset \mathcal{I}(\mathcal{V}(I))$. We show the other inclusion using the 'trick of Rabinowitsch'. Suppose that we have a polynomial $f \in \mathcal{I}(\mathcal{V}(I))$. Let us introduce a new variable t. Then the variety $\mathcal{V}(I, tf-1) \subset \mathbb{K}^{n+1}$ defined by I and $tf-1$ is empty. Indeed, if $(a_1, \ldots, a_n, b)$ were a point of this variety, then $(a_1, \ldots, a_n)$ would be a point of $\mathcal{V}(I)$. But then $f(a_1, \ldots, a_n) = 0$, and so the polynomial $tf-1$ evaluates to 1 (and not 0) at the point $(a_1, \ldots, a_n, b)$.

By the weak Nullstellensatz, $\langle I, tf-1\rangle = \mathbb{K}[x,\, t]$. In particular, $1 \in \langle I, tf-1\rangle$, and so there exist polynomials $f_1, \ldots, f_m \in I$ and $g_1, \ldots, g_m, g \in \mathbb{K}[x,\, t]$ such that

$$1 \;=\; f_1(x)g_1(x,t) + f_2(x)g_2(x,t) + \cdots + f_m(x)g_m(x,t) \;+\; (tf(x)-1)g(x,t)\,.$$

If we apply the substitution $t = \frac{1}{f}$, then the last term with factor $tf-1$ vanishes and each polynomial $g_i(x,t)$ becomes a rational function in $x_1, \ldots, x_n$ whose denominator is a power of f. Clearing these denominators gives an expression of the form

$$f^N \;=\; f_1(x)G_1(x) + f_2(x)G_2(x) + \cdots + f_m(x)G_m(x)\,,$$

where $G_1, \ldots, G_m \in \mathbb{K}[x]$. But this shows that $f \in \sqrt{I}$, and completes the proof of the Nullstellensatz. $\square$

Corollary 1.2.10 (Algebra-Geometry Dictionary I). *Over any field $\mathbb{K}$, the maps $\mathcal{V}$ and $\mathcal{I}$ give an inclusion reversing correspondence*

$$\{\textit{Radical ideals } I \textit{ of } \mathbb{K}[x]\} \quad \underset{\mathcal{I}}{\overset{\mathcal{V}}{\rightleftarrows}} \quad \{\textit{Subvarieties } X \textit{ of } \mathbb{K}^n\} \tag{1.5}$$

with $\mathcal{V}(\mathcal{I}(X)) = X$. When $\mathbb{K}$ is algebraically closed, the maps $\mathcal{V}$ and $\mathcal{I}$ are inverses, and this correspondence is a bijection.

Proof. First, we already observed that $\mathcal{I}$ and $\mathcal{V}$ reverse inclusions and these maps have the domain and range indicated. Let X be a subvariety of $\mathbb{K}^n$. In Lemma 1.2.4 we showed that $X = \mathcal{V}(\mathcal{I}(X))$. Thus $\mathcal{V}$ is onto and $\mathcal{I}$ is one-to-one.

Now suppose that $\mathbb{K}$ is algebraically closed. By the Nullstellensatz, if I is radical then $\mathcal{I}(\mathcal{V}(I)) = I$, and so $\mathcal{I}$ is onto and $\mathcal{V}$ is one-to-one. Thus $\mathcal{I}$ and $\mathcal{V}$ are inverse bijections. □

Corollary 1.2.10 is only the beginning of the algebra-geometry dictionary. Many natural operations on varieties correspond to natural operations on their ideals. The *sum $I + J$* and *product $I \cdot J$* of ideals I and J are defined to be

$$\begin{aligned} I + J &:= \{f + g \mid f \in I \quad \text{and} \quad g \in J\} \\ I \cdot J &:= \langle fg \mid f \in I \quad \text{and} \quad g \in J\rangle\,. \end{aligned}$$

Note that $I + J$ is the ideal $\langle I, J\rangle$ generated by $I \cup J$, and that $I \cap J$ is also an ideal.

Lemma 1.2.11. *Let I, J be ideals in $\mathbb{K}[x]$ and set $X := \mathcal{V}(I)$ and $Y := \mathcal{V}(J)$ to be their corresponding varieties. Then*

1. *$\mathcal{V}(I + J) = X \cap Y$,*
2. *$\mathcal{V}(I \cdot J) = \mathcal{V}(I \cap J) = X \cup Y$,*

If $\mathbb{K}$ is algebraically closed, then by the Nullstellensatz we also have

3. *$\mathcal{I}(X \cap Y) = \sqrt{I + J}$, and*
4. *$\mathcal{I}(X \cup Y) = \sqrt{I \cap J} = \sqrt{I \cdot J}$.*

You are asked to prove this in Exercise 8.

Example 1.2.12. It can happen that $I \cdot J \neq I \cap J$. For example, if $I = \langle xy - x^3\rangle$ and $J = \langle y^2 - x^2y\rangle$, then $I \cdot J = \langle xy(y - x^2)^2\rangle$, while $I \cap J = \langle xy(y - x^2)\rangle$. ⋄

The correspondence (1.5) will be further refined in Section 1.3 to include maps between varieties. Because of this correspondence, each geometric concept has a corresponding algebraic concept, and *vice-versa*, when $\mathbb{K}$ is algebraically closed. When $\mathbb{K}$ is not algebraically closed, this correspondence is not exact. In that case we will often use algebra to guide our geometric definitions.

A polynomial $f \in \mathbb{K}[x]$ has an essentially unique factorization $f = f_1 \cdots f_s$ into irreducible polynomials. It is unique in that any other factorization into irreducible polynomials will have the same length, and after permuting factors, the corresponding factors in each factorization are proportional. Collecting proportional factors and extracting a constant α if necessary, have $f = \alpha g_1^{n_1} \cdots g_r^{n_r}$ with each $n_i \geq 1$, where, if $i \neq j$, then g_i is not proportional to g_j. The *square-free* part of f is $\sqrt{f} = g_1 \cdots g_r$, and we have

$$\sqrt{\langle f \rangle} \;=\; \left\langle \sqrt{f}\, \right\rangle \;=\; \langle g_1 \rangle \cap \langle g_2 \rangle \cap \cdots \cap \langle g_r \rangle\,,$$

so that $\mathcal{V}(f) = \mathcal{V}(g_1) \cup \cdots \cup \mathcal{V}(g_r)$.

Exercises

1. Show that the map $\mathcal{I}$ reverses inclusions so that $Z \subset Y$ implies $\mathcal{I}(Z) \supset \mathcal{I}(Y)$.

2. Verify the claim that the smallest ideal containing a set $S \subset \mathbb{K}[x]$ of polynomials consists of all expressions of the form

$$h_1 f_1 + \cdots + h_m f_m$$

   where $f_1, \ldots, f_m \in S$ and $h_1, \ldots, h_m \in \mathbb{K}[x]$.

3. Let I be an ideal of $\mathbb{K}[x]$. Show that

$$\sqrt{I} \;:=\; \{f \in \mathbb{K}[x] \mid f^N \in I,\ \text{for some } N \in \mathbb{N}\}$$

   is an ideal, is radical, and is the smallest radical ideal containing I.

4. If $Y \subsetneq X$ are varieties, show that $\mathcal{I}(X) \subsetneq \mathcal{I}(Y)$.

5. Suppose that I and J are radical ideals. Show that $I \cap J$ is also a radical ideal.

6. Give radical ideals I and J for which $I + J$ is not radical.

7. Let I be an ideal in $\mathbb{K}[x]$, where $\mathbb{K}$ is a field. Prove or find counterexamples to the following statements. Make your assumptions clear.

   (a) If $\mathcal{V}(I) = \mathbb{K}^n$ then $I = \langle 0 \rangle$.

   (b) If $\mathcal{V}(I) = \emptyset$ then $I = \mathbb{K}[x]$.

8. Give a proof of Lemma 1.2.11. Hint: Statements 1. and 2. are set-theoretic.

9. Give two algebraic varieties Y and Z such that $\mathcal{I}(Y \cap Z) \neq \mathcal{I}(Y) + \mathcal{I}(Z)$.

10. (a) Let I be an ideal of $\mathbb{K}[x]$. Show that if $\mathbb{K}[x]/I$ is a finite dimensional $\mathbb{K}$-vector space then $\mathcal{V}(I)$ is a finite set.

(b) Let $J = \langle xy, yz, xz\rangle$ be an ideal in $\mathbb{K}[x, y, z]$. Find the generators of $\mathcal{I}(\mathcal{V}(J))$. Show that J cannot be generated by two polynomials in $\mathbb{K}[x, y, z]$. Describe $V(I)$ where $I = \langle xy, xz - yz\rangle$. Show that $\sqrt{I} = J$.

11. Prove that there are three points p, q, and r in $\mathbb{K}^2$ such that

$$\sqrt{\langle x^2 - 2xy^4 + y^6, y^3 - y\rangle} \;=\; I(\{p\}) \cap I(\{q\}) \cap I(\{r\})\,.$$

Show directly that the ideal $\langle x^2 - 2xy^4 + y^6, y^3 - y\rangle$ is not radical.

12. Deduce the weak Nullstellensatz from the statement of the Strong Nullstellensatz, showing that they are equivalent.

1.3 Maps and homomorphisms

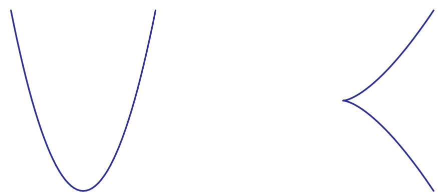
We strengthen the algebra-geometry dictionary of Section 1.2 in two ways. We first replace affine space $\mathbb{K}^n$ by an affine variety X and the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ by the ring $\mathbb{K}[X]$ of regular functions on X and establish a correspondence between subvarieties of X and radical ideals of $\mathbb{K}[X]$. Next, we establish a correspondence between regular maps of varieties and homomorphisms of their coordinate rings.

We have used that a polynomial $f \in \mathbb{K}[x] = \mathbb{K}[x_1, \ldots, x_n]$ gives a function $f \colon \mathbb{K}^n \to \mathbb{K}$, defined by evaluation at points of $\mathbb{K}^n$. When $\mathbb{K}$ is infinite, the function is identically zero if and only if f is the zero polynomial, so this representation of polynomials by functions is faithful. Further suppose that $X \subset \mathbb{K}^n$ is an affine variety. Any polynomial function $f \in \mathbb{K}[x]$ restricts to give a *regular function* on X, $f \colon X \to \mathbb{K}$. We may add and multiply regular functions, and the set of all regular functions on X forms a ring, $\mathbb{K}[X]$, called the *coordinate ring* of the affine variety X or the ring of regular functions on X. The coordinate ring of an affine variety X is a basic invariant of X, which we will show is in fact equivalent to X itself.

The restriction of polynomial functions on $\mathbb{K}^n$ to regular functions on X defines a surjective ring homomorphism $\mathbb{K}[x] \twoheadrightarrow \mathbb{K}[X]$. The kernel of this restriction homomorphism is the set of polynomials that vanish identically on X, that is, the ideal $\mathcal{I}(X)$ of X. Under the correspondence between ideals, quotient rings, and homomorphisms, this restriction map gives an isomorphism between $\mathbb{K}[X]$ and the quotient ring $\mathbb{K}[x]/\mathcal{I}(X)$.

Example 1.3.1. The coordinate ring of the parabola $y = x^2$ is $\mathbb{K}[x, y]/\langle y - x^2\rangle$, which is isomorphic to $\mathbb{K}[x]$, the coordinate ring of $\mathbb{K}^1$. To see this, observe that substituting x^2 for y rewrites any polynomial $f(x, y) \in \mathbb{K}[x, y]$ as a polynomial $g(x) = f(x, x^2)$ in x alone. The resulting map $\mathbb{K}[x, y]/\langle y - x^2\rangle \to \mathbb{K}[x]$ is well-defined and surjective. Since

$y - x^2$ divides the difference $f(x,y) - g(x)$, the map is injective.



Parabola Cuspidal cubic

On the other hand, the coordinate ring of the cuspidal cubic $y^2 = x^3$ is $\mathbb{K}[x,y]/\langle y^2 - x^3\rangle$. This ring is not isomorphic to $\mathbb{K}[x,y]/\langle y - x^2\rangle$. Indeed, the element $y^2 = x^3$ has two distinct factorizations into irreducible elements, while polynomials $f(x)$ in one variable have a unique factorization into irreducible polynomials. $\diamond$

Let $X \subset \mathbb{K}^n$ be a variety. Its coordinate ring $\mathbb{K}[X] = \mathbb{K}[x]/\mathcal{I}(X)$ has the structure of a vector space over $\mathbb{K}$, where addition is defined by the addition in the ring and scalar multiplication is defined by multiplication with an element in $\mathbb{K}$.

Definition 1.3.2. A $\mathbb{K}$-*algebra* is a ring that contains the field $\mathbb{K}$ as a subring. $\diamond$

A $\mathbb{K}$-algebra has the structure of a vector space over $\mathbb{K}$. The coordinate ring $\mathbb{K}[X]$ of a variety X is a $\mathbb{K}$-algebra. Observe that $\mathbb{K}[X] = \mathbb{K}[x]/\mathcal{I}(X)$ is finitely generated as a $\mathbb{K}$-algebra by the images of the variables x_i. Since $\mathcal{I}(X)$ is radical, Exercise 4 implies that the coordinate ring $\mathbb{K}[X]$ has no nilpotent elements (elements f such that $f^N = 0$ for some N). Such a ring with no nilpotent elements is called *reduced*. When $\mathbb{K}$ is algebraically closed, these two properties characterize coordinate rings of algebraic varieties.

Theorem 1.3.3. *Suppose that $\mathbb{K}$ is algebraically closed. A $\mathbb{K}$-algebra R is the coordinate ring of an affine variety if and only if R is finitely generated and reduced.*

Proof. We need only show that a finitely generated reduced $\mathbb{K}$-algebra R is the coordinate ring of some affine variety. Suppose that the reduced $\mathbb{K}$-algebra R has generators $r_1, \ldots, r_n$ for some $n \in \mathbb{N}$. Then there is a surjective ring homomorphism

$$\varphi : \mathbb{K}[x_1, \ldots, x_n] \twoheadrightarrow R$$

given by $x_i \mapsto r_i$. Let $I \subset \mathbb{K}[x]$ be the kernel of φ. This identifies R with $\mathbb{K}[x]/I$. Since R is reduced, we have that I is radical. Indeed, a polynomial $f \notin I$ with $f^N \in I$ gives a nonzero element $\varphi(f) \in R$ with $(\varphi(f))^N = 0$.

