Algebra Autumn 2017
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Fourth Homework

Write your answers neatly, in complete sentences. Recopy your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Hand in for the grader Tuesday 26 September:

12. Let G be a group and $C(G) := \{g \in G \mid gh = hg \text{ for all } h \in G\}$ be its *center*. Prove that if $G/C(G)$ is cyclic, then G is abelian.
13. Let G be a finite group of order n and let $\varphi : \hookrightarrow S_n$ be the right regular representation of G on itself (the Cayley embedding). Find necessary and sufficient conditions on G so that its image under φ is a subgroup of the alternating group, A_n.
14. Suppose that G and K are groups with respective normal subgroups $H \triangleleft G$ and $L \triangleleft K$. Give examples showing that each of the following statements do not hold for all groups. You have already encountered these groups.
    (a) $G \simeq K$ and $H \simeq L$ implies that $G/H \simeq K/L$.
    (b) $G \simeq K$ and $G/H \simeq K/L$ implies that $H \simeq L$.
    (c) $G/H \simeq K/L$ and $H \simeq L$ implies that $G \simeq K$.
15. Determine the sizes of the different conjugacy classes in S_4. Use this to prove that D_{12}, the dihedral group of the 12-gon (which has order 24) is not isomorphic to S_4.
16. Let $SL_2(\mathbb{Z}_3)$ be the group of 2×2 matrices of determinant 1 with entries in the field $\mathbb{Z}_3$ with three elements. Show that $SL_2(\mathbb{Z}_3)$ has order 24, and that it is not isomorphic to S_4. Is it isomorphic to D_{24}?