members of this school were known, is the idea that mathematical entities such as numbers and geometrical figures are in fact abstractions distinct from the material world. To the Pythagoreans the concept of number was the key to understanding the mysteries of the universe. By numbers they always meant whole numbers. Although they worked with ratios of whole numbers, which we call fractions, they did not consider them numbers per se. The Pythagoreans are credited with discovering the Pythagorean Theorem but probably did not give a formal proof of it. And as the records left behind by the Babylonians indicate, the Pythagoreans were not the first to use it.

Exercises 1.1

1. Determine whether each of the following sentences is a statement, an open sentence, or neither.
   (a) The Boston Celtics have won 16 NBA championships.
   (b) The plane is leaving in five minutes.
   (c) Get a note from your doctor.
   (d) Is that the best you can do?
   (e) Excessive exposure to the sun may cause skin cancer.
   (f) $5 + 2 = 6$.
   (g) Someone in this room is the murderer.
   (h) $x^2 + 1 \neq 0$.
   (i) For every real number x, $x^2 + 1 \neq 0$.
   (j) The equation of a circle of radius 1 with center at the origin is $x^2 + y^2 = 1$.
   (k) If n and m are even integers, then nm is even.

2. For each of the following statements, determine if it has any universal or existential quantifiers. If it has universal quantifiers, rewrite it in the form "for all . . . ." If it has existential quantifiers, rewrite it in the form, "there exists . . . such that . . . ." Introduce variables where appropriate.
   (a) The area of a rectangle is its length times its width.
   (b) A triangle may be equilateral.
   (c) $8 - 8 = 0$.
   (d) The sum of an even integer and an odd integer is even.
   (e) For every even integer, there is an odd integer such that the sum of the two is odd.
   (f) A function that is continuous on the closed interval $[a, b]$ is integrable on $[a, b]$.
   (g) A function is continuous on $[a, b]$ whenever it is differentiable on $[a, b]$.
   (h) A real-valued function that is continuous at 0 is not necessarily differentiable at 0.
   (i) All positive real numbers have a square root.
   (j) 1 is the smallest positive integer.

3. Write the negation of each statement in Exercise 2.

4. Write the negation of each of the following statements.
   (a) All triangles are isosceles.
   (b) Some even numbers are multiples of three.
   (c) Every door in the building was locked.
   (d) All new cars have something wrong with them.
   (e) Some angles of a triangle are greater than 90 degrees.
   (f) There are sets that contain infinitely many elements.

5. Write the negation of each of the following statements.
   (a) There is a real number x such that $x^2 + x + 1 = 0$.
   (b) Every real number is less than 100.
   (c) If f is a polynomial function, then f is continuous at 0.
   (d) If f is a polynomial function, then f is continuous everywhere.
   (e) $\forall x$, x real, $\exists$ a real number $y \ni y = x^3$.
   (f) There is a real-valued function $f(x)$ such that $f(x)$ is not continuous at any real number x.

6. Consider the following statement P: "The square of an even integer is divisible by 4."
   (a) Write P as a statement in the form, "for all . . . , if . . . , then . . . ."
   (b) Write the negation of P.
   (c) Prove P or $\neg P$. Explain any inductive reasoning you use to conjecture that P is true or that P is false.

7. Consider the following statement P: "The sum of two even integers is divisible by 4."
   (a) Write P as a statement in the form "for all . . . , if . . . , then . . . ."
   (b) Write the negation of P.
   (c) Prove P or $\neg P$. Explain any inductive reasoning you use to conjecture that P is true or that P is false.

8. In Example 14 the definition of a bounded function was given.
   (a) Write the negation of this definition; that is, complete the following statement: "A real-valued function $f(x)$ is *not bounded* on the closed interval $[a, b]$ if . . . ."
   (b) Give an example of a bounded function on $[0, 1]$. Justify your answer by determining a value for M.
   (c) Give an example of an unbounded function on $[0, 1]$. Justify your answer.
   (d) Suppose the definition of bounded function were worded this way: "A real-valued function $f(x)$ is said to be *bounded* on the closed interval $[a, b]$ if for all $x \in [a, b]$, there exists a positive real number M such that $|f(x)| \leq M$." Does this definition mean the same as the one given in Example 14? If not, explain how they differ. Could this new definition make sense as the definition of a bounded function? Explain.

9. A real-valued function $f(x)$ is said to be *increasing* on the closed interval $[a, b]$ if for all $x_1, x_2 \in [a, b]$, if $x_1 < x_2$, then $f(x_1) < f(x_2)$.
   (a) Write the negation of this definition.
   (b) Give an example of an increasing function on $[0, 1]$.
   (c) Give an example of a function that is not increasing on $[0, 1]$.
10. (a) State a definition for a real-valued function $f(x)$ to be *decreasing* on a closed interval $[a, b]$.
   (b) Give the negation of this definition.
   (c) Give an example of a decreasing function on $[0, 1]$.
   (d) Give an example of a function on $[0, 1]$ that is neither increasing nor decreasing.
11. Prove the following corollary of the Archimedean Principle. (See Example 28 for the statement.) For every positive real number ε, there exists a positive integer N such that $1/n < \varepsilon$ for all $n \geq N$. (Note: this exercise is the basis for the formal proof that the sequence $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ converges to 0.)
12. Use the Archimedean Principle to prove the following: if x is a real number, then there exists a positive integer n such that $-n < x < n$.
13. Prove that if x is a positive real number, then there exists a positive integer n such that $\frac{1}{n} < x < n$.

