Algebra Autumn 2017
Frank Sottile
24 October 2017

Eighth Homework

Write your answers neatly, in complete sentences. Recopy your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Hand in to Frank Tuesday 31 October: (Have this on a separate sheet of paper.)

29. The wreath product $S_m \wr S_n$ of symmetric groups is the semidirect product $(S_m)^n \rtimes_\varphi S_n$ where φ is the action of S_n on $(S_m)^n$ permuting the factors of $(S_m)^n$.

    (a) For $(\pi_1, \ldots, \pi_n\,,\, \omega) \in S_m \wr S_n$ ($\pi_i \in S_n$ and $\omega \in S_n$) define the map from $[m] \times [n]$ to itself by
    $$(\pi_1, \ldots, \pi_n\,,\, \omega).(i,j) \;:=\; (\pi_{\omega(j)}(i)\,,\, \omega(j))\,.$$
    (Here, $[m] := \{1, \ldots, m\}$ and the same for $[n]$.) Show that this defines an imprimitive action of $S_m \wr S_n$ on $[m] \times [n]$.

    (b) Using this action or any other methods show that $S_2 \wr S_2 \simeq D_8$, the dihedral group with 8 elements.

    (c) This action realizes $S_3 \wr S_2$ as a sugroup of S_6. What are the cycle types of permutations of $S_3 \wr S_2$? For each cycle type, how many elements of $S_3 \wr S_2$ have that cycle type?

Hand in for the grader Tuesday 31 October:

30. Determine the derived series for the symmetric group S_4. (This is the series of iterated commutator subgroups).

31. Use semidirect products to classify all groups of order 30 up to isomorphism.

32. Use semidirect products to classify all groups of order 18 up to isomorphism.

33. Let p be a prime number. How many simple subgroups are there in $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$?

34. Prove the Fundamental Theorem of Arithmetic by applying the Jordan-Hölder Theorem to the cyclic group $\mathbb{Z}_n$, where $n \in \mathbb{N}$.

35. Let p and q be prime numbers. Prove that any group of order p^2q is solvable.