As $\mathbb{K}$ is algebraically closed, the algebra-geometry dictionary of Corollary 1.2.10 shows that $I = \mathcal{I}(\mathcal{V}(I))$ and so $R \simeq \mathbb{K}[x]/I \simeq \mathbb{K}[\mathcal{V}(I)]$. $\square$

A different choice $s_1, \ldots, s_m$ of generators for R in this proof will give a different affine variety with the same coordinate ring R. We seek to understand this apparent ambiguity.

Example 1.3.4. The finitely generated $\mathbb{K}$-algebra $R := \mathbb{K}[t]$ is the coordinate ring of the affine line $\mathbb{K}$. Note that if we set $x := t+1$ and $y := t^2+3t$, these generate R. As $y = x^2+x-2$, this choice of generators realizes R as $\mathbb{K}[x,y]/\langle y-x^2-x+2\rangle$, which is the coordinate ring of a parabola in $\mathbb{K}^2$. $\diamond$

Among the coordinate rings $\mathbb{K}[X]$ of affine varieties are the polynomial algebras $\mathbb{K}[x]$. Many properties of polynomial algebras, including the algebra-geometry dictionary of Corollary 1.2.10 and the Hilbert Theorems hold for these coordinate rings $\mathbb{K}[X]$.

Given regular functions $f_1,\dots,f_m \in \mathbb{K}[X]$ on an affine variety $X \subset \mathbb{K}^n$, their set of common zeroes

$$\mathcal{V}(f_1,\dots,f_m) \;:=\; \{x \in X \mid f_1(x) = \cdots = f_m(x) = 0\}\,,$$

is a subvariety of X. To see this, let $F_1,\dots,F_m \in \mathbb{K}[x]$ be polynomials which restrict to the functions $f_1,\dots,f_m$ on X. Then

$$\mathcal{V}(f_1,\dots,f_m) \;=\; X \,\cap\, \mathcal{V}(F_1,\dots,F_m)\,,$$

and by Lemma 1.2.11 intersections of varieties are again varieties. As in Section 1.2, we may extend this notation and define $\mathcal{V}(I)$ for an ideal I of $\mathbb{K}[X]$. If $Y \subset X$ is a subvariety of X, then $\mathcal{I}(X) \subset \mathcal{I}(Y)$ and so $\mathcal{I}(Y)/\mathcal{I}(X)$ is an ideal in the coordinate ring $\mathbb{K}[X] = \mathbb{K}[x]/\mathcal{I}(X)$ of X. (Recall that from abstract algebra, ideals of a quotient ring R/I have the form J/I, where J is an ideal of R which contains I.) Write $\mathcal{I}(Y) \subset \mathbb{K}[X]$ for the ideal of Y in $\mathbb{K}[X]$.

Both Hilbert's Basis Theorem and Hilbert's Nullstellensätze have analogs for affine varieties X and their coordinate rings $\mathbb{K}[X]$. These consequences of the original Hilbert Theorems follow from the surjection $\mathbb{K}[x] \twoheadrightarrow \mathbb{K}[X]$ and corresponding inclusion $X \hookrightarrow \mathbb{K}^n$.

Theorem 1.3.5 (Hilbert Theorems for $\mathbb{K}[X]$)**.** *Let X be an affine variety. Then*

1. *Any ideal of $\mathbb{K}[X]$ is finitely generated.*
2. *If Y is a subvariety of X then $\mathcal{I}(Y) \subset \mathbb{K}[X]$ is a radical ideal.*
3. *Suppose that $\mathbb{K}$ is algebraically closed. An ideal I of $\mathbb{K}[X]$ defines the empty set if and only if $I = \mathbb{K}[X]$.*

As in Section 1.2 we obtain a version of the algebra-geometry dictionary between subvarieties of an affine variety X and radical ideals of $\mathbb{K}[X]$. The proofs are nearly the same, and we leave them to you in Exercise 2.

Theorem 1.3.6. *Let X be an affine variety. Then the maps $\mathcal{V}$ and $\mathcal{I}$ give an inclusion reversing correspondence*

$$\{\textit{Radical ideals } I \textit{ of } \mathbb{K}[X]\} \quad \underset{\mathcal{I}}{\overset{\mathcal{V}}{\rightleftarrows}} \quad \{\textit{Subvarieties } Y \textit{ of } X\} \tag{1.6}$$

with $\mathcal{I}$ injective and $\mathcal{V}$ surjective. When $\mathbb{K}$ is algebraically closed, the maps $\mathcal{V}$ and $\mathcal{I}$ are inverse bijections.

We enrich this correspondence by studying maps between varieties.

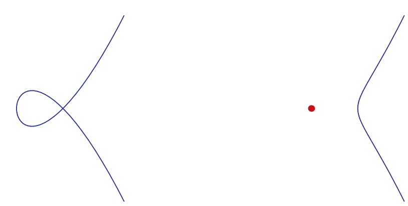
Definition 1.3.7. A list $f_1, \ldots, f_m \in \mathbb{K}[X]$ of regular functions on an affine variety X defines a function

$$\begin{array}{rcl} \varphi : X & \longrightarrow & \mathbb{K}^m \\ x & \longmapsto & (f_1(x), f_2(x), \ldots, f_m(x)), \end{array}$$

which we call a *regular map*. ◇

Example 1.3.8. The elements $t^2, t, -t^3 \in \mathbb{K}[t]$ define the map $\mathbb{K}^1 \to \mathbb{K}^3$ whose image is the cubic curve of Figure 1.2. The elements t^2, t^3 of $\mathbb{K}[t]$ define a map $\mathbb{K}^1 \to \mathbb{K}^2$ whose image is the cuspidal cubic that we saw in Example 1.3.1.

Let $x = t^2 - 1$ and $y = t^3 - t$, which are elements of $\mathbb{K}[t]$. These define a map $\mathbb{K}^1 \to \mathbb{K}^2$ whose image is the nodal cubic curve $\mathcal{V}(y^2 - (x^3 + x^2))$ on the left below. If we instead take $x = t^2 + 1$ and $y = t^3 + t$, then we get a different map $\mathbb{K}^1 \to \mathbb{K}^2$ whose image is the curve $\mathcal{V}(y^2 - (x^3 - x^2))$ on the right below. Both are singular at the origin.



In the curve on the right, the image of $\mathbb{R}^1$ is the arc, while the isolated or *solitary point* is the image of the points $\pm\sqrt{-1}$.

Another regular map is matrix multiplication, $\mathbb{K}^{m\times n} \times \mathbb{K}^{n\times p} \to \mathbb{K}^{m\times p}$, because the product of two matrices $(a_{i,j}) \in \mathbb{K}^{m\times n}$ and $(b_{k,l}) \in \mathbb{K}^{n\times p}$ is the matrix in $\mathbb{K}^{m\times p}$ whose (i,l)-entry is $\sum_{j=1}^{n} a_{i,j} b_{j,l}$. Similarly, Cramer's rule (Exercise 9) shows that operation of taking the inverse of a matrix is a regular map from $GL_n(\mathbb{K})$ to itself. ◇

Suppose that X is an affine variety and we have a regular map $\varphi \colon X \to \mathbb{K}^m$ given by regular functions $f_1, \ldots, f_m \in \mathbb{K}[X]$. A polynomial $g \in \mathbb{K}[x_1, \ldots, x_m]$ *pulls back* along φ to give the regular function $\varphi^* g$, which is defined by

$$\varphi^* g \ := \ g(f_1, \ldots, f_m).$$

This element of the coordinate ring $\mathbb{K}[X]$ of X is the usual pull back of a function. For $x \in X$ we have

$$(\varphi^* g)(x) \ = \ g(\varphi(x)) \ = \ g(f_1(x), \ldots, f_m(x)).$$

The resulting map $\varphi^* \colon \mathbb{K}[x_1, \ldots, x_m] \to \mathbb{K}[X]$ is a homomorphism of $\mathbb{K}$-algebras. Conversely, given a homomorphism $\psi \colon \mathbb{K}[x_1, \ldots, x_m] \to \mathbb{K}[X]$ of $\mathbb{K}$-algebras, if we set $f_i := \psi(x_i)$, then $f_1, \ldots, f_m \in \mathbb{K}[X]$ define a regular map φ with $\varphi^* = \psi$.

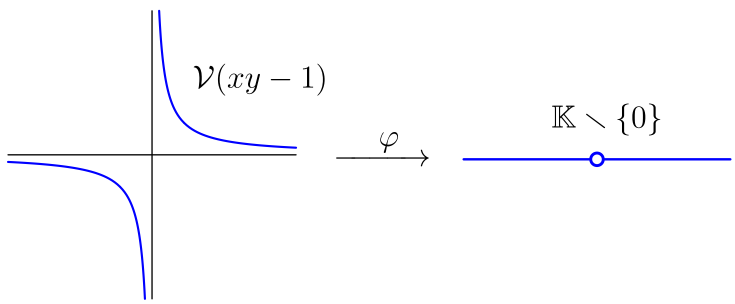
We have just shown the following basic fact.

Lemma 1.3.9. *The association $\varphi \mapsto \varphi^*$ defines a bijection*

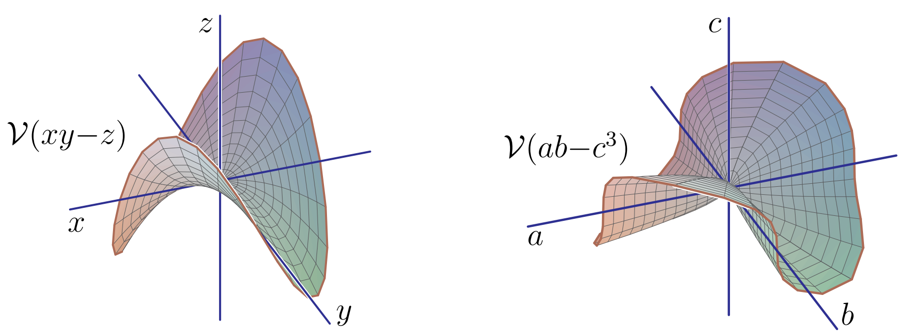
$$\left\{\begin{array}{c}\textit{Regular maps}\\ \varphi\colon X \to \mathbb{K}^m\end{array}\right\} \quad \longleftrightarrow \quad \left\{\begin{array}{c}\mathbb{K}\textit{-algebra homomorphisms}\\ \psi\colon \mathbb{K}[x_1,\ldots,x_m] \to \mathbb{K}[X]\end{array}\right\}$$

In each of the regular maps of Example 1.3.8, the image $\varphi(X)$ of X under φ was equal to a subvariety. This is not always the case.

Example 1.3.10. Let $X = \mathcal{V}(xy-1)$ be the hyperbola in $\mathbb{K}^2$ and $\varphi\colon \mathbb{K}^2 \to \mathbb{K}$ the map which forgets the second coordinate. Then $\varphi(X) = \mathbb{K} \smallsetminus \{0\} \subsetneq \mathbb{K}$.



For a more interesting example, let $X = \mathcal{V}(xy-z) \subset \mathbb{K}^3$, the hyperbolic paraboloid. Consider the map $\varphi\colon X \to \mathbb{K}^3$ given by the three regular functions on X which are the images in $\mathbb{K}[X]$ of yx, xz, xy. Let (a,b,c) be coordinates for the image $\mathbb{K}^3$. Then $\varphi^*(a) = yz$, $\varphi^*(b) = xz$, and $\varphi^*(c) = xy = z$, as $xy = z$ in $\mathbb{K}[X]$. But then $\varphi^*(ab-c^3) = xyz^2 - z^3 = 0$ as again $xy = z$ in $\mathbb{K}[X]$. Consequently, $\varphi(X) \subset \mathcal{V}(ab-c^3)$. We show these two varieties $\mathcal{V}(xy-z)$ and $\mathcal{V}(ab-c^3)$.

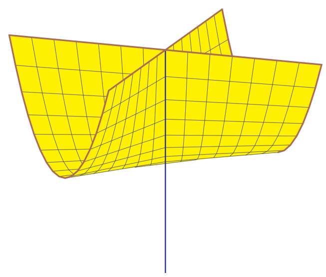


We do not have $\varphi(X) = \mathcal{V}(ab-c^3)$. Let $(a,b,c) \in \mathcal{V}(ab-c^3)$. If $c \neq 0$, then you may check that $(b/c, a/c, c) \in \mathcal{V}(xy-z)$, and $\varphi(b/c, a/c, c) = (a,b,c)$. However, if $c = 0$, then either $a = 0$ or $b = 0$. If $(a,b) \neq (0,0)$, then the point (a,b,c) does not lie in the image of φ, but $(0,0,0) = \varphi(0,0,0)$. Thus the image of X under φ is the complement of the a- and b-axes in $\mathcal{V}(ab-c^3)$, together with the origin. This image is neither a subvariety, nor the complement of a subvariety. $\diamond$

For the rest of this section, suppose that $\mathbb{K}$ is algebraically closed. Impose this condition more surgically.

Lemma 1.3.11. *Let X be an affine variety, $\varphi\colon X \to \mathbb{K}^m$ a regular map, and $Y \subset \mathbb{K}^m$ a subvariety. Then $\varphi(X) \subset Y$ if and only if $\mathcal{I}(Y) \subset \ker\varphi^*$.*

In particular, $\mathcal{V}(\ker\varphi^*)$ is the smallest subvariety of $\mathbb{K}^m$ that contains the image $\varphi(X)$ of X under φ. We call this the subvariety of $\mathbb{K}^m$ *parameterized* by φ. Determining its ideal $\ker\varphi^* \subset \mathbb{K}[y_1,\dots,y_m]$ is called the implictization problem as it seeks implicit equations that define the the image $\varphi(X) \subset \mathbb{K}^m$. For example, consider the map $\mathbb{K}^2 \to \mathbb{K}^3$ defined by $\varphi(u,v) = (uv, v, v^2)$. Its image is the Whitney umbrella



which has implicit equation $x^2 - y^2 z$, where (x,y,z) are the coordinates of $\mathbb{K}^3$.