Discussion and Discovery Exercises

D1. Consider the following question. What positive integers n can be written as the difference of two squares? For example, $5 = 3^2 - 2^2$ and $24 = 5^2 - 1^2$. The following table lists the expression $n = x^2 - y^2$ for varying values of x and y. Since n is positive, we assume that $x > y \geq 0$.

x	y	$x^2 - y^2$	x	y	$x^2 - y^2$	x	y	$x^2 - y^2$
1	0	1	5	4	9	8	0	64
2	0	4	6	0	36	8	1	63
2	1	3	6	1	35	8	2	60
3	0	9	6	2	32	8	3	55
3	1	8	6	3	27	8	4	48
3	2	5	6	4	20	8	5	39
4	0	16	6	5	11	8	6	28
4	1	15	7	0	49	8	7	15
4	2	12	7	1	48	9	0	81
4	3	7	7	2	45	9	1	80
5	0	25	7	3	40	9	2	77
5	1	24	7	4	33	9	3	72
5	2	21	7	5	24	9	4	65
5	3	16	7	6	13	9	5	56

(a) Based on the table, conjecture a theorem that states exactly which positive integers can be written as the difference of two squares.
(b) Try to prove your conjecture.
(c) Notice that some integers appear more than once as the difference of two squares. What accounts for this? Is it possible that an integer will appear infinitely often on this list? Give a reason for your answer.

D2. Reread Example 1. Explain what part of that example involves inductive reasoning and what part involves deductive reasoning. See the introduction to this chapter for a discussion of the difference between inductive and deductive reasoning.

D3. Use inductive reasoning to find a statement about whether or not the sum of two consecutive even integers is divisible by 4. Then prove your statement.

D4. Criticize the following "statement" of the Fundamental Theorem of Calculus: $\int_a^b f(x)\,dx = F(b) - F(a)$.

D5. Criticize the following "proof" of the fact that if n and m are even then $n + m$ is even.

We know that $n = 2t$ and $m = 2t$, so $n + m = 2t + 2t = 4t$. Therefore, $n + m$ is even.

Write out a correct proof.

D6. Is the following a valid proof that every integer multiple of 4 is even? If not, explain what you think is wrong with it and then write your own proof.

Every multiple of 4 has a 4 in it and 4 has a 2 in it.
Therefore, every multiple of 4 has a 2 in it and therefore a multiple of 4 is even.

D7. Consider the following statement P: If n is an integer and n^2 is a multiple of 4, then n is a multiple of 4. The following is a "proof" that P is a false statement:

$6^2 = 36$ and 36 is a multiple of 4 but 6 is not a multiple of 4. Therefore, P is false.

Is this proof valid? Give your reasons and if you think it is not a valid proof, write a correct one.

D8. Consider the following "proof" of the fact that if n is an integer and n^2 is even, then n is even.

Exercises 1.2

1. Let P be a statement form. Prove that P and $\neg(\neg P)$ are logically equivalent.

2. Let P and Q be statement forms. Write the truth tables for the following statement forms.
   (a) $(\neg P) \vee Q$. (b) $\neg(P \vee Q)$.
   (c) $\neg((\neg P) \wedge Q))$. (d) $((\neg P) \wedge (\neg Q)) \vee Q$.

3. Prove that the statement forms $\neg((\neg P) \vee Q)$ and $P \wedge (\neg Q)$ are logically equivalent.

4. Write the negation of the following statements.
   (a) August is a hot month and September is sometimes cool.
   (b) Every member of the baseball team is complaining and not hitting.
   (c) Some cars are comfortable and not expensive.
   (d) Math tests are long or difficult.

5. Write the negation of the following statements.
   (a) If x and y are real numbers such that $xy = 0$, then $x = 0$ or $y = 0$.
   (b) For every integer x, x^2 is odd and $x^3 - 1$ is divisible by 4.
   (c) $\forall n$, n an integer, $\exists$ an integer k such that $n = 2k$ or $n = 2k + 1$.
   (d) $\exists$ a rational number $r \ni 1 < r < 2$.
   (e) A real number can be greater than 2 or less than 1.
   (f) Some functions are neither differentiable at 0 nor continuous at 0.

6. Let P be the statement "Every multiple of 6 is even and is not a multiple of 4" of Example 11 in this section.
   (a) Write P in the form, "for all . . . , if . . . , then. . . ." Use variables.
   (b) Write the negation of P. Use variables.
   (c) Prove P or $\neg P$.

