Algebra Autumn 2017
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31 November 2017

Ninth Homework

Write your answers neatly, in complete sentences. Recopy your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Hand in to Frank Tuesday 7 November: (Have this on a separate sheet of paper.)

36. Let R be a ring, I an ideal of R, and $n \geq 1$ an integer. Define $M_n(I)$ to be the set of $n \times n$ matrices with entries in I.
    (1) Prove that for any ideal I of R, $M_n(I)$ is an ideal of $M_n(R)$.
    (2) Prove that any ideal of $M_n(R)$ has the form $M_n(I)$ for I an ideal of R.

Hand in for the grader Tuesday 7 November:

32. Classify all groups of order 18 up to isomorphism.
37. Let S be a *subset* of a ring R. Show that the intersection of all ideals of R that contain S is the set
$$\Big\{\sum_{i=1}^{n} r_i s_i t_i \mid r_1, t_1, \ldots, r_n, t_n \in R \quad s_1, \ldots, s_n \in S \quad n \in \mathbb{N}\Big\}.$$
38. Let $C \subset M_2(\mathbb{R})$ be the set of matrices of the form
$$C \;=\; \left\{\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \mid a, b \in \mathbb{R}\right\}.$$
    Prove that C is a subring of $M_2(\mathbb{R})$, and that C is a field. Can you identify C with any field you know of?
39. A ring R in which every element is idempotent ($\forall a \in R, a^2 = a$) is a Boolean ring. Prove that every Boolean ring is commutative and has characteristic 2.
40. Let a, b be elements of a ring R. Prove that $1 - ab$ is invertible in R if and only if $1 - ba$ is invertible in R.
41. Define the binomial coefficient $\binom{n}{k}$ to be $n!/k!/(n-k)!$ for integers $0 \leq k \leq n$. Let R be a commutative ring. Prove the binomial theorem:
$$\forall a, b \in R\ \forall n \in \mathbb{N} \qquad (a+b)^n \;=\; \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}\,.$$
42. Suppose that R is a commutative ring of characteristic p, a prime number. Prove that the map $a \mapsto a^p$ defined for $a \in R$ is a ring homomorphism. (This is called the *Frobenius homomorphism*.)