Proof of Lemma 1.3.11. First suppose that $\varphi(X) \subset Y$. If $f \in \mathcal{I}(Y)$ then f vanishes on Y and hence on $\varphi(X)$. But then $\varphi^* f$ is the zero function, and so $\mathcal{I}(Y) \subset \ker\varphi^*$.

For the other direction, suppose that $\mathcal{I}(Y) \subset \ker\varphi^*$ and let $x \in X$. If $f \in \mathcal{I}(Y)$, then $\varphi^* f = 0$ and so $0 = \varphi^* f(x) = f(\varphi(x))$. This implies that $\varphi(x) \in Y$, and so we conclude that $\varphi(X) \subset Y$. $\square$

Definition 1.3.12. Affine varieties X and Y are *isomorphic* if there are regular maps $\varphi\colon X \to Y$ and $\psi\colon Y \to X$ such that both $\varphi \circ \psi$ and $\psi \circ \varphi$ are the identity maps on Y and X, respectively. In this case, we say that φ and ψ are isomorphisms. $\diamond$

Corollary 1.3.13. *Let X be an affine variety, $\varphi\colon X \to \mathbb{K}^m$ a regular map, and $Y \subset \mathbb{K}^m$ a subvariety. Then*

(1) $\ker\varphi^*$ *is a radical ideal.*

(2) $\mathcal{V}(\ker\varphi^*)$ *is the smallest affine variety containing $\varphi(X)$.*

(3) *If $\varphi\colon X \to Y$, then $\varphi^*\colon \mathbb{K}[x_1,\dots,x_m] \to \mathbb{K}[X]$ factors through $\mathbb{K}[Y]$ inducing a homomorphism $\varphi^*\colon \mathbb{K}[Y] \to \mathbb{K}[X]$.*

(4) *φ is an isomorphism of varieties if and only if $\varphi^*\colon \mathbb{K}[Y] \to \mathbb{K}[X]$ is an isomorphism of $\mathbb{K}$-algebras.*

(5) *$\varphi^{-1}(Y) = \mathcal{V}(\varphi^*\mathcal{I}(Y))$, and if $Z \subset X$ is a subvariety, then $\mathcal{I}(\varphi(Z)) = (\varphi^*)^{-1}\mathcal{I}(Z)$.*

Proof. For (1), suppose that $f^N \in \ker \varphi^*$, so that $0 = \varphi^*(f^N) = (\varphi^*(f))^N$. Since $\mathbb{K}[X]$ has no nilpotent elements, we conclude that $\varphi^*(f) = 0$ and so $f \in \ker \varphi^*$.

Suppose that Y is an affine variety containing $\varphi(X)$. By Lemma 1.3.11, $\mathcal{I}(Y) \subset \ker \varphi^*$ and so $\mathcal{V}(\ker \varphi^*) \subset Y$. Statement (2) follows as we also have $X \subset \mathcal{V}(\ker \varphi^*)$.

For (3), we have $\mathcal{I}(Y) \subset \ker \varphi^*$ and so the map $\varphi^* \colon \mathbb{K}[x_1, \ldots, x_m] \to \mathbb{K}[X]$ factors through the quotient map $\mathbb{K}[x_1, \ldots, x_m] \twoheadrightarrow \mathbb{K}[x_1, \ldots, x_m]/\mathcal{I}(Y) = \mathbb{K}[Y]$.

Statement (4) is immediate from the definitions.

For (5), observe that $x \in \varphi^{-1}(Y)$ if and only if $\varphi(x) \in Y$, if and only if $0 = f(\varphi(x)) = \varphi^* f(x)$ for all $f \in \mathcal{I}(Y)$. By part (2), $\mathcal{I}(\varphi(Z))$ is the kernel of the composition of φ^* with the surjection $\mathbb{K}[X] \twoheadrightarrow \mathbb{K}[Z] = \mathbb{K}[X]/\mathcal{I}(Z)$, which is $\mathcal{I}(Z)$. $\square$

Thus we may refine the correspondence of Lemma 1.3.9. Let X and Y be affine varieties. Then the association $\varphi \mapsto \varphi^*$ gives a bijective correspondence

$$\left\{ \begin{array}{c} \text{Regular maps} \\ \varphi \colon X \to Y \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \mathbb{K}\text{-algebra homomorphisms} \\ \psi \colon \mathbb{K}[Y] \to \mathbb{K}[X] \end{array} \right\} .$$

This map $X \mapsto \mathbb{K}[X]$ from affine varieties to finitely generated reduced $\mathbb{K}$-algebras not only sends objects to objects, but it induces an isomorphism on maps between objects (reversing their direction however). In mathematics, such an association is called a *contravariant equivalence of categories*. The point here of this equivalence is that an affine variety and its coordinate ring are different packages for the same information. Each one determines and is determined by the other. Whether we study algebra or geometry, we are studying the same thing.

As observed in Example 1.3.10, the image of a variety under a regular map need not be a variety. We consider a class of maps that sends varieties to varieties.

Let $X \subset \mathbb{K}^n$ be a variety and $\varphi \colon X \to \mathbb{K}^m$ be a regular map. Set Y to be the smallest variety containing $\varphi(X)$. By construction, the map $\varphi^* \colon \mathbb{K}[Y] \to \mathbb{K}[X]$ is an injection. Identifying $\mathbb{K}[Y]$ with the image of φ^*, we may consider $\mathbb{K}[Y] \subset \mathbb{K}[X]$. We say that the map $\varphi \colon X \to Y$ is *finite* if there exist $u_1, \ldots, u_s \in \mathbb{K}[X]$ such that

$$\mathbb{K}[x] \;=\; u_1\mathbb{K}[Y] + u_2\mathbb{K}[Y] + \cdots + u_s\mathbb{K}[Y] \,. \tag{1.7}$$

That is, every element of $\mathbb{K}[X]$ is a $\mathbb{K}[Y]$-linear combination of $u_1, \ldots, u_s$. In other words, $\mathbb{K}[X]$ is finitely generated as a $\mathbb{K}[Y]$-module. (See A.1.3 in the Appendix.) We present the main consequence of finite maps.

Theorem 1.3.14. *A finite map $\varphi \colon X \to Y$ of affine varieties is surjective. If $Z \subset X$ is a subvariety, then $\varphi(Z)$ is a subvariety of Y.*

The second statement is left to you as Exercise 8. Before proving this theorem, we explain why such a map is called finite.

Corollary 1.3.15. *Let $\varphi \colon X \to Y$ be a finite map. Then every fiber $\varphi^{-1}(y)$ for $y \in Y$ is a nonempty finite set.*

Proof. Let $t \in \mathbb{K}[X]$. Since $\mathbb{K}[X]$ is finitely generated as a module over $\mathbb{K}[Y]$, it is Noetherian and there is a number $k \geq 1$ such that t^k lies in the submodule generated by $1, t, \ldots, t^{k-1}$. (Indeed, for $i \in \mathbb{N}$, let M_i be the $\mathbb{K}[Y]$-submodule generated by $1, t, \ldots, t^i$. Then $M_0 \subset M_1 \subset \cdots \subset \mathbb{K}[X]$ is an increasing chain of submodules. As $\mathbb{K}[X]$ is Noetherian, there is some k such that $M_k = M_{k-1}$.) This implies that there exist $c_0, c_1, \ldots, c_{k-1} \in \mathbb{K}[Y]$ such that in $\mathbb{K}[X]$ we have

$$t^k + c_{k-1}t^{k-1} + \cdots + c_1 t + c_0 \;=\; 0\,.$$

For $y \in Y$ the value of t at any point $x \in \varphi^{-1}(y)$ is solution of the equation

$$t^k + c_{k-1}(y)t^{k-1} + \cdots + c_1(y)t + c_0(y) \;=\; 0\,. \tag{1.8}$$

That is, t takes on only finitely many values on $\varphi^{-1}(y)$. As $X \subset \mathbb{K}^n$, doing this for all n coordinate functions shows that the fiber $\varphi^{-1}(y)$ is finite. $\square$

This proof illustrates a useful characterization of finite extensions $\mathbb{K}[Y] \subset \mathbb{K}[X]$. Suppose that $S \subset R$ are $\mathbb{K}$-algebras. An element $t \in R$ is *integral* over S if there are $c_1, \ldots, c_k \in S$ such that

$$t^k + c_1 t^{k-1} + c_2 t^{k-2} + \cdots + c_{k-1}t + c_k \;=\; 0\,.$$

That is, t satisfies a monic polynomial equation with coefficients in S. A map $\varphi\colon X \to Y$ is finite if and only if $\varphi^*\colon \mathbb{K}[Y] \to \mathbb{K}[X]$ is an injection and every element $t \in \mathbb{K}[X]$ is integral over $\mathbb{K}[Y]$. Corollary 1.3.15 gives one direction, the other is discussed in the Appendix A.1.3.

Another meaningful interpretation of the adjective finite is that as y moves in Y, none of the points of $\varphi^{-1}(y)$ may disappear by going to infinity as in Example 1.3.10. Indeed, if t is a coordinate function on $\mathbb{K}^n$, then on $\varphi^{-1}(y)$ it satisfies (1.8), and no root of this polynomial can tend to infinity as the coefficient of the leading term is 1. Consequently, as y moves in Y, the points in the fiber may merge, but they will not disappear.

Typical proofs of Theorem 1.3.14 use standard results in commutative algebra, such as Nakayama's Lemma. To stress the elementary nature of the argument, we give a complete treatment. Let us first recall Cramer's rule, which is a consequence of the standard formula for determinant. Let R be a ring and $M \in \mathrm{Mat}_{n\times n}(R)$, an $n \times n$ matrix with entries from R. We define the determinant of $M = (m_{i,j})_{i,j=1}^n$ by the usual formula,

$$\det(M) \;:=\; \sum_{\pi \in S_n} \mathrm{sgn}(\pi) m_{1,\pi(1)} \cdot m_{2,\pi(2)} \cdots m_{n,\pi(n)}\ , \tag{1.9}$$

where S_n is the group of permutations of $[n] := \{1, \ldots, n\}$, and for $\pi \in S_n$, its sign is $\mathrm{sgn}(\pi) := (-1)^{\ell(\pi)}$, where

$$\ell(\pi) \;:=\; \#\{i < j \mid \pi(i) > \pi(j)\}\,.$$

For $i, j \in [n]$, let $\widehat{M}_{i,j}$ be the matrix obtained from M by deleting its ith row and jth column. Define the *adjoint* of the matrix M to be $\mathrm{adj}M \in \mathrm{Mat}_{n\times n}(R)$ to be the matrix whose (i,j)-th entry is

$$\mathrm{adj}M_{i,j} \ := \ (-1)^{i+j}\det(\widehat{M}_{j,i})\,.$$

Exercise 9 asks you to prove Cramer's rule,

$$\mathrm{adj}M \cdot M \ = \ \det(M) I_n\,. \tag{1.10}$$

Here I_n is the identity matrix. When $\det(M)$ is invertible, this gives a formula for the inverse of a matrix M.

The next step is a result from commutative algebra.

Lemma 1.3.16. *Let $S \subset R$ be $\mathbb{K}$-algebras and suppose that R is finitely generated as an S-module. If $I \subsetneq S$ is a proper ideal of S, then $IR \neq R$.*

Proof. There exist $r_1, \ldots, r_n \in R$ such that $R = r_1S + \cdots + r_nS$. Suppose that $IR = R$. Then there are elements $m_{i,j} \in I$ for $i, j \in [n]$ such that for each $i = 1, \ldots, n$,

$$r_i \ = \ \sum_{j=1}^{n} m_{i,j} r_j\,. \tag{1.11}$$

Writing M for the matrix $(m_{i,j})_{i,j=1}^n$ and $\vec{r}$ for the vector $(r_1, \ldots, r_n)^T$, (1.11) becomes $(I_n - M)\vec{r} = 0$ in R^n.

Applying Cramer's rule to the matrix $I_n - M$ and writing μ for $\det(I_n - M)$, we have $\mu\vec{r} = 0$. That is, $\mu r_i = 0$ for each i, and so $\mu R = 0$. As $1 \in R$ (it is a $\mathbb{K}$-algebra), this implies that $\mu = 0$. As $m_{i,j} \in I$, the formula (1.9) for $\det(I_n - M)$ shows that $\mu = 1 + m$ for some $m \in I$, which implies that $-1 \in I$ and so $I = S$, a contradiction. $\square$

Proof of Theorem 1.3.14. Let $\varphi\colon X \to Y$ be a finite map of varieties. Then $\mathbb{K}[X]$ is a finitely generated $\mathbb{K}[Y]$-module. Let $y \in Y$ and $\mathfrak{m}_y \subset \mathbb{K}[Y]$ its (maximal) ideal. By Corollary 1.3.13(5), the ideal of $\varphi^{-1}(y)$ is $\mathfrak{m}_y\mathbb{K}[X]$. By Lemma 1.3.16, $\mathfrak{m}_y\mathbb{K}[X] \neq \mathbb{K}[X]$, as $\mathfrak{m}_y \neq \mathbb{K}[Y]$. By the Nullstellensatz, $\emptyset \neq \mathcal{V}(\mathfrak{m}_y\mathbb{K}[X]) = \varphi^{-1}(y)$, which completes the proof. $\square$

Exercises

1. Suppose that $\mathbb{K}$ is an infinite field. Show that $f \in \mathbb{K}[x_1, \ldots, x_n]$ defines the zero function $f\colon \mathbb{K}^n \to \mathbb{K}$ if and only if f is the zero polynomial. (Hint: For the interesting direction, consider first the case when $n = 1$ and then use induction.)

2. Give a proof of Theorem 1.3.5.

3. Let $V = \mathcal{V}(y - x^2) \subset \mathbb{K}^2$ and $W = \mathcal{V}(xy - 1) \subset \mathbb{K}^2$. Show that

$$\begin{aligned}\mathbb{K}[V] &:= \mathbb{K}[x, y]/\mathcal{I}(V) \cong \mathbb{K}[t]\\ \mathbb{K}[W] &:= \mathbb{K}[x, y]/\mathcal{I}(W) \cong \mathbb{K}[t, t^{-1}]\end{aligned}$$

   Conclude that the hyperbola $V(xy - 1)$ is not isomorphic to the affine line.

4. Let $I \subset \mathbb{K}[x_1, \ldots, x_n]$ be an ideal. Show that the quotient ring $\mathbb{K}[x_1, \ldots, x_n]/I$ has nilpotent elements if and only if I is not a radical ideal.