7. Repeat Exercise 6 if P is the statement "If the product of two integers is even, then both of the integers are even."

8. Let $n, m \in \mathbf{Z}$. Write the negation of the statement "Exactly one of the integers n or m is odd."

9. Consider the following statement P: "If n is an odd integer, then there exists an integer x such that $n = 4x + 1$ or $n = 4x + 3$."
   (a) Write the negation of P.
   (b) Prove P or $\neg P$.

10. Let P and Q be statement forms.
    (a) Prove that $P \wedge Q$ is logically equivalent to $Q \wedge P$.
    (b) Prove that $P \vee Q$ is logically equivalent to $Q \vee P$.

11. Let $P(x)$ and $Q(x)$ be open sentences containing the variable x. In each part of this problem, determine if the given statements S and T are

logically equivalent. If they are, give a proof. If they are not, give an example of open sentences $P(x)$ and $Q(x)$ for which S and T are not logically equivalent.

(a) S: $\forall x, (P(x) \wedge Q(x))$.
T: $(\forall x, P(x)) \wedge (\forall x, Q(x))$.

(b) S: $\exists x \ni (P(x) \wedge Q(x))$.
T: $(\exists x \ni P(x)) \wedge (\exists x \ni Q(x))$.

12. Let P, Q, and R be statement forms. Write the truth tables for the following statement forms.
(a) $(P \wedge \neg Q) \vee (\neg R)$. (b) $(\neg P) \wedge (R \wedge (\neg Q))$.
(Reminder: To do this problem, it is necessary to list all possible combinations of truth values for P, Q, and R. There are eight of them.)

13. Let P, Q, and R be statement forms.
(a) Prove that $(P \vee Q) \vee R$ and $P \vee (Q \vee R)$ are equivalent statement forms.
(b) Prove that $(P \wedge Q) \wedge R$ and $P \wedge (Q \wedge R)$ are equivalent statement forms.
(c) Prove that $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge (P \vee R)$ are equivalent statement forms.

14. Characterize all real numbers x such that $x > 1$ or $|x| < 3$. Express your answer in the simplest possible way without using absolute value signs.

15. Let P, Q, R, and S be statement forms.
(a) Prove that $P \vee ((Q \wedge R) \wedge S))$ and $(P \vee Q) \wedge (P \vee R) \wedge (P \vee S)$ are equivalent statement forms.
(b) Prove that $P \wedge ((Q \vee R) \vee S))$ and $(P \wedge Q) \vee (P \wedge R) \vee (P \wedge S)$ are equivalent statement forms.

16. Let P and Q be statement forms.
(a) Prove that $(P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$ is a tautology.
(b) Prove that $(P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg Q)$ is a contradiction.

Discussion and Discovery Exercises

D1. Consider the following "proof" that if n or m is an odd integer, then nm is an even integer.

Suppose that n is odd and m is even. Then $m = 2t$ for some integer t. Therefore $nm = n(2t) = 2nt$, which is even. Next suppose that n is even and m is odd. We can write $n = 2s$ for some integer s. Thus $nm = 2sm$, which is also even. In both cases, we get that nm is even. Thus the statement is proved.

Is the proof valid? If you think it is not, explain.

D2. Consider the following two statements:

P: For every real number x, $x^2 \geq 0$.
Q: Lyndon Johnson was elected president in 1964.

Are these statements logically equivalent? Explain.

D3. Using the five statements below as clues, match Sarah, Ann, and Bob with their respective occupations (teacher, entomologist, or poet) and the color of their houses (brown, white, or green). Explain how you arrived at your answer and mention any rules of logic that you use. Assume that no two people have the same occupation or the same color house.

The first three clues are *true* statements:

1. Sarah or Ann is the poet.
2. Sarah's or Bob's house is green.
3. Ann's house is green or white.

The next two clues are *false* statements:

4. The teacher's house is green.
5. Sarah is the poet or Bob is the entomologist.

D4. Consider the following two statements:

P: For every even integer n, there is an odd integer m such that $n + m$ is odd.
Q: There is an odd integer m such that for every even integer n, $n + m$ is odd.

(a) Do these statements mean the same thing? If not, explain the difference.
(b) Write the negations of P and Q.
(c) Discuss the truth or falsity of statements P and Q. Give proofs.
(d) Are these statements logically equivalent? Give reasons for your answer.

D5. Repeat the previous problem for the following statements:

P: For every real number x between 0 and 1, there is a real number y between 1 and 2 such that $x + y < 2$.
Q: There is a real number y between 1 and 2 such that for every real number x between 0 and 1, $x + y < 2$.

D6. Euclid's fourth postulate says: "All right angles are equal to one another." Isn't it obvious that all right angles are equal? What do you think Euclid meant by right angle? Why does he consider it necessary to include this postulate?