Algebra Autumn 2017
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1 September 2017

First Homework

Write your answers neatly, in complete sentences. I highly recommend recopying your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Hand in to Frank Thursday 7 September: (Have this on a separate sheet of paper.)

1. Linear algebra is part of the algebra qualifying exam. Prove the following identity, where A is an $n \times n$ matrix over a field $\mathbb{K}$ (it is OK to work over $\mathbb{C}$ if that makes you more comfortable) and $u, v \in \mathbb{K}^n$ are (column) vectors,

$$\det(A) u^T A^{-1} v \;=\; \det(A + vu^T) - \det(A)\,.$$

(Hint: Suppose that u and v are vectors in an ordered basis and use Cramer's rule, then argue the general form from this.)

Note: The earlier versions of this problem were incorrect. This was an algebraic version of the dreaded sign error.

Hand in for the grader Thursday 7 September: (Have this separate from #1.)

A *subgroup* of a group G is a subset H of G which is a group in its own right, under the group operations of G. For example, the set $2\mathbb{Z}$ of even integers is a subgroup of the additive group of integers.

2. Show that a group G cannot be the union of two proper subgroups.
3. Show that the additive group of ordered pairs of integers $\mathbb{Z} \oplus \mathbb{Z}$ is the union of three proper subgroups. (There is a story here: A TAMU graduate student found an error in an important paper in tropical geometry, where this was the counterexample to a key assertion in the key lemma.)
4. Suppose that G is an abelian group with elements a and b of respective orders m and n. What is the order of the element ab?

   This problem was incorrectly formulated, so it was unassigned.
5. Let $GL(2, \mathbb{Z})$ be the collection of 2×2 matrices with integer entries and determinant ± 1. This is a group under multiplication of matrices, with identity $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Let $A := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $B := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$. What is the order of A ? Of B ? Of AB?