5. Suppose that $I \subset \mathbb{K}[x_1, \ldots, x_n]$ is a radical ideal and that $X := \mathcal{V}(I)$ is a finite set. Prove that the restriction of polynomial functions to X is a surjective map from the ring of polynomials $\mathbb{K}[x_1, \ldots, x_n]$ to the finite vector space of functions from $X \to \mathbb{K}$.

6. Verify the claims about the parametrizations in Example 1.3.8, that the image of $\mathbb{K}$ under the map $t \mapsto (t^2 - 1, t^3 - t)$ is $\mathcal{V}(y^2 - (x^3 + x^2))$ and its image under $t \mapsto (t^2 + 1, t^3 + t)$ is $\mathcal{V}(y^2 - (x^3 - x^2))$.

7. Show that $A \mapsto A^{-1}$ is a regular map on $GL_m(\mathbb{K})$. (You may need Exercise 9.)

8. Prove the second statement of Theorem 1.3.14, by using (1.7) and Corollary 1.3.13(5) to show that $\mathbb{K}[Z]$ is finitely generated as a module over an appropriate subalgebra.

9. Prove Cramer's rule (1.10). You may use the formula (1.9) for the determinant, or any other properties of determinant. Under what conditions does this give a formula for the inverse of a matrix?

1.4 Projective varieties

Projective space and projective varieties are of central importance in algebraic geometry. We motivate projective space with an example.

Example 1.4.1. Consider the intersection of the parabola $y = x^2$ in the affine plane $\mathbb{K}^2$ with a line, $\ell := \mathcal{V}(ay + bx + c)$. Solving these implied equations gives

$$ax^2 + bx + c = 0 \qquad \text{and} \qquad y = x^2 . \tag{1.12}$$

There are several cases to consider, illustrated below (1.13).

(i) $a \neq 0$ and $b^2 - 4ac > 0$. Then ℓ meets the parabola in two distinct real points.

(i′) $a \neq 0$ and $b^2 - 4ac < 0$. While ℓ does not appear to meet the parabola, that is because we have drawn the picture in $\mathbb{R}^2$. In $\mathbb{C}^2$, ℓ meets it in two complex conjugate points.

When $\mathbb{K}$ is algebraically closed, then cases (i) and (i′) coalesce to the case of $a \neq 0$ and $b^2 - 4ac \neq 0$. These two points of intersection are predicted by Bézout's Theorem in the plane (Theorem 2.1.17).

(ii) $a \neq 0$ but $b^2 - 4ac = 0$. Then ℓ is tangent to the parabola and we solve the equations (1.12) to get

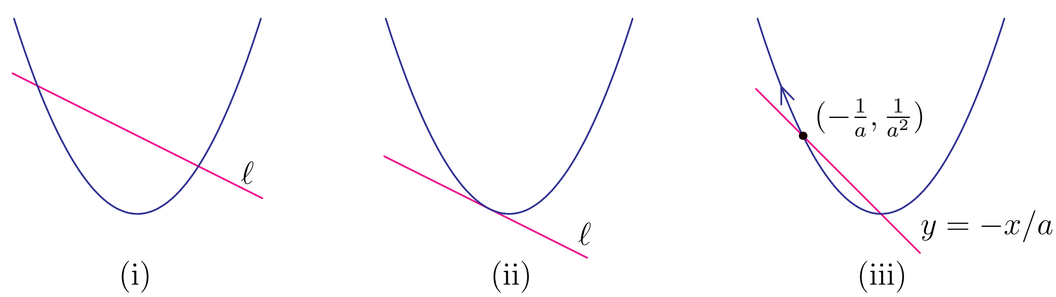
$$a(x - \tfrac{b}{2a})^2 \;=\; 0 \qquad \text{and} \qquad y \;=\; x^2\,.$$

Thus there is one solution, $(\frac{b}{2a}, \frac{b^2}{4a^2})$. As $x = \frac{b}{2a}$ is a root of multiplicity 2 in the first equation, it is reasonable to say that this one solution to our geometric problem occurs with multiplicity 2.

(iii) $a = 0$, so that the line ℓ is vertical. There is a single, unique solution, $x = -c/b$ and $y = c^2/b^2$.

Let us examine this passage to a vertical line. Suppose now that $c = 0$ and let $b = 1$. For $a \neq 0$, there are two solutions $(0,0)$ and $(-\frac{1}{a}, \frac{1}{a^2})$. In the limit as $a \to 0$, the second solution disappears off to infinity.

We illustrate these three possibilities.



(1.13)

One purpose of projective space is to prevent this last phenomenon from occurring. ◇

Definition 1.4.2. The set of all 1-dimensional linear subspaces of $\mathbb{K}^{n+1}$ is called *n-dimensional projective space* and written $\mathbb{P}^n$ or $\mathbb{P}^n_{\mathbb{K}}$. If V is a finite-dimensional vector space, then $\mathbb{P}(V)$ is the set of all 1-dimensional linear subspaces of V. Note that $\mathbb{P}(V) \simeq \mathbb{P}^{\dim V - 1}$. If $V \subset \mathbb{K}^{n+1}$ is a linear subspace, then $\mathbb{P}(V) \subset \mathbb{P}^n$ is a *linear subspace* of $\mathbb{P}^n$. ◇

Example 1.4.3. The projective line $\mathbb{P}^1$ is the set of lines through the origin in $\mathbb{K}^2$. When $\mathbb{K} = \mathbb{R}$, the line $x = ay$ through the origin intersects the circle $\mathcal{V}(x^2 + (y-1)^2 - 1)$ in the origin and in the second point $(\frac{2a}{1+a^2}, \frac{2}{1+a^2})$, as shown in Figure 1.4. Identifying the nonhorizontal line $x = ay$ with this point $(\frac{2a}{1+a^2}, \frac{2}{1+a^2})$ and the horizontal x-axis with the origin, this identifies $\mathbb{P}^1_{\mathbb{R}}$ with the circle. ◇

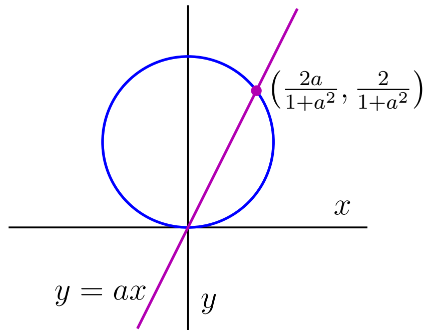


Figure 1.4: Lines through the origin meet the circle in a second point.

This definition of $\mathbb{P}^n$ leads to a system of coordinates for $\mathbb{P}^n$. We may represent a point, ℓ, of $\mathbb{P}^n$ by the coordinates $[a_0, a_1, \ldots, a_n]$ of any non-zero vector lying on the one-dimensional linear subspace $\ell \subset \mathbb{K}^{n+1}$. These coordinates are not unique. If $\lambda \neq 0$, then $[a_0, a_1, \ldots, a_n]$ and $[\lambda a_0, \lambda a_1, \ldots, \lambda a_n]$ both represent the same point. This non-uniqueness is the reason that we use rectangular brackets $[\ldots]$ in our notation for these *homogeneous coordinates*. Some authors prefer the notation $[a_0 : a_1 : \cdots : a_n]$.

Example 1.4.4. When $\mathbb{K} = \mathbb{R}$, observe that a 1-dimensional subspace of $\mathbb{R}^{n+1}$ meets the unit sphere S^n in two antipodal points, v and $-v$. The group $S^0 = \{-1, 1\}$ of real numbers of absolute value 1 acts on S^n by scalar multiplication interchanging antipodal points. This identifies real projective space $\mathbb{P}^n_{\mathbb{R}}$ with the quotient $S^n/\{\pm 1\}$, showing that $\mathbb{P}^n_{\mathbb{R}}$ is a compact manifold in the usual (Euclidean) topology.

Suppose that $\mathbb{K} = \mathbb{C}$. Given a point $a \in \mathbb{P}^n_{\mathbb{C}}$, after scaling, we may assume that $|a_0|^2 + |a_1|^2 + \cdots + |a_n|^2 = 1$. Identifying $\mathbb{C}$ with $\mathbb{R}^2$, this is the set of points a on the $(2n+1)$-sphere $S^{2n+1} \subset \mathbb{R}^{2n+2}$. If $[a_0, \ldots, a_n] = [b_0, \ldots, b_n]$ with $a, b \in S^{2n+1}$, then there is some $\zeta \in S^1$, the unit circle in $\mathbb{C}$, such that $a_i = \zeta b_i$. This identifies $\mathbb{P}^n_{\mathbb{C}}$ with the quotient of S^{2n+1}/S^1, showing that $\mathbb{P}^n_{\mathbb{C}}$ is a compact manifold. This is a version of the Hopf fibration. Since $\mathbb{P}^n_{\mathbb{R}} \subset \mathbb{P}^n_{\mathbb{C}}$, we again see that $\mathbb{P}^n_{\mathbb{R}}$ is compact. $\diamond$

Homogeneous coordinates of a point are not unique. Uniqueness may be restored, but at the price of non-uniformity. Let $A_i \subset \mathbb{P}^n$ be the set of points $[a_0, a_1, \ldots, a_n]$ in projective space $\mathbb{P}^n$ with $a_i \neq 0$, but $a_{i+1} = \cdots = a_n = 0$. Given a point $a \in A_i$, we may divide by its ith coordinate to get a representative of the form $[a_0, \ldots, a_{i-1}, 1, 0, \ldots, 0]$. These i numbers $(a_0, \ldots, a_{i-1})$ provide coordinates for A_i, identifying it with the affine space $\mathbb{K}^i$. This decomposes projective space $\mathbb{P}^n$ into a disjoint union of $n+1$ affine spaces

$$\mathbb{P}^n \;=\; \mathbb{K}^n \sqcup \cdots \sqcup \mathbb{K}^1 \sqcup \mathbb{K}^0 .$$

When a variety admits a decomposition as a disjoint union of affine spaces, we say that it is *paved by affine spaces*. Many important varieties admit such a decomposition, such as the Grassmannians of Section 10.1.

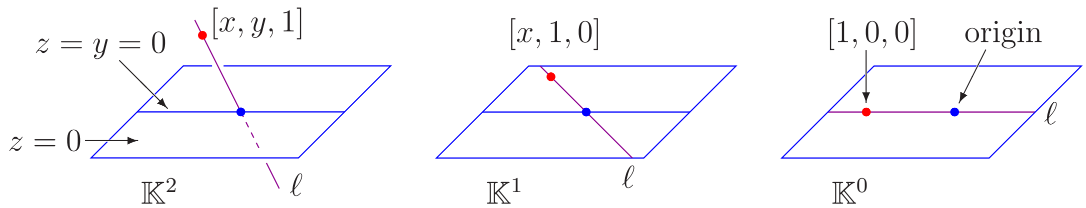


Figure 1.5: Affine paving of $\mathbb{P}^2$.

It is instructive to look at this closely for $\mathbb{P}^2$. Figure 1.5 shows the possible positions of a one-dimensional linear subspace $\ell \subset \mathbb{K}^3$ with respect to the x, y-plane $z = 0$, the x-axis $z = y = 0$, and the origin in $\mathbb{K}^3$. Note that the last two charts give $\mathbb{P}^1$, so we have $\mathbb{P}^2 = \mathbb{K}^2 \sqcup \mathbb{P}^1$, which is the familiar decomposition of the projective plane as the plane plus the line at infinity.

Projective space also admits systems of local coordinates. For $i = 0, \ldots, n$, let U_i be the set of points $a \in \mathbb{P}^n$ in projective space whose ith coordinate is non-zero. Dividing by this ith coordinate, we obtain a representative of the point having the form

$$[a_0, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n]\,.$$

The n coordinates $(a_0, \ldots, a_{i-1},\ a_{i+1}, \ldots, a_n)$ determine this point, identifying U_i with affine n-space, $\mathbb{K}^n$. Geometrically, U_i is the set of lines in $\mathbb{K}^{n+1}$ that meet the affine plane defined by $x_i = 1$, with the point of intersection identifying U_i with this affine plane. Every point of $\mathbb{P}^n$ lies in some U_i, so that we have

$$\mathbb{P}^n \;=\; U_0 \cup U_1 \cup \cdots \cup U_n\,.$$

When $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, these U_i are coordinate charts for $\mathbb{P}^n$ as a manifold. For any field $\mathbb{K}$, these affine sets U_i provide coordinate charts for $\mathbb{P}^n$.

These affine charts have a coordinate-free description. Let $\Lambda\colon \mathbb{K}^{n+1} \to \mathbb{K}$ be a linear map, and let $H \subset \mathbb{K}^{n+1}$ be the set $\{x \in \mathbb{K}^{n+1} \mid \Lambda(x) = 1\}$. Then $H \simeq \mathbb{K}^n$, and the map

$$H \ni x \;\longmapsto\; [x] \in \mathbb{P}^n$$

identifies H with the complement $U_\Lambda := \mathbb{P}^n - \mathbb{P}(\mathcal{V}(\Lambda))$ of the hyperplane defined by Λ.

Example 1.4.5 (Probability simplex)**.** This second and more general description of affine charts leads to an application of algebraic geometry to statistics. Here $\mathbb{K} = \mathbb{R}$, the real numbers and we set $\Lambda(x) := x_0 + \cdots + x_n$. If we consider those points x where $\Lambda(x) = 1$ which have nonnegative coordinates, we obtain the *probability simplex*

$$\triangle^n \;:=\; \{(p_0, p_1, \ldots, p_n) \in \mathbb{R}^{n+1}_+ \mid p_0 + p_1 + \cdots + p_n = 1\}\,,$$

where $\mathbb{R}^{n+1}_+$ is the *nonnegative orthant*, the points of $\mathbb{R}^{n+1}$ with nonnegative coordinates. Here p_i represents the probability that event i occurs, and the condition $p_0 + \cdots + p_n = 1$ reflects that every event does occur. Figure 1.6 shows this when $n = 2$. $\diamond$

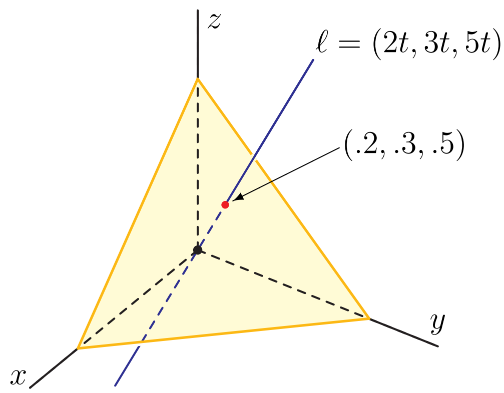


Figure 1.6: Probability simplex when $n = 2$.

