Moreover, if a problem has been reduced to the integration of a rational function, it is then certain that an elementary primitive exists, even when the difficulty or impossibility of finding the factors of the denominator may preclude writing this primitive explicitly.

PROBLEMS

1. This problem contains some integrals which require little more than algebraic manipulation, and consequently test your ability to discover algebraic tricks, rather than your understanding of the integration processes. Nevertheless, any one of these tricks might be an important preliminary step in an honest integration problem. Moreover, you want to have some feel for which integrals are easy, so that you can see when the end of an integration process is in sight. The answer section, if you resort to it, will only reveal what algebra you should have used.

(i) $\displaystyle\int \frac{\sqrt[5]{x^3}+\sqrt[6]{x}}{\sqrt{x}}\,dx.$

(ii) $\displaystyle\int \frac{dx}{\sqrt{x-1}+\sqrt{x+1}}.$

(iii) $\displaystyle\int \frac{e^x+e^{2x}+e^{3x}}{e^{4x}}\,dx.$

(iv) $\displaystyle\int \frac{a^x}{b^x}\,dx.$

(v) $\displaystyle\int \tan^2 x\,dx.$ (Trigonometric integrals are always very touchy, because there are so many trigonometric identities that an easy problem can easily look hard.)

(vi) $\displaystyle\int \frac{dx}{a^2+x^2}.$

(vii) $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}}.$

(viii) $\displaystyle\int \frac{dx}{1+\sin x}.$

(ix) $\displaystyle\int \frac{8x^2+6x+4}{x+1}\,dx.$

(x) $\displaystyle\int \frac{1}{\sqrt{2x-x^2}}\,dx.$

2. The following integrations involve simple substitutions, most of which you should be able to do in your head.

(i) $\displaystyle\int e^x \sin e^x\,dx.$

(ii) $\int x e^{-x^2}\,dx.$

(iii) $\int \frac{\log x}{x}\,dx.$ (In the text this was done by parts.)

(iv) $\int \frac{e^x\,dx}{e^{2x}+2e^x+1}.$

(v) $\int e^{e^x} e^x\,dx.$

(vi) $\int \frac{x\,dx}{\sqrt{1-x^4}}.$

(vii) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx.$

(viii) $\int x\sqrt{1-x^2}\,dx.$

(ix) $\int \log(\cos x)\tan x\,dx.$

(x) $\int \frac{\log(\log x)}{x\log x}\,dx.$

3. Integration by parts.

(i) $\int x^2 e^x\,dx.$

(ii) $\int x^3 e^{x^2}\,dx.$

(iii) $\int e^{ax}\sin bx\,dx.$

(iv) $\int x^2\sin x\,dx.$

(v) $\int (\log x)^3\,dx.$

(vi) $\int \frac{\log(\log x)}{x}\,dx.$

(vii) $\int \sec^3 x\,dx.$ (This is a tricky and important integral that often comes up. If you do not succeed in evaluating it, be sure to consult the answers.)

(viii) $\int \cos(\log x)\,dx.$

(ix) $\int \sqrt{x}\log x\,dx.$

(x) $\int x(\log x)^2\,dx.$

4. The following integrations can all be done with substitutions of the form $x = \sin u$, $x = \cos u$, etc. To do some of these you will need to remember that

$$\int \sec x \, dx = \log(\sec x + \tan x)$$

as well as the following formula, which can also be checked by differentiation:

$$\int \csc x \, dx = -\log(\csc x + \cot x).$$

In addition, at this point the derivatives of all the trigonometric functions should be kept handy.

(i) $\displaystyle\int \frac{dx}{\sqrt{1-x^2}}$. (You already know this integral, but use the substitution $x = \sin u$ anyway, just to see how it works out.)

(ii) $\displaystyle\int \frac{dx}{\sqrt{1+x^2}}$. (Since $\tan^2 u + 1 = \sec^2 u$, you want to use the substitution $x = \tan u$.)

(iii) $\displaystyle\int \frac{dx}{\sqrt{x^2-1}}$.

(iv) $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$. (The answer will be a certain inverse function that was given short shrift in the text.)

(v) $\displaystyle\int \frac{dx}{x\sqrt{1-x^2}}$.

(vi) $\displaystyle\int \frac{dx}{x\sqrt{1+x^2}}$.

(vii) $\displaystyle\int x^3\sqrt{1-x^2}\,dx$.

(viii) $\displaystyle\int \sqrt{1-x^2}\,dx$.

} You will need to remember the methods for integrating powers of sin and cos.

(ix) $\displaystyle\int \sqrt{1+x^2}\,dx$.

(x) $\displaystyle\int \sqrt{x^2-1}\,dx$.

5. The following integrations involve substitutions of various types. There is no substitute for cleverness, but there is a general rule to follow: substitute for an expression which appears frequently or prominently; if two different troublesome expressions appear, try to express them both in terms of some new expression. And don't forget that it usually helps to express x directly in terms of u, to find out the proper expression to substitute for dx.

(i) $\displaystyle\int \frac{dx}{1+\sqrt{x+1}}$.

(ii) $\displaystyle\int \frac{dx}{1+e^x}$.

(iii) $\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$.

