Algebra Autumn 2017
Frank Sottile
17 October 2017

Seventh Homework

Write your answers neatly, in complete sentences. Recopy your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Hand in to Frank Tuesday 24 October: (Have this on a separate sheet of paper.)

22. I want the best, slickest, and most complete proof of the problem from the exam:

    Let G be any group. Set M to be the intersection of all subgroups of finite index in G. Prove that M is normal in G.

23. Suppose that G is a finite p-group. Show that G contains a normal subgroup of order q for every positive integer q dividing the order of G. Describe such subgroups for the Heisenberg group (from Problem 19 on the sixth Homework).

Hand in for the grader Tuesday 24 October:

24. Show that the dihedral group D_n acting on the vertices of the n-gon is primitive if and only if n is a prime number.

25. Let G be a finite group acting faithfully on a set S. Prove that if G is 2-transitive, then G is a primitive permutation group.

26. Let $G \subset S_{12}$ be the subgroup of the group of permutations of $[12]$ generated by the following two permutations

    $$\sigma := (1,2)(3,4)(5,7)(6,8)(9,11)(10,12) \quad \text{and} \quad \tau := (1,2,3)(4,5,6)(8,9,10)\,.$$

    Show that G is 2-transitive. (In fact, G is 5-transitive, it is the Mathieu group M_{12}.)

27. Let G be a group with an element x having exactly three distinct conjugates. Prove that G is not simple.

    Prove that if G has an element with exactly four distinct conjugates, then G is not simple.

28. Show that any group of order 200 must contain a normal Sylow p-subgroup, and hence is not simple.