Algebra Autumn 2017
Frank Sottile
21 November 2017

Eleventh Homework

Write your answers neatly, in complete sentences. Recopy your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Hand in to Frank Thursday 30 November: (Have this on a separate sheet of paper.)

52. Using, for example, that a polynomial over a field of degree d has at most d roots and the structure of cylcic groups (or any other legitimate methods), prove that any finite multiplicative subgroup of a field is cyclic.

Hand in for the grader Thursday 30 November:

53. Suppose that $S \subset R$ is a multiplicatively closed subset of an integral domain R that does not contain 0. Prove that if R is an principal ideal domain, then so is $R[S^{-1}]$, and the same implication for unique factorization domains.
54. Let R be an integral domain, and for each maximal ideal $\mathfrak{m}$ of R, show that the localization $R_{\mathfrak{m}}$ is a subring of the quotient field of R.
55. Continuing the previous problem, show that the intersection of the rings $R_{\mathfrak{m}}$, as $\mathfrak{m}$ ranges over all maximal ideals of R, is R itself
56. Show that the equation $x^2 + 1 = 0$ has infinitely many solutions in Hamilton's Quaternions, $\mathbb{H}H$, which is $\mathbb{R} \oplus i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R}$, where $ij = k$, $ji = -k$, etc. These are defined in the Example on page 117 of my copy of Hungerford in Section III.1.
57. Let F be a field, and consider the ring of formal power series $R := F[[x]]$ in one variable. Show that $f \in R$ is a unit if and only if it has a nonzero constant term. Use this to show that the only ideals in R are $\{\langle x^n \rangle \mid n \in \mathbb{N}\}$.
58. Continuing the previous problem, show that the subring $F[[x]][x^{-1}]$ of the quotient field of $F[[x]]$ is a field. This is the field of formal Laurent series in x.