Math 220 Section 902
First Test: Answer key

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Full credit is given only for complete and correct answers.
No aids allowed on the exam. Please write your answers in blue books.
I have extra blue books, and blank paper for scratch work.
Point totals are in brackets next to each problem.

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1. [30] You have watched the videos I assigned about Klein bottles, and hopefully some more. Write a couple of coherent paragraphs about Klein bottles. There are many different topics you could choose to address, among them could be:

   - What is a Klein bottle?
   - What is a Möbius loop?
   - Who was Felix Klein?
   - Is a Klein bottle a bottle?
   - Who is Cliff Stoll?
   - What is the problem with three dimensions?
   - How are Möbius loops related to Klein bottles?
   - What is a non-orientable surface?
   - What is the volume of a Klein bottle?
   - What do you get if you cut a Klein bottle in half?

   You are not limited to these topics and should not address all of these questions, they are only to help you get started. Your writing will be judged on writing criteria: structure of paragraphs, coherence, clarity, relevance to Klein bottles, etc.

   For marking this, I gave full credit for a mostly coherent, organized paragraph, took a few points off for those who did not quite do that, and somewhat more for non-coherent paragraphs, or no paragraph at all. It was fairly lenient for writing, but as this was on a test, leniency was called for.

2. [15] Let $P(x)$ and $Q(x)$ be predicates, with x taking values in some universe U. Are the two statements

   $$(\exists x \in U)\,(P(x) \wedge Q(x)) \qquad \text{and} \qquad ((\exists x \in U)P(x)) \wedge ((\exists x \in U)Q(x))$$

   logically equivalent? If so, provide a proof. If not, then given an example of predicates $P(x)$ and $Q(x)$ for which they are not logically equivalent, together with a convincing demonstration.

   These two statements are **not** logically equivalent. For example, let $U := \mathbb{R}$ and the predicates be $P(x) : x < 0$ and $Q(x) : x > 0$. Then $\neg\big(\exists x(P(x) \wedge Q(x))\big)$, as there is no number that is both negative and positive, however, $(\exists x P(x)) \wedge (\exists x Q(x))$ is true, as there is a negative number and there is positive number (these numbers are necessarily different).

   While these are not logically equivalent, there is always the implication

   $$\big(\exists x(P(x) \wedge Q(x)\big) \;\Rightarrow\; \big((\exists x(P(x)) \wedge (\exists x Q(x))\big)\,.$$

   I gave 7 points for essentially getting this implication.

3. [15] Consider the statement: "If the math test is long, then it is difficult". Give (a) its converse, (b) its contrapositive, and (c) a useful negation of this statement.

   (a) If the math test is difficult, then it is long.

   (b) If the math test is easy, then it is short.

   (c) The math test is long and easy.

4. [15] Recall that a real number x is *rational* if it is a quotient of integers, $x = m/n$, where $m, n \in \mathbb{Z}$, and otherwise it is *irrational*.

   Prove the following statement: For all real numbers a, b, if $a + b$ is irrational, then either a is irrational or b is irrational.

   Let a, b be real numbers. We prove the contrapositive of the implication. Suppose that both a and b are rational numbers. (This is the negation of a is irrational or b is irrational.) Since the rational numbers are closed under addition, (or direct computation) $a + b$ is rational. This proves the contrapositive.

5. [10] In Question 4, if the universe for the universal quantifier on a, b is restricted to the rational numbers (instead of the real numbers) is the statement true or false? Give a valid reason for your answer.

   This is actually true. If a, b are rational numbers, then the hypothesis "$a + b$ is irrational" is false, and an implication with a false hypothesis is valid (true). I had not meant to trip people up on this one.

6. [15] Let $x \in \mathbb{R}$ and assume that for all $\epsilon > 0$, $|x| < \epsilon$. Prove that $x = 0$. (Hint: use proof by contradiction, and find a specific $\epsilon > 0$ that contradicts the hypothesis.)

   Let $x \in \mathbb{R}$ and assume that for all real numbers ϵ, if $\epsilon > 0$, then $|x| < \epsilon$. We will prove that $x = 0$, arguing by contradiction.

   Suppose on the contrary that $x \neq 0$. From the definition of $|x|$, we have that $|x| > 0$. Set $\epsilon := |x|$, which is positive, $\epsilon > 0$. By our hypothesis and *modus ponens*, we have that $|x| < \epsilon$. But then $|x| < |x|$, which is a contradiction.

   We see that $x \neq 0$ is impossible, and so we conclude that $x = 0$.

   Here are at least two alternatives. We could argue that as $x \neq 0$ either $x < 0$ or $x > 0$, and then use the definition of absolute value to conclude that $|x| > 0$. Another more definitive contradiction would be to set $\epsilon := |x|/2$, which is less than $|x|$ as $|x| > 0$. Either way leads to a contradiction, which proves the desired statement.