Algebra Autumn 2017
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8 November 2017

Tenth Homework

Write your answers neatly, in complete sentences. Recopy your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Hand in to Frank Tuesday 14 November: (Have this on a separate sheet of paper.)

43. Let G be a finite group. Show that the center of the group algebra $\mathbb{C}[G]$ has dimension as a $\mathbb{C}$-vector space equal to the number of conjugacy classes of G. (The expected answer is a huge hint for a possible basis for $Z(\mathbb{C}[G])$.)

Hand in for the grader Tuesday 14 November: (several of these are very simple)

44. The *center* $Z(R)$ of a ring R is the set of elements that (multiplicatively) commute with every element of R. Show that the center of a ring R is a subring.

45. Let $n \in \mathbb{N}$ be any square-free integer. Show that, for each choice of sign, $\pm$ the ring $\mathbb{Z}[\sqrt{\pm n}]$ consists of elements of the form $a + b\sqrt{\pm n}$ for $a, b \in \mathbb{Z}$.

46. Show that the ring $\mathbb{Z}[\sqrt{2}]$ has infinitely many units.

47. Show that the ring $\mathbb{Z}[\sqrt{10}]$ is not a UFD.

48. Consider the ring $\mathbb{Z}[x]$ of univariate polynomials with integer coefficients. Show that its I generated by 2 and x is not principal.

49. For $\alpha = a + b\sqrt{-1} \in \mathbb{Z}[\sqrt{-1}]$, (the Gaußian integers) set $N(\alpha) := a^2 + b^2$. Determine the units in $\mathbb{Z}[\sqrt{-1}]$. Show that if $N(\alpha)$ is prime, then α is irreducible. Show that the same conclusion holds if $N(\alpha) = p^2$, where p is a prime in $\mathbb{Z}$ that is congruent to 3 modulo 4.

50. Show that $\mathbb{Z}[\sqrt{-1}]$ is a unique factorization domain. (Hint: Show that it is a Euclidean domain.)

51. Using that $\mathbb{Z}[\sqrt{-1}]$ is a unique factorization domain, show that every prime p that is congruent to 1 modulo 4 is the sum of two squares.

    Hint: use the cyclicity of the group of units of $\mathbb{Z}_p^*$ (you know how to prove this, don't you?) to show that there is a number $n \in \mathbb{Z}$ with $n^2 \cong -1 \pmod{p}$. Then use this to show that p is reducible in $\mathbb{Z}[\sqrt{-1}]$.