Algebra Autumn 2017
Frank Sottile
27 September 2017

Fifth Homework

Write your answers neatly, in complete sentences. Recopy your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Hand in for the grader Tuesday 3 October:

17. Which conjugacy class in $S_6$ is the largest? Justify your answer.
18. Given an example to show that the weak direct product is not a coproduct in the category of all groups. It suffices to consider the case of two factors. That is, find a group $G$ and groups $H, K$ that have homomorphisms $f_H\colon H \to G$ and $f_K\colon K \to G$ for which there is no homomorphism $f\colon H \times K \to G$ such that $f|_H = f_H$ and $f|_K = f_K$.
19. Following this last question up, show that weak product is a coproduct in the category of abelian groups. That is, suppose $\{H_\alpha \mid \alpha \in I\}$ is a family of abelian groups indexed by a set $I$, and $G$ is an abelian group such that there are homomorphisms $f_\alpha\colon H_\alpha \to G$ for $\alpha \in I$. Prove there is a unique map $f\colon {}^w\prod\{H_\alpha \mid \alpha \in I\} \to G$ such that for each $\alpha \in I$ we have $f_\alpha = f \circ \iota_\alpha$, where $\iota_\alpha\colon H_\alpha \hookrightarrow {}^w\prod\{H_\alpha \mid \alpha \in I\}$ is the canonical injection.

    Deduce that this property determines the weak product ${}^w\prod\{H_\alpha \mid \alpha \in I\}$ of abelian grops up to unique automorphism.
20. Let $G$ be a group with normal subgroups $H$ and $K$ such that both $G/H$ and $G/K$ are abelian. Prove that $G/(H \cap K)$ is abelian.
21. Let $F$ be a free group. Prove that the subgroup generated by all $n$th powers, $\{x^n \mid x \in F\}$, is a normal subgroup of $F$.