

Write your answers neatly, in complete sentences. I highly recommend recopying your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Hand in to Frank Tuesday 12 September: (Have this on a separate sheet of paper.)

4. Suppose that G is an abelian group with elements a and b of respective orders m and n . Show that G has an element (not necessarily ab !) of order the least common multiple of m and n . Hint: First do the case when m and n are relatively prime.

Bonus: What orders are possible for the element ab ?

Hand in for the grader Tuesday 12 September: (Have this separate from #4.)

6. Suppose that $\varphi: G \rightarrow H$ is a group homomorphism. Fill in the details of the assertion in class: If φ is a bijection, then the inverse function $\varphi^{-1}: H \rightarrow G$ is also a homomorphism. Do this by checking that it preserves the identity, sends products to products, and sends inverses to inverses.
7. Let D_4 be the group under matrix multiplication generated by the real 2×2 matrices $S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $R := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Show that D_4 is a nonabelian group of order 8. Let \square be the square with vertices $(\pm 1, \pm 1)$ in \mathbb{R}^2 . Show that D_4 acts on \square , and is its group of symmetries, called the *dihedral group* of order 8.
8. Let Q_8 be the group generated by the complex 2×2 matrices $\mathbf{i} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\mathbf{j} := \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$. Show that Q_8 is a nonabelian group of order 8. Hint: Observe that $\mathbf{i}\mathbf{j} = \mathbf{j}\mathbf{i}^3$, so that every element of Q_8 has the form $\mathbf{i}^a\mathbf{j}^b$. Note further that $\mathbf{i}^4 = \mathbf{j}^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the identity. This is the *quaternion* group.

Comparing subgroups, or the number of elements of different orders, show that Q_8 is not isomorphic to D_4 .