We wish to extend the definitions and structures of affine algebraic varieties to projective space. One problem arises immediately: given a polynomial $f \in \mathbb{K}[x_0, \ldots, x_n]$ and a point $a \in \mathbb{P}^n$, we cannot in general define $f(a) \in \mathbb{K}$. To see why this is the case, for each natural number d, let f_d be the sum of the terms of f of degree d. We call f_d the dth *homogeneous component* of f. If $[a_0, \ldots, a_n]$ and $[\lambda a_0, \ldots, \lambda a_n]$ are two representatives of a point $a \in \mathbb{P}^n$, and f has degree m, then

$$f(\lambda a_0, \ldots, \lambda a_n) \;=\; f_0(a_0, \ldots, a_n) + \lambda f_1(a_0, \ldots, a_n) + \cdots + \lambda^m f_m(a_0, \ldots, a_n)\,, \qquad (1.14)$$

since we can factor λ^d from every monomial $(\lambda x)^\alpha$ of degree d. Thus $f(a)$ is a well-defined number only if the polynomial (1.14) in λ is constant. That is, if and only if

$$f_i(a_0, \ldots, a_n) \;=\; 0 \qquad i = 1, \ldots, \deg(f)\,.$$

For a particular case, observe that a polynomial f vanishes at a point $a \in \mathbb{P}^n$ if and only if every homogeneous component f_d of f vanishes at a. A polynomial f is *homogeneous* of degree d when $f = f_d$. We also use the term *form* for a homogeneous polynomial.

Definition 1.4.6. Let $f_1, \ldots, f_m \in \mathbb{K}[x_0, \ldots, x_n]$ be forms. These define a *projective variety*

$$\mathcal{V}(f_1, \ldots, f_m) \;:=\; \{a \in \mathbb{P}^n \mid f_i(a) = 0,\ i = 1, \ldots, m\}\,. \qquad \diamond$$

An ideal $I \subset \mathbb{K}[x_0, \ldots, x_n]$ is *homogeneous* if whenever $f \in I$ then all homogeneous components of f lie in I. Thus projective varieties are defined by homogeneous ideals. Given a subset $Z \subset \mathbb{P}^n$ of projective space, its ideal is the collection of polynomials which vanish on Z,

$$\mathcal{I}(Z) \;:=\; \{f \in \mathbb{K}[x_0, x_1, \ldots, x_n] \mid f(z) = 0 \text{ for all } z \in Z\}\,.$$

In Exercise 2, you are asked to show that this ideal is homogeneous.

It is often convenient to work in an affine space when treating projective varieties. The *(affine) cone* $CZ \subset \mathbb{K}^{n+1}$ over a subset Z of projective space $\mathbb{P}^n$ is the union of the one-dimensional linear subspaces $\ell \subset \mathbb{K}^{n+1}$ corresponding to points of Z. The ideal $\mathcal{I}(X)$ of a projective variety X is equal to the ideal $\mathcal{I}(CX)$ of the affine cone over X.

Example 1.4.7. Let $\Lambda := a_0x_0 + a_1x_1 + \cdots + a_nx_n$ be a linear form. Then $\mathcal{V}(\Lambda)$ is a *hyperplane*. Let $V \subset \mathbb{K}^{n+1}$ be the kernel of Λ which is an n-dimensional linear subspace. It is also the affine variety defined by Λ. We have $\mathcal{V}(\Lambda) = \mathbb{P}(V) \subset \mathbb{P}^n$. $\diamond$

Example 1.4.8. Let $[x, y, z]$ be homogeneous coordinates for the projective plane $\mathbb{P}^2$, and consider the two subvarieties $\mathcal{V}(yz - x^2)$ and $\mathcal{V}(x + ay)$. In the affine patch U_z where $z \neq 0$, these subvarieties are the parabola and the line $x = -ay$ of Example 1.4.1. Their intersection, $\mathcal{V}(x + ay, yz - x^2)$, consists of the points $[0, 0, 1]$ and $[-a, 1, a^2]$. We see that as $a \to 0$, the second point approaches $[0, 1, 0]$, and does not “disappear off to infinity” as in Example 1.4.1(iii). $\diamond$

The weak Nullstellensatz does not hold for projective space, as $\mathcal{V}(x_0, x_1, \ldots, x_n) = \emptyset$. We call this ideal, $\mathfrak{m}_0 := \langle x_0, x_1, \ldots, x_n \rangle$, the *irrelevant ideal*.

Lemma 1.4.9. *Let $I \subset \mathbb{K}[x]$ be a homogeneous ideal. Then $\mathcal{V}(I) = \emptyset$ if and only if there is some $d \geq 0$ such that $I \supset \mathfrak{m}_0^d$.*

Proof. Note that $\mathcal{V}(I) = \emptyset$ in projective space if and only if, in the affine cone $\mathbb{K}^{n+1}$ over projective space, we have either $\mathcal{V}(I) = \emptyset$ or $\mathcal{V}(I) = \{0\}$. This is equivalent to either $I = \mathbb{K}[x]$ or $\sqrt{I} = \mathfrak{m}_0$, which is in turn equivalent to $I \supset \mathfrak{m}_0^d$ for some $d \geq 0$. $\square$

The irrelevant ideal plays a special role in the projective algebra-geometry dictionary.

Theorem 1.4.10 (Projective Algebra-Geometry Dictionary)**.** *Over any field $\mathbb{K}$, the maps $\mathcal{V}$ and $\mathcal{I}$ give an inclusion reversing correspondence*

$$\left\{\begin{array}{c}\textit{Radical homogeneous ideals } I \textit{ of}\\ \mathbb{K}[x_0, \ldots, x_n] \textit{ properly contained in } \mathfrak{m}_0\end{array}\right\} \quad \underset{\mathcal{I}}{\overset{\mathcal{V}}{\rightleftarrows}} \quad \{\textit{Subvarieties } X \textit{ of } \mathbb{P}^n\}$$

with $\mathcal{V}(\mathcal{I}(X)) = X$. When $\mathbb{K}$ is algebraically closed, the maps $\mathcal{V}$ and $\mathcal{I}$ are inverses, and this correspondence is a bijection.

This follows from Lemma 1.4.9 and the algebra-geometry dictionary for affine varieties (Corollary 1.2.10), if we replace a subvariety X of projective space by its affine cone CX.

If we relax the condition that an ideal be radical, then the corresponding geometric objects are *projective schemes*. This comes at a price, for many homogeneous ideals will define the same projective scheme (and even the same projective variety), which is not the case for their affine cousins. This non-uniqueness comes from the irrelevant ideal, $\mathfrak{m}_0$. Recall the construction of colon ideals from commutative algebra. Let I be an ideal and

g a polynomial. Then the colon ideal $(I : g)$ is $\{f \mid fg \in I\}$. If J is an ideal, then the *colon ideal* (or *ideal quotient* of I by J) is

$$(I : J) \ := \ \{f \mid fJ \subset I\} \ = \ \bigcap\{(I : g) \mid g \in J\}\,.$$

The *saturation* of I by J is

$$(I : J^\infty) \ = \ \bigcup_{m\geq 0} (I : J^m)\ .$$

One reason for these definitions are the following results for affine varieties.

Lemma 1.4.11. *Let I be an ideal and $g \in \mathbb{K}[x]$ a polynomial. Then $\mathcal{V}(I : g^\infty)$ is the smallest affine variety containing $\mathcal{V}(I) \smallsetminus \mathcal{V}(g)$.*

Proof. First note that as $I \subset (I : g^\infty)$, we have $\mathcal{V}(I : g^\infty) \subset \mathcal{V}(I)$. Let $x \in \mathcal{V}(I) \smallsetminus \mathcal{V}(g)$. If $f \in (I : g^\infty)$, then there is some $m \in \mathbb{N}$ with $fg^m \in I$, so that $fg^m(x) = 0$. Since $x \notin \mathcal{V}(g)$, we conclude that $f(x) = 0$ as $g(x) \neq 0$. Thus $\mathcal{V}(I) \smallsetminus \mathcal{V}(g) \subset \mathcal{V}(I : g^\infty)$.

For the other inclusion, let $x \in \mathcal{V}(I : g^\infty)$. If $g(x) \neq 0$, then $x \in \mathcal{V}(I) \smallsetminus \mathcal{V}(g)$. Suppose now that $g(x) = 0$. Let $f \in \mathcal{I}(\mathcal{V}(I) \smallsetminus \mathcal{V}(g))$. Note that fg vanishes on $\mathcal{V}(I)$. By the Nullstellensatz, there is some m such that $f^m g^m \in I$, and so $f^m \in (I : g^\infty)$. But then $f^m(x) = 0$ and so $f(x) = 0$. Thus $x \in \mathcal{V}(\mathcal{I}(\mathcal{V}(I) \smallsetminus \mathcal{V}(g)))$, which completes the proof. $\square$

Corollary 1.4.12. *Let I and J be ideals in $\mathbb{K}[x]$. Then $\mathcal{V}(I : J^\infty)$ is the smallest variety containing $\mathcal{V}(I) \smallsetminus \mathcal{V}(J)$.*

A homogeneous ideal $I \subset \mathbb{K}[x_0, x_1, \ldots, x_n]$ is *saturated* if

$$I \ = \ (I : \mathfrak{m}_0) \ = \ \{f \mid x_i f \in I \text{ for } i = 0, 1, \ldots, n\}\,.$$

The reason for this definition is that I and $(I : \mathfrak{m}_0)$ define the same projective variety, by Corollary 1.4.12 applied to the affine cones these varieties define in $\mathbb{K}^{n+1}$.

Given a projective variety $X = \mathcal{V}(I) \subset \mathbb{P}^n$, consider its intersection with an affine chart $U_i = \{x \in \mathbb{P}^n \mid x_i \neq 0\}$. For simplicity of notation, suppose that $i = 0$. Then

$$X \cap U_0 \ = \ \{x \in U_0 \mid f(x) = 0 \text{ for all } f \in I\}\,.$$

If we identify U_0 with $\mathbb{K}^n$ by $U_0 = \{[1, x_1, \ldots, x_n] \mid (x_1, \ldots, x_n) \in \mathbb{K}^n\}$, this is

$$X \cap U_0 \ = \ \{x \in \mathbb{K}^n \mid f(1, x_1, \ldots, x_n) = 0 \text{ for all } f \in I\}\,. \tag{1.15}$$

We call the polynomial $f(1, x_1, \ldots, x_n)$ the *dehomogenization* of the homogeneous polynomial f with respect to x_0. The calculation (1.15) shows that $X \cap U_0$ is the affine variety defined by the ideal generated by the dehomogenizations of forms in I.

This proves the forward implication of the following characterization of projective varieties in terms of their intersections with these affine charts.

Lemma 1.4.13. *A subset $X \subset \mathbb{P}^n$ is a projective variety if and only if $X \cap U_i$ is an affine variety, for each $i = 0, \ldots, n$.*

Proof. For the reverse implication, suppose that $X \subset \mathbb{P}^n$ is a subset such that for each $i = 0, \ldots, n$, $X \cap U_i$ is an affine variety. For each i, let $H_i = \mathcal{V}(x_i)$ be the hyperplane that is the complement of U_i. Then $X \subset (X \cap U_i) \cup H_i = X \cup H_i$. We claim that $X \cup H_i$ is a projective variety. This will imply the lemma, as

$$\bigcap_{i=0}^{n}(X \bigcup H_i) \;=\; X \cup \bigcap_{i=0}^{n} H_i \;=\; X\,,$$

as $H_0 \cap H_1 \cap \cdots \cap H_n = \mathcal{V}(x_0, \ldots, x_n) = \emptyset$ in $\mathbb{P}^n$.

To prove the claim, let $i = 0$ for simplicity and identify U_0 with $\mathbb{K}^n$ whose coordinate ring is $\mathbb{K}[x_1, \ldots, x_n]$. For a polynomial $g \in \mathbb{K}[x_1, \ldots, x_n]$ of degree d, we have the homogeneous form g_+ of degree $d+1$ defined by

$$g_+(x_0, x_1, \ldots, x_n) \;:=\; x_0^{d+1} g(\tfrac{x_1}{x_0}, \ldots, \tfrac{x_n}{x_0})\,.$$

Let I_+ be the homogeneous ideal generated by $\{g_+ \mid g \in \mathcal{I}(X \cap U_0)\}$. Since x_0 always divides g_+, we have that $H_0 \subset \mathcal{V}(I_+)$. Since the dehomogenization of g_+ is g, the dehomogenization of I_+ is $\mathcal{I}(X \cap U_0)$. Then our previous calculations show that $X \cap U_0 = \mathcal{V}(I_+) \cap U_0$, which completes the proof. □

Corollary 1.4.14. *Let $X \subset \mathbb{P}^n$ be a projective variety and Λ a linear form. Then $X_\Lambda := X \smallsetminus \mathcal{V}(\Lambda)$ is an affine variety. Regular functions on X_Λ have the form f/Λ^d, where $f \in \mathbb{K}[x_0, \ldots, x_n]$ is a form of degree d.a*

Proof. Applying a linear change of coordinates, it suffices to prove this for $\Lambda = x_0$, in which case X_Λ becomes $X_0 = X \cap U_0$, which is affine. A regular function on X_0 is the restriction of a regular function on U_0. Such a function is the dehomogenization $f(1, x_1, \ldots, x_n)$ of a form $f \in \mathbb{K}[x_0, \ldots, x_n]$. Let d be the degree of f. Then, as a function on U_0,

$$f(1, x_1, \ldots, x_n) \;=\; f\left(\tfrac{x_0}{x_0}, \tfrac{x_1}{x_0}, \ldots, \tfrac{x_n}{x_0}\right) \;=\; \tfrac{1}{x_0^d} f(x_0, x_1, \ldots, x_n)\,,$$

which completes the proof. □

The point of Lemma 1.4.13 is that every projective variety X is naturally a union of affine varieties

$$X \;=\; \bigcup_{i=0}^{n}(X \cap U_i)\,.$$

Consequently, we may often prove results for projective varieties by arguing locally on each of these affine sets that cover it. It also illustrates a relationship between varieties and manifolds: Affine varieties are to varieties as open subsets of $\mathbb{R}^n$ are to manifolds.

