Algebra Autumn 2017
Frank Sottile
10 October 2017

Sixth Homework

Write your answers neatly, in complete sentences. Recopy your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Hand in for the grader Tuesday 17 October:

17. Free abelian groups and the rational numbers.
    (a) Show that the additive group of the rational numbers is not a free abelian group.
    (b) Show that the multiplicative group of the positive rational numbers is a free abelian group of countable rank.
18. Suppose that G is a group of order p^2, p a prime number. Show that G is either isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$ or to $\mathbb{Z}_{p^2}$.
19. Let p be a prime number and consider the group

$$U \ := \ \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{Z}_p \right\} \ ,$$

    Show that U is a subgroup of $GL(3, \mathbb{Z}_p)$, and that its order is p^3.

    If $p \geq 3$, show that every non-identity element of U has order p. What if $p = 2$?

    What is the centre $Z(U)$ of U? Show that $U/Z(U) \simeq \mathbb{Z}_p \oplus \mathbb{Z}_p$.

    When $p = 2$, which group of order 8 (you are familiar with all 5) is U isomorphic to?
20. Show that if $n \neq 6$, then the symmetric group S_n has only inner automorphisms.

    Hint: Any automorphism of a group permutes the conjugacy classes. Determine the numbers in the different conjugacy classes of involutions (permutations $\sigma \neq e$ with $\sigma^2 = e$).

    What happens when $n = 6$?
21. Let H be a subgroup of a group G and define the core of H to be

$$\text{core}(H) \ := \ \bigcap \{ H^g \mid g \in G \} \,,$$

    the intersection of all conjugates of H by elements of G.

    Let $S := \{xH \mid x \in G\}$ be the set of left cosets of H in G. For each $g \in G$, define $g^* \colon S \to S$ by $g^*(xH) = gxH$. Show that the kernel of the group homomorphism $G \to \text{Sym}(S)$ given by $g \mapsto g^*$ is the core of H.