Just as with affine varieties, projective varieties have coordinate rings. Let $X \subset \mathbb{P}^n$ be a projective variety. Its *homogeneous coordinate ring* $\mathbb{K}[X]$ is the quotient

$$\mathbb{K}[X] \ := \ \mathbb{K}[x_0, x_1, \ldots, x_n]/\mathcal{I}(X)\,.$$

If we set $\mathbb{K}[X]_d$ to be the image of all degree d homogeneous polynomials, $\mathbb{K}[x_0, \ldots, x_n]_d$, then this ring is graded,

$$\mathbb{K}[X] \ = \ \bigoplus_{d \geq 0} \mathbb{K}[X]_d\,,$$

where if $f \in \mathbb{K}[X]_d$ and $g \in \mathbb{K}[X]_e$, then $fg \in \mathbb{K}[X]_{d+e}$. More concretely, we have

$$\mathbb{K}[X]_d \ = \ \mathbb{K}[x_0, \ldots, x_n]_d/\mathcal{I}(X)_d\,,$$

where $\mathcal{I}(X)_d = \mathcal{I}(X) \cap \mathbb{K}[x_0, \ldots, x_n]_d$.

This differs from the coordinate ring of an affine variety in that its elements are not functions on X. Indeed, we already observed that, apart from constant polynomials, elements of $\mathbb{K}[x_0, \ldots, x_n]$ do not give functions on any subset of $\mathbb{P}^n$. Despite this, they will be used to define maps of projective varieties, and the homogeneous coordinate ring plays another role which will be developed in Section 3.5.

Let Λ be a linear form on $\mathbb{P}^n$ and $X \subset \mathbb{P}^n$ a subvariety. A consequence of Corollary 1.4.14 is that elements of the coordinate ring $\mathbb{K}[X_\Lambda]$ of the affine variety X_Λ have the form $f/\Lambda^{\deg(f)}$ for f a homogebeous element of $\mathbb{K}[X]$. The ring $\mathbb{K}[X][\frac{1}{\Lambda}]$ is graded by $\deg(g/\Lambda^d) = \deg(g) - d$. This gives another description of $\mathbb{K}[X_\Lambda]$.

Corollary 1.4.15. *The coordinate ring of the affine variety X_Λ is the degree 0 homogeneous component of the graded ring $\mathbb{K}[X][\frac{1}{\Lambda}]$.*

Exercises

1. Verify the claim in Example 1.4.4 that if a, b lie on the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$ and define the same point in $\mathbb{P}^n$, then $a = \zeta b$ for some unit complex number ζ.

2. Let $Z \subset \mathbb{P}^n$. Show that $\mathcal{I}(Z)$ is a homogeneous ideal.

3. A transition function $\varphi_{i,j}$ expresses how to change from the local coordinates from U_i of a point $p \in U_i \cap U_j$ to the local coordinates from U_j. Write down the transition functions for $\mathbb{P}^n$ provided by the affine charts $U_0, \ldots, U_n$.

4. Show that an ideal I is homogeneous if and only if it is generated by homogeneous polynomials.

5. Show that a radical homogeneous ideal is saturated.

6. Show that the homogeneous ideal $\mathcal{I}(Z)$ of a subset $Z \subset \mathbb{P}^n$ is equal to the ideal $\mathcal{I}(CZ)$ of the affine cone over Z.

7. Remove Zariski closure from this! Verify the claim concerning the relation between the ideal of an affine subvariety $Y \subset U_0$ and of its Zariski closure $\overline{Y} \subset \mathbb{P}^n$:

$$\mathcal{I}(\overline{Y}) \;=\; \{x_0^{\deg(g)+m} g(\tfrac{x_1}{x_0}, \ldots, \tfrac{x_n}{x_0}) \mid g \in \mathcal{I}(Y) \subset \mathbb{K}[x_1, \ldots, x_n],\ m \geq 0\}\,.$$

8. Show that if $X \subset \mathbb{P}^n$ is a projective variety, then the smallest projective variety containing its intersection with the principal affine set U_{x_0}, $\overline{X \cap U_{x_0}}$, has ideal the saturation $(\mathcal{I}(X) \colon x_0^\infty)$.

9. Show that if I is a homogeneous ideal and $J = (I : \mathfrak{m}_0^\infty)$ is its saturation with respect to the irrelevant ideal $\mathfrak{m}_0$, then there is some integer N such that

$$J_d \;=\; I_d \qquad \text{for} \quad d \geq N\,.$$

10. Verify the claim in the text that if $X \subset \mathbb{P}^n$ is a projective variety, then its homogeneous coordinate ring is graded with

$$\mathbb{K}[X]_d \;=\; \mathbb{K}[x_0, \ldots, x_n]_d / \mathcal{I}(X)_d\,.$$

1.5 Maps of projective varieties

Many properties of a projective variety X are inherited from the affine cone CX over X, but with some changes. The same is true for maps from X to a projective space. Elements of its homogeneous coordinate ring give maps, but care must be taken for the map to be well-defined. We explain this and describe some important maps of projective varieties. This leads to the product of projective varieties, and one of the most important properties of projective varieties; that the image of a projective variety under a map is a subvariety. We conclude by extending finite maps to projective varieties.

Suppose that $X \subset \mathbb{P}^n$ is a projective variety. Let $f_0, \ldots, f_m \in \mathbb{K}[X]$ be elements of its homogeneous coordinate ring. Under what circumstances does

$$X \ni x \longmapsto [f_0(x), f_1(x), \ldots, f_m(x)] \in \mathbb{P}^m$$

define a map $X \to \mathbb{P}^m$? (It always defines a map on affine cones $CX \to \mathbb{K}^{m+1}$.) Already the evaluation $f_i(x)$ of f_i at $x \in \mathbb{P}^n$ is a problem as the value of $f_i(x)$ is ambiguous. When f is homogeneous of degree d we saw that $f(\lambda x) = \lambda^d f(x)$ for $\lambda \in \mathbb{K}$. Thus when $f_0, \ldots, f_m$ are all homogeneous of the same degree d, their values at $x \in \mathbb{P}^n$ share the same ambiguity. In fact, as long as $x \notin \mathcal{V}(f_0, \ldots, f_m)$, then

$$\varphi(x) \;:=\; [f_0(x), f_1(x), \ldots, f_m(x)] \tag{1.16}$$

is a well-defined element of $\mathbb{P}^m$. Indeed, for $\lambda \in \mathbb{K}$, $\varphi(\lambda x) = \lambda^d \varphi(x)$ in $\mathbb{K}^{m+1}$, so that $\varphi(\lambda x) = \varphi(x)$ in $\mathbb{P}^m$ for $\lambda \neq 0$. When $\mathcal{V}(f_0, \ldots, f_m) = \emptyset$, so that the f_i have no common zeroes on X, then (1.16) defines a *regular map* $\varphi \colon X \to \mathbb{P}^m$.

Example 1.5.1. Suppose that $\mathbb{P}^1$ has homogeneous coordinates $[s,t]$. Then s^2, st, t^2 are homogeneous elements of its coordinate ring of the same degree, 2, with $\mathcal{V}(s^2,st,t^2)=\emptyset$. These define a regular map $\varphi\colon \mathbb{P}^1 \to \mathbb{P}^2$ where

$$\varphi \;:\; \mathbb{P}^1 \;\ni\; [s,t] \;\longmapsto\; [s^2,st,t^2] \;\in\; \mathbb{P}^2\,.$$

If $[x,y,z]$ are coordinates for $\mathbb{P}^2$, then the image of φ is $\mathcal{V}(xz-y^2)$. Indeed, the image is a subset of $\mathcal{V}(xz-y^2)$ as $(s^2)(t^2)-(st)^2=0$. Let $[x,y,z]\in\mathcal{V}(xz-y^2)$. If $x=0$, then $y=0$ and $[0,0,z]=[0,0,1]=\varphi([0,1])$. If $x\neq 0$, then $z=y^2/x$, and we have

$$[x,y,z] \;=\; [1,y/x,z/x] \;=\; [1,y/x,y^2/x^2] \;=\; \varphi([1,y/x]) \;=\; \varphi([x,y])\,. \tag{1.17}$$

Thus $\mathcal{V}(xz-y^2)$ is the image of φ. This is the parabola of Examples 1.4.1 and 1.4.8. ⋄

The map φ of Example 1.5.1 is injective, and we would like to have that $\mathbb{P}^1$ is isomorphic to its image, C. For that, we need a map $C\to\mathbb{P}^1$ that is inverse to φ. For this, we extend and refine our notion of regular map of projective varieties. Let X be a projective variety and suppose that $f_0,\dots,f_m\in\mathbb{K}[X]$ are homogeneous elements of the same degree with $\mathcal{V}(f_0,\dots,f_m)=\emptyset$ which define a regular map $\varphi\colon X\to\mathbb{P}^m$ (1.16). A second list $g_0,\dots,g_m\in\mathbb{K}[X]$ of elements of the same degree (possible different from the degree of the f_i) with $\mathcal{V}(g_0,\dots,g_m)=\emptyset$ defines the same regular map if we have

$$\operatorname{rank}\begin{pmatrix} f_0 & f_1 & \cdots & f_m\\ g_0 & g_1 & \cdots & g_m\end{pmatrix} \;=\; 1\,,\quad \text{i.e., if}\;\; f_ig_j-f_jg_i \;\in\; \mathcal{I}(X)\;\;\text{for } i\neq j\,. \tag{1.18}$$

Indeed, the conditions $\mathcal{V}(f_0,\dots,f_m)=\mathcal{V}(g_0,\dots,g_m)=\emptyset$ and (1.18) imply that for any $x\in X$, $[f_0(x),\dots,f_m(x)]=[g_0(x),\dots,g_m(x)]$ in $\mathbb{P}^m$.

More interesting is when $f_0,\dots,f_m,g_0,\dots,g_m$ satisfy (1.18), but we do not have that $\mathcal{V}(f_0,\dots,f_m)=\mathcal{V}(g_0,\dots,g_m)=\emptyset$. In Example 1.5.1, (1.17) shows that if $[x,y,z]\in C=\mathcal{V}(xz-y^2)$, then $[x,y,z]=\varphi([x,y])$, but this requires that $(x,y)\neq(0,0)$. Similarly, if $(y,z)\neq(0,0)$, then $[x,y,z]=\varphi([y,z])$. We may understand this in terms of (1.18), at least when $xyz\neq 0$ as $\det\left(\begin{smallmatrix} x & y\\ y & z\end{smallmatrix}\right)=xz-y^2$, which vanishes on C. On C, $\mathcal{V}(x)=[0,0,1]$, $\mathcal{V}(z)=[1,0,0]$, and $\mathcal{V}(y)=\{[1,0,0],[0,0,1]\}$, so that at every point of C, at least one of (x,y) or (y,z) can be used to define a map to $\mathbb{P}^1$ which is the inverse of φ.

Definition 1.5.2. A map $\varphi\colon X\to\mathbb{P}^m$ from a projective variety X is a *regular map* if for every $x\in X$, there are elements $f_0,\dots,f_m\in\mathbb{K}[X]$ of the same degree with $x\notin\mathcal{V}(f_0,\dots,f_m)$ such that for every $y\in X\smallsetminus\mathcal{V}(f_0,\dots,f_m)$,

$$\varphi(y) \;=\; [f_0(y),f_1(y),\dots,f_m(y)]\,.$$

That is, φ has the form (1.16), but the elements $f_0,\dots,f_m$ may change for different parts of X (but any two choices satisfy (1.18), where both are defined). ⋄

Perhaps the simplest regular map of projective varieties is a linear projection. Let $X \subset \mathbb{P}^n$ be a projective variety and suppose that $L \subset \mathbb{P}^n$ is a linear subspace disjoint from X. Let $\Lambda_0, \ldots, \Lambda_m$ be linearly independent forms that vanish on L. Then $L = \mathcal{V}(\Lambda_0, \ldots, \Lambda_m)$ and L has dimenion $n-m-1$. We also have $X \cap \mathcal{V}(\Lambda_0, \ldots, \Lambda_m) = \emptyset$, so that $(\Lambda_0, \ldots, \Lambda_m)$ defined a regular map $\pi_L \colon X \to \mathbb{P}^m$, which is called a *linear projection* with center L.

Projective varieties $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ are *isomorphic* if we have regular maps $\varphi \colon X \to Y$ and $\psi \colon Y \to X$ for which the compositions $\psi \circ \varphi$ and $\varphi \circ \psi$ are the identity maps on X and Y, respectively.

The map φ of Example 1.5.1 is an isomorphism between $\mathbb{P}^1$ and its image C. More generally, the set $V_{n,d}$ of all $\binom{n+d}{n}$ homogeneous monomials in $x_0, \ldots, x_n$ of degree d is a basis for the degree d component of the irrelevant ideal, and thus generates $\mathfrak{m}_0^d$. By Lemma 1.4.9, $\mathcal{V}(V_{n,d}) = \emptyset$, and thus this list of monomials gives a regular map,

$$\nu_{n,d} \ : \ \mathbb{P}^n \ \longrightarrow \ \mathbb{P}^{\binom{n+d}{n}-1} \,,$$

called the *dth Veronese map*. The map φ of Example 1.5.1 is $\nu_{1,2}$. Let us study the image of ν_d. We adopt a useful convention from Section 8.1 and label the coordinates of $\mathbb{P}^{\binom{n+d}{n}-1}$ by the exponents of monomials in $V_{n,d}$,

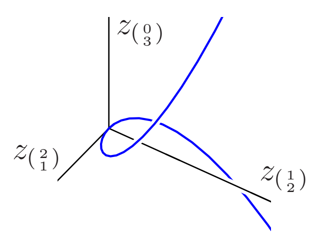
$$\mathcal{A}_{n,d} \ := \ \{(a_0, \ldots, a_n) \mid a_i \in \mathbb{N} \text{ and } a_0 + \cdots + a_n = d\}\,.$$

Then $V_{n,d} = \{x^\alpha \mid \alpha \in \mathcal{A}_{n,d}\}$ and $[z_\alpha \mid \alpha \in \mathcal{A}_{n,d}]$ are homogeneous coordinates of $\mathbb{P}^{\binom{n+d}{n}-1}$, with the z_αth coordinate of the Veronese map $\nu_{n,d}$ equal to x^α.

Observe that if $\alpha, \beta, \gamma, \delta \in \mathcal{A}_{n,d}$ satisfy $\alpha + \beta = \gamma + \delta$ (as integer vectors), then $z_\alpha z_\beta - z_\gamma z_\delta$ vanishes on the image $\nu_d(\mathbb{P}^n)$ as $\nu_d^*(z_\alpha z_\beta - z_\gamma z_\delta) = x^\alpha x^\beta - x^\gamma x^\delta = 0$. This is the equation $xz - y^2 = 0$ that we found for $\nu_{1,2}$ in Example 1.5.1. When $n = 1$ and $d = 3$, we have $\mathcal{A}_{1,3} = \{\binom{3}{0}, \binom{2}{1}, \binom{1}{2}, \binom{0}{3}\}$, $\nu_{1,3}([s,t]) = [s^3, s^2t, st^2, t^3]$, and the quadratic polynomials that vanish on the image include

$$z_{\binom{3}{0}} z_{\binom{1}{2}} - z_{\binom{2}{1}}^2 \,, \ z_{\binom{3}{0}} z_{\binom{0}{3}} - z_{\binom{2}{1}} z_{\binom{1}{2}} \,, \quad \text{and} \quad z_{\binom{2}{1}} - z_{\binom{0}{3}} z_{\binom{1}{2}}^2 \,.$$

The image $\nu_{1,3}(\mathbb{P}^1)$ is the rational normal (or monomial) curve, depicted in $U_{\binom{3}{0}}$ below.



Theorem 1.5.3. *The image $\nu_{n,d}(\mathbb{P}^n) \subset \mathbb{P}^{\binom{n+d}{n}-1}$ is the subvariety defined by the vanishing of the quadratic polynomials*

$$z_\alpha z_\beta \ - \ z_\gamma z_\delta \quad \textit{for} \quad \alpha, \beta, \gamma, \delta \in \mathcal{A}_{n,d} \quad \textit{with} \quad \alpha + \beta \ = \ \gamma + \delta\,, \tag{1.19}$$

and it is isomorphic to $\mathbb{P}^n$.

The image of $\nu_{n,d}$ is called the *Veronese variety*, and $\nu_{n,d}$ is the *Veronese embedding*.

Proof. We observed that these quadratics (1.19) vanish on $\nu_{n,d}(\mathbb{P}^n)$.

Let $X \subset \mathbb{P}^{\binom{n+d}{n}-1}$ be the variety defined by the vanishing of the quadratics (1.19) and let $z \in X$. In Exercise 3, you will show that there is at least one $i = 0, \dots, n$ such that $z_{de_i} \neq 0$. (Here, e_i is the ith standard basis vector so that de_i is the exponent of x_i^d. In Example 1.5.1 this was that one of $x = z_{\binom{2}{0}}$ or $z = z_{\binom{0}{2}}$ did not vanish.) Thus $z \in U_{de_i}$. Define φ_i on the affine patch $X \cap U_{de_i}$ by

$$\varphi_i(z) \;=\; [z_{e_j+(d-1)e_i} \mid j = 0, \dots, n]\,.$$

(Here, the subscript $e_j+(d-1)e_i$ is the exponent of $x_j x_i^{d-1}$.) Then $\varphi_i \colon X \cap U_{de_i} \to U_i \subset \mathbb{P}^n$ is an inverse to $\nu_{n,d}$ on U_i, showing that $X \cap U_{de_i} = \nu_{n,d}(U_i) = \nu_{n,d}(\mathbb{P}^n) \cap U_{de_i}$. Thus these φ_i piece together to define a regular map $\varphi \colon \nu_{n,d}(\mathbb{P}^n) \to \mathbb{P}^n$ that is inverse to $\nu_{n,d}$. This completes the proof. $\square$

The value of the Veronese embedding is that if $f \in \mathbb{K}[x]$ is any form of degree d, then there is a linear form Λ_f on $\mathbb{P}^{\binom{n+d}{n}-1}$ such that $f = \nu_{n,d}^*(\Lambda_f)$. More precisely,

$$f \;=\; \sum_{\alpha \in \mathcal{A}_{n,d}} c_\alpha x^\alpha \;=\; \nu_{n,d}^*\Big(\sum_{\alpha \in \mathcal{A}_{n,d}} c_\alpha z_\alpha\Big)\,. \tag{1.20}$$

Then $\nu_{n,d}(\mathcal{V}(f)) = \nu_{n,d}(\mathbb{P}^n) \cap \mathcal{V}(\Lambda_f)$. Extending this to a basis of $\mathcal{I}(X)_d$ for d large enough shows that any subvariety X of $\mathbb{P}^n$ is isomorphic to a linear section of some Veronese variety. Consequently, any projective variety is isomorphic to a variety defined by equations of degree at most two.

Furthermore, if f is a degree d form on $\mathbb{P}^n$ with corresponding linear form Λ_f on $\mathbb{P}^{\binom{n+d}{n}-1}$, then $U_f = \mathbb{P}^n \smallsetminus \mathcal{V}(f)$ is an affine variety as it is isomorphic to $\nu_{n,d}(\mathbb{P}^n) \cap U_{\Lambda_f}$. Consequently, for any projective variety $X \subset \mathbb{P}^n$ and any homogeneous element f of its coordinate ring, the set $X_f := X \smallsetminus \mathcal{V}(f) = X \cap U_f$ is an affine variety. Lemma 1.4.13 extends to these more general affine charts U_f of $\mathbb{P}^n$. That is, a subset $X \subset \mathbb{P}^n$ is a variety if and only if $X_f \subset U_f$ is an affine variety for every homogeneous form f.

This is related to maps of projective varieties. Suppose that $\varphi \colon X \to \mathbb{P}^m$ is a regular map defined on part of X by $\varphi(x) = [f_0(x), \dots, f_m(x)]$ for $f_0, \dots, f_m$ homogeneous elements of $\mathbb{K}[X]$ of the same degree. Then φ is defined as a map of affine varieties on each affine patch X_{f_i}.

The product of affine varieties required no special treatment as the product $\mathbb{K}^m \times \mathbb{K}^n$ of two affine spaces is again an affine space, $\mathbb{K}^{m+n}$. This is not the case with projective spaces. To remedy this, we identify $\mathbb{P}^m \times \mathbb{P}^n$ with a subvariety of the projective space $\mathbb{P}^{mn+m+n}$, and use this identification to help understand subvarieties of $\mathbb{P}^m \times \mathbb{P}^n$.

Let $x_0, \dots, x_m$ and $y_0, \dots, y_n$ be homogeneous coordinates for $\mathbb{P}^m$ and $\mathbb{P}^n$, respectively. Let $z_{i,j}$ for $i = 0, \dots, m$ and $j = 0, \dots, n$ be homogeneous coordinates for $\mathbb{P}^{mn+m+n}$. (Note that $(m+1)(n+1)-1 = mn+m+n$.) Define a map $\sigma_{m,n} \colon \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{mn+m+n}$ by

$$\sigma_{m,n}(x, y) \;=\; z\,, \quad \text{where} \quad z_{i,j} = x_i y_j\,.$$

This map becomes more clear when lifted to the affine cones over these projective spaces, where it is the map $\mathbb{K}^{m+1} \times \mathbb{K}^{n+1} \to \mathrm{Mat}_{m+1,n+1}(\mathbb{K})$ that sends a pair of column vectors (x, y) to their outer product $xy^T \in \mathrm{Mat}_{m+1,n+1}(\mathbb{K})$. The image is the set of rank 1 matrices, which is defined by the vanishing of the quadratic polynomials,

$$\det\begin{pmatrix} z_{i,j} & z_{i,l} \\ z_{k,j} & z_{k,l} \end{pmatrix} \;=\; z_{i,j}z_{k,l} \;-\; z_{i,l}z_{k,j} \quad \text{for} \quad 0 \le i < k \le m \;\text{ and }\; 0 \le j < l \le n\,. \tag{1.21}$$

This is a special case of Exercise 9(a) in Section 1.1.

Theorem 1.5.4. *The image $\sigma_{m,n}(\mathbb{P}^m \times \mathbb{P}^n) \subset \mathbb{P}^{mn+m+n}$ is the subvariety defined by the vanishing of the quadratic polynomials* (1.21). *The map $\sigma_{m,n}$ admits an inverse.*

Call the map $\sigma_{m,n}$ the *Segre map* and its image the *Segre variety*. Exercise 5 explores the Segre variety $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$.

Proof. We sketch the proof, which is similar to that of Theorem 1.5.3, and leave the details as Exercise 6. For the inverse to $\sigma_{m,n}$, suppose that $X \subset \mathbb{P}^{mn+m+n}$ satisfies the equations (1.21). For each index k, l of a coordinate of $\mathbb{P}^{mn+m+n}$, we have an affine patch $U_{k,l} := \{z \in \mathbb{P}^{mn+m+n} \mid z_{k,l} \neq 0\}$. Define a map to $\mathbb{P}^m \times \mathbb{P}^n$ on the affine patch $X \cap U_{k,l}$ by

$$\varphi_{k,l}(z) \;=\; \big([z_{i,l} \mid i = 0, \ldots, m]\,,\, [z_{k,j} \mid j = 0, \ldots, n]\big)\,.$$

Then $\varphi_{k,l}$ is an isomorphism between the affine varieties $X \cap U_{k,l}$ and $U_k \times U_l$. $\square$

This proof identifies affine patches $X \cap U_{k,l}$ with affine spaces $U_k \times U_l \simeq \mathbb{K}^m \times \mathbb{K}^n \subset \mathbb{P}^m \times \mathbb{P}^n$, and could be used to put the structure of an algebraic variety on the product $\mathbb{P}^m \times \mathbb{P}^n$, much as in differential geometry. Another approach is intrinsic: define subvarieties of $\mathbb{P}^m \times \mathbb{P}^n$ directly as we did with projective space. A third approach is extrinsic: use the Segre embedding to define subvarieties of $\mathbb{P}^m \times \mathbb{P}^n$. The first, using a covering by affine varieties to give $\mathbb{P}^m \times \mathbb{P}^n$ the structure of an algebraic variety, is the starting point for the general development of algebraic schemes that we do not pursue here. We develop the second and third approaches and show they coincide. That they give the same notion of subvariety as the first follows from the application of Lemma 1.4.13 to $\mathbb{P}^{mn+m+n}$ and $\sigma_{m,n}(\mathbb{P}^m \times \mathbb{P}^n)$.

Both the intrinsic and extrinsic approaches begin with the same definition. A monomial $x^\alpha y^\beta$ in $\mathbb{K}[x_0, \ldots, x_n, y_0, \ldots, y_m] = \mathbb{K}[x; y]$ has *bidegree* (a, b) where $a = \deg(x^\alpha)$ and $b = \deg(y^\beta)$. For example, the bidegree of $x_0x_1^3y_0y_2y_3$ is $(4, 3)$. A polynomial $g(x; y) \in \mathbb{K}[x; y]$ is *bihomogeneous* of *bidegree* (a, b) if each of its monomials has bidegree (a, b). The same discussion from Section 1.4 that led us to understand the role of homogeneous ideals for projective varieties leads to bihomogeneous ideals defining subsets of $\mathbb{P}^m \times \mathbb{P}^n$.

We may also ask what are the subsets X of $\mathbb{P}^m \times \mathbb{P}^n$ whose image $\sigma_{m,n}(X)$ is a subvariety of $\mathbb{P}^{mn+m+n}$? As $\sigma_{m,n}$ is given by bilinear monomials, the pullback $\sigma^*(f)$ of a form of degree d in z is a form of degree $2d$ that is bihomogeneous of bidegree (d, d). Consequently, a subset X of $\mathbb{P}^m \times \mathbb{P}^n$ whose image $\sigma_{m,n}(X)$ is a subvariety is defined by bihomogeneous polynomials $g(x; y)$ with *diagonal* bidegree (a, a).

To reconcile these two approaches, let $g(x; y)$ be a bihomogeneous polynomial with a non-diagonal bidegree (a, b) and suppose that $a = b + k$ with $k > 0$. Observe that

$$\mathcal{V}(g(x;y)) \;=\; \mathcal{V}(y_j^k g(x;y) \mid j = 0, \ldots, n)\,. \tag{1.22}$$

The same observation, but with x when $a < b$ shows that any subset of $\mathbb{P}^m \times \mathbb{P}^n$ defined by bihomogeneous polynomials may also be defined by bihomogeneous polynomials with a diagonal bidegree.

We follow the discussion leading up to Lemma 1.4.13 to define subvarieties of $\mathbb{P}^m \times \mathbb{K}^n$. If we restrict the second factor of $\mathbb{P}^m \times \mathbb{P}^n$ to $U_0 \simeq \mathbb{K}^n$ and dehomogenize bihomogeneous forms with respect to y_0, we see that subvarieties of $\mathbb{P}^m \times \mathbb{K}^n$ are given by polynomials $f(x; y) \in \mathbb{K}[x_0, \ldots, x_m, y_1, \ldots, y_n]$ that are homogeneous in x, but with no restriction on y. We have shown the following characterization of subvarieties of products.

Proposition 1.5.5. *A subvariety $X \subset \mathbb{P}^m \times \mathbb{P}^n$ is defined by a system of bihomogeneous polynomials $f_1(x; y), \ldots, f_r(x; y)$. A subvariety $X \subset \mathbb{P}^m \times \mathbb{K}^n$ is defined by a system of polynomials $g_1(x; y), \ldots, g_r(x; y)$ that are homogeneous in x.*

Let X, Y be varieties. Then $X \times Y$ is a variety, as it is defined by the set of polynomials $\{f(x)g(y) \mid f \in \mathcal{I}(X), g \in \mathcal{I}(Y)\}$. When both X and Y are affine, this was discussed in Section 1.1, when both are projective this is a set of bihomogeneous polynomials, and if X is projective and Y affine, then these are homogeneous in the first set of variables. This description of subvarieties of products of two projective spaces or of a projective space and an affine space extends in a natural way to arbitrary finite products of projective spaces with affine space; we leave the details to the reader.

In Example 1.4.4 we remarked that when $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, projective space is compact in the usual (Euclidean) topology, and consequently the projection maps $\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^n$ and $\mathbb{P}^m \times \mathbb{K}^n \to \mathbb{K}^n$ are proper in that the image of a closed set is also closed. This remains true, whatever the field, if we replace the property of being closed by that of being a subvariety. This will be a consequence of the following theorem.

Theorem 1.5.6. *The image of a subvariety $X \subset \mathbb{P}^m \times \mathbb{P}^n$ under the projection to $\mathbb{P}^n$ is again a subvariety, and the same for a subvariety of $\mathbb{P}^m \times \mathbb{K}^n$ under projection to $\mathbb{K}^n$.*

Before proving Theorem 1.5.6, we use it to show that the image of a projective variety under a map is a variety. That is, maps from projective varieties have the same property as finite maps of affine varieties. You will see in the proof of Theorem 1.5.6 that the reason is similar; points in a fiber cannot disappear.

A first step is to introduce a general construction. Let $\varphi\colon X \to Y$ be a regular map of algebraic varieties (projective or affine). The *graph* of φ is the set

$$\Gamma \;:=\; \{(x, y) \in X \times Y \mid x \in X \text{ and } \varphi(x) = y\}\,.$$

Let π_X and π_Y be the maps that project Γ to the first and second factors of $X \times Y$, respectively, and $\iota\colon x \mapsto (x, f(x)) \in X \times Y$ the natural map from X to Γ. You are asked to prove the following in Exercise 9.

Lemma 1.5.7. *The graph Γ of φ is a subvariety of $X \times Y$. The projection π_1 to X is an isomorphism and φ is the composition $\pi_2 \circ \iota$.*

Corollary 1.5.8. *If $\varphi\colon X \to Y$ is a regular map of projective varieties, then its image $\varphi(X)$ is a subvariety of Y.*

Proof. Let $\Gamma \subset X \times Y$ be the graph of φ. Suppose that X is a subvariety of $\mathbb{P}^m$ and Y is a subvariety of $\mathbb{P}^n$. Then Γ is a subvariety of $\mathbb{P}^m \times \mathbb{P}^n$. By Theorem 1.5.6 the projection of Γ to $\mathbb{P}^n$, which is the image $\varphi(X)$, is a subvariety of $\mathbb{P}^n$, and hence of Y. $\square$

Proof of Theorem 1.5.6. By Lemma 1.4.13, it suffices to prove the statement about projection to $\mathbb{K}^n$, as we may argue locally on the affine patches $U_0, \dots, U_n$ of $\mathbb{P}^n$. Let $X \subset \mathbb{P}^m \times \mathbb{K}^n$ be a subvariety. By Proposition 1.5.5, X is defined by the vanishing of finitely many polynomials

$$g_1(x;y)\,,\ g_2(x;y)\,,\ \dots\,,\ g_s(x;y)\ \in\ \mathbb{K}[x_0,\dots,x_m\,,\,y_1,\dots,y_n]\,,$$

where each g_i is homogeneous of degree d_i in the x variables and with no condition on y.

Let $\pi\colon \mathbb{P}^m \times \mathbb{K}^n \to \mathbb{K}^n$ be the projection. A point $b \in \mathbb{K}^m$ lies in the image $\pi(X)$ if and only if the system of homogeneous polynomials

$$g_1(x;b)\ =\ g_2(x;b)\ =\ \cdots\ =\ g_s(x;b)\ =\ 0\,,$$

has a solution in $\mathbb{P}^m$. By Lemma 1.4.9 this holds if and only if the ideal $I(b)$ these polynomials generate does not contain $\mathfrak{m}_0(x)^d$ for any d. Since $\mathfrak{m}_0(x)^d$ is generated by the vector space $\mathbb{K}[x]_d$ of all forms of degree d, this is equivalent to $I(b)_d \neq \mathbb{K}[x]_d$, for all d.

This degree d component $I(b)_d$ of $I(b)$ is the image of the linear map

$$\Lambda_d(b)\ :\ \mathbb{K}[x]_{d-d_1} \oplus \cdots \oplus \mathbb{K}[x]_{d-d_s}\ \longrightarrow\ \mathbb{K}[x]_d\,,$$

given by $(f_1, \dots, f_s) \mapsto f_1 g_1(x;b) + \cdots + f_s g_s(x;b)$. If we write the linear map $\Lambda_d(b)$ in terms of the bases of monomials of $\mathbb{K}[x]_d$ and $\mathbb{K}[x]_{d-d_i}$, we obtain a matrix $M_d(b)$ with entries the coefficients of the monomials in x in the $g_i(x;b)$, which are polynomials in b. Thus $I(b)_d \neq \mathbb{K}[x]_d$ if and only if $\Lambda_d(b)$ is not surjective if and only if the maximal minors of $M_d(b)$ vanish.

We conclude that b lies in $\pi(X)$ if and only if all the maximal minors of $M_d(b)$ vanish for all d. But this is a collection of polynomials in $\mathbb{K}[y_1, \dots, y_n]$, which shows that $\pi(X)$ is an affine subvariety of $\mathbb{K}^n$. $\square$

We close with an extension of finite maps from affine varieties to projective varieties. If $\varphi\colon X \to \mathbb{P}^m$ is a regular map from a projective variety X, then its image $\varphi(X)$ is a subvariety of $\mathbb{P}^m$. This map is *finite* if for every linear form $\Lambda \in \mathbb{K}[y_0, \dots, y_m]$ on $\mathbb{P}^m$, the corresponding map of affine varieties

$$\varphi \;:\; X \setminus \mathcal{V}(\varphi^*(\Lambda)) \;\longrightarrow\; \varphi(X) \setminus \mathcal{V}(\Lambda)$$

is a finite map. We show that linear projections are finite maps.

Theorem 1.5.9. *Let $X \subset \mathbb{P}^n$ be a projective variety and $L \subset \mathbb{P}^n$ be a linear subspace disjoint from X. Then the linear projection with center L is a finite map $\pi\colon X \to \pi(X)$.*

Proof. Let $m := n - \dim(L) - 1$, so that $\pi\colon X \to \mathbb{P}^m$, and let Λ be a linear form on $\mathbb{P}^m$. Linear forms on $\mathbb{P}^m$ pull back to linear forms on $\mathbb{P}^n$ that vanish along L. Choosing coordinates $y_0, \dots, y_m$ on $\mathbb{P}^m$, we may assume that $\Lambda = \Lambda_0, \Lambda_1, \dots, \Lambda_m$ are independent linear forms on $\mathbb{P}^n$ vanishing along L with π defined by $y_i = \Lambda_i$. We show that

$$\pi \;:\; X_{\Lambda_0} = X \setminus \mathcal{V}(\Lambda_0) \;\longrightarrow\; \pi(X) \setminus \mathcal{V}(y_0)$$

is a finite map. That is, we show that the corresponding extension of coordinate rings is a finite extension. We will do this by showing that every element of $\mathbb{K}[X_{\Lambda_0}]$ is integral over the coordinate ring of $\pi(X) \smallsetminus \mathcal{V}(y_0)$, which is its subring generated by $\frac{\Lambda_1}{\Lambda_0}, \dots, \frac{\Lambda_m}{\Lambda_0}$.

By Corollary 1.4.14, an element of $\mathbb{K}[X_{\Lambda_0}]$ is the restriction of a rational function $\frac{f}{\Lambda_0^d}$, where f is a form of degree d on $\mathbb{P}^n$. Since $\mathcal{V}(\Lambda_0^d, \dots, \Lambda_m^d) = L$ is disjoint from X, the degree d forms $(\Lambda_0^d, \dots, \Lambda_m^d, f)$ define a regular map $\psi\colon X \to \mathbb{P}^{m+1}$. Let $g_1, \dots, g_r \in \mathbb{K}[z_0, \dots, z_{m+1}]$ be homogeneous forms that define the image $\psi(X)$ as a subvariety of $\mathbb{P}^{m+1}$.

We just observed that $\emptyset = L \cap X = \mathcal{V}(\Lambda_0^d, \dots, \Lambda_m^d) \cap X = \psi^{-1}(\mathcal{V}(z_0, \dots, z_m))$. This implies that $\emptyset = \mathcal{V}(z_0, \dots, z_m) \cap \psi(X) = \mathcal{V}(z_0, \dots, z_m, g_1, \dots, g_r)$ in $\mathbb{P}^{m+1}$. By Lemma 1.4.9, there is some $N \in \mathbb{N}$ such that $\mathfrak{m}_0^N \subset \langle z_0, \dots, z_m, g_1, \dots, g_r\rangle$, where $\mathfrak{m}_0 = \langle z_0, \dots, z_{m+1}\rangle$ is the irrelevant ideal of $\mathbb{P}^{m+1}$. As $z_{m+1}^N \in \mathfrak{m}_0^N$, there are forms $p_0, \dots, p_m, q_1, \dots, q_r \in \mathbb{K}[z_0, \dots, z_{m+1}]$ such that

$$z_{m+1}^N \;=\; \sum_{i=0}^{m} z_i p_i \;+\; \sum_{j=1}^{r} g_j q_j\,.$$

Restricting to the homogeneous part of this expression, we may assume that $\deg(p_i) = N - 1$. Set

$$F \;:=\; z_{m+1}^N \;-\; \sum_{i=0}^{m} z_i p_i \;\in\; \mathbb{K}[z_0, \dots, z_{m+1}]\,.$$

Then $F = 0$ in $\mathbb{K}[\psi(X)]$, and thus $0 = \psi^*(F)$ in $\mathbb{K}[X]$.

Since $\deg(p_i) = N - 1$, z_{m+1} has degree at most $N - 1$ in the sum $\sum_{i=0}^m z_i p_i$. Thus, if we write F as a polynomial in z_{m+1}, we obtain

$$F \;=\; z_{m+1}^N \;+\; \sum_{i=0}^{N-1} z_{m+1}^i A_{N-i}(z_0, \dots, z_m)\,,$$

where A_i is a form of degree i. Then the pullback $\psi^*(F)$ is

$$f^N \;+\; \sum_{i=0}^{N-1} f^i A_{N-i}(\Lambda_0^d, \ldots, \Lambda_m^d)\,.$$

Dividing this expression by Λ_0^{Nd}, we obtain

$$\left(\tfrac{f}{\Lambda_0^d}\right)^N \;+\; \sum_{i=0}^{N-1} \left(\tfrac{f}{\Lambda_0^d}\right)^i A_{N-i}\left(1, \left(\tfrac{\Lambda_1}{\Lambda_0}\right)^d, \ldots, \left(\tfrac{\Lambda_m}{\Lambda_0}\right)^d\right)\,.$$

As this is 0 in $\mathbb{K}[X_{\Lambda_0}]$, we have shown that $\frac{f}{\Lambda_0^d} \in \mathbb{K}[X_{\Lambda_0}]$ is integral over its subring generated by $\frac{\Lambda_1}{\Lambda_0}, \ldots, \frac{\Lambda_m}{\Lambda_0}$, which is $\mathbb{K}[\pi(X) \cap U_0]$. This completes the proof that $\pi\colon X \to \pi(X)$ is a finite map of projective varieties. $\square$

Exercises

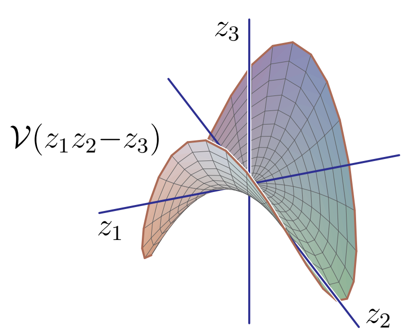
1. Show that if $f_0, \ldots, f_m \in \mathbb{K}[x]$ are forms of the same degree that do not simultaneously vanish and if $\mathcal{V}(f_0, \ldots, f_m) = \emptyset$, then (1.16) defines a map $\varphi\colon \mathbb{P}^n \to \mathbb{P}^m$.

2. Show that the number of monomials in $x_0, \ldots, x_n$ of degree d is $\binom{n+d}{n} = \binom{n+d}{d}$.

3. Complete the proof of Theorem 1.5.3, verifying the claims made.

4. The quadratic Veronese map $\nu_{n,2}\colon \mathbb{P}^n \to \mathbb{P}^{\binom{n+2}{2}-1}$ may be written as $z_{i,j} = x_i x_j$ for $0 \leq i \leq j \leq n$. Identify $\mathbb{P}^{\binom{n+2}{2}-1}$ with the projective space on $(n{+}1) \times (n{+}1)$ symmetric matrices and show that the quadratic Veronese variety is the projectivization of the set of symmetric matrices of rank 1.

5. Show that the image of the Segre map $\sigma_{1,1}\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ is the projectivization of hyperbolic paraboloid $z_3 = z_1 z_2$.



6. Complete the proof of Theorem 1.5.4.

7. Explain why the set $\mathcal{V}(f) \subset \mathbb{P}^m \times \mathbb{P}^n$ is well-defined for a bihomogeneous polynomial $f(x, y) \in \mathbb{K}[x; y]$ and prove that for any subset $Z \subset \mathbb{P}^m \times \mathbb{P}^n$ its ideal $\mathcal{I}(Z) \subset \mathbb{K}[x; y]$ is bihomogeneous.

8. Prove the equality (1.22). Hint: saturate with respect to $\mathfrak{m}_0(y)$.

9. Prove Lemma 1.5.7.

1.6 Notes

This needs to be rewritten. Most of the material in this chapter is standard material within courses of algebraic geometry or related courses. User-friendly, introductory texts to these topics include the books of Beltrametti, Carletti, Gallarati, and Monti Bragadin [6], Cox, Little, O'Shea [25], Holme [62], Hulek [63], Perrin [102], Smith, Kahanpää, Kekäläinen, and Traves [129]. Advanced, in-depth treatments from the viewpoint of modern, abstract algebraic geometry can be found in the books of Eisenbud [35], Harris [48], Hartshorne [49], and Shafarevich [126]. Our treatment here and in Chapter 3 is most influenced by Shafarevich.

If the polynomials $f_1, \ldots, f_m \in \mathbb{K}[x_1, \ldots, x_n]$ over an algebraically closed K do not have a common zero, then Hilbert's Nullstellensatz implies a polynomial identity of the form $\sum g_i f_i = 1$ with $g_1, \ldots, g_m \in \mathbb{K}[x_1, \ldots, x_n]$. However, the degrees of the polynomials in such a representation can grow doubly exponentially in the number n of variables, see Kollár [72].