(iv) $\int \frac{dx}{\sqrt{1+e^x}}$. (The substitution $u = e^x$ leads to an integral requiring yet another substitution; this is all right, but both substitutions can be done at once.)

(v) $\int \frac{dx}{2 + \tan x}$.

(vi) $\int \frac{dx}{\sqrt{\sqrt{x}+1}}$. (Another place where one substitution can be made to do the work of two.)

(vii) $\int \frac{4^x + 1}{2^x + 1}\, dx$.

(viii) $\int e^{\sqrt{x}}\, dx$.

(ix) $\int \frac{\sqrt{1-x}}{1-\sqrt{x}}\, dx$. (In this case two successive substitutions work out best; there are two obvious candidates for the first substitution, and either will work.)

*(x) $\int \sqrt{\frac{x-1}{x+1}} \cdot \frac{1}{x^2}\, dx$.

6. The previous problem provided gratis a haphazard selection of rational functions to be integrated. Here is a more systematic selection.

(i) $\int \frac{2x^2 + 7x - 1}{x^3 + x^2 - x - 1}\, dx$.

(ii) $\int \frac{2x + 1}{x^3 - 3x^2 + 3x - 1}\, dx$.

(iii) $\int \frac{x^3 + 7x^2 - 5x + 5}{(x-1)^2(x+1)^3}\, dx$.

(iv) $\int \frac{2x^2 + x + 1}{(x+3)(x-1)^2}\, dx$.

(v) $\int \frac{x + 4}{x^2 + 1}\, dx$.

(vi) $\int \frac{x^3 + x + 2}{x^4 + 2x^2 + 1}\, dx$.

(vii) $\int \frac{3x^2 + 3x + 1}{x^3 + 2x^2 + 2x + 1}\, dx$.

(viii) $\int \frac{dx}{x^4 + 1}$.

(ix) $\int \frac{2x}{(x^2 + x + 1)^2}\, dx$.

(x) $\int \frac{3x}{(x^2 + x + 1)^3}\, dx$.

***7.** Potpourri. (No holds barred.) The following integrations involve all the methods of the previous problems

(i) $\displaystyle\int \frac{\arctan x}{1+x^2}\,dx.$

(ii) $\displaystyle\int \frac{x \arctan x}{(1+x^2)^3}\,dx.$

(iii) $\displaystyle\int \log\sqrt{1+x^2}\,dx.$

(iv) $\displaystyle\int x \log\sqrt{1+x^2}\,dx.$

(v) $\displaystyle\int \frac{x^2-1}{x^2+1}\cdot\frac{1}{\sqrt{1+x^4}}\,dx.$

(vi) $\displaystyle\int \arcsin\sqrt{x}\,dx.$

(vii) $\displaystyle\int \frac{x}{1+\sin x}\,dx.$

(viii) $\displaystyle\int e^{\sin x}\cdot\frac{x\cos^3 x-\sin x}{\cos^2 x}\,dx.$

(ix) $\displaystyle\int \sqrt{\tan x}\,dx.$

(x) $\displaystyle\int \frac{dx}{x^6+1}.$ (To factor x^6+1, first factor y^3+1, using Problem 1-1.)

The following two problems provide still more practice at integration, if you need it (and can bear it). Problem 8 involves algebraic and trigonometric manipulations and integration by parts, while Problem 9 involves substitutions. (Of course, in many cases the resulting integrals will require still further manipulations.)

8. Find the following integrals.

(i) $\displaystyle\int \log(a^2+x^2)\,dx.$

(ii) $\displaystyle\int \frac{1+\cos x}{\sin^2 x}\,dx.$

(iii) $\displaystyle\int \frac{x+1}{\sqrt{4-x^2}}\,dx.$

(iv) $\displaystyle\int x \arctan x\,dx.$

(v) $\displaystyle\int \sin^3 x\,dx.$

(vi) $\displaystyle\int \frac{\sin^3 x}{\cos^2 x}\,dx.$

(vii) $\int x^2 \arctan x \, dx.$

(viii) $\int \frac{x \, dx}{\sqrt{x^2 - 2x + 2}}.$

(ix) $\int \sec^3 x \tan x \, dx.$

(x) $\int x \tan^2 x \, dx.$

9. Find the following integrals.

(i) $\int \frac{dx}{(a^2 + x^2)^2}.$

(ii) $\int \sqrt{1 - \sin x} \, dx.$

(iii) $\int \arctan \sqrt{x} \, dx.$

(iv) $\int \sin \sqrt{x + 1} \, dx.$

(v) $\int \frac{\sqrt{x^3 - 2}}{x} \, dx.$

(vi) $\int \log(x + \sqrt{x^2 - 1}) \, dx.$

(vii) $\int \log(x + \sqrt{x}) \, dx.$

(viii) $\int \frac{dx}{x - x^{3/5}}.$

(ix) $\int (\arcsin x)^2 \, dx.$

(x) $\int x^5 \arctan(x^2) \, dx.$

10. If you have done Problem 18-9, the integrals (ii) and (iii) in Problem 4 will look very familiar. In general, the substitution $x = \cosh u$ often works for integrals involving $\sqrt{x^2 - 1}$, while $x = \sinh u$ is the thing to try for integrals involving $\sqrt{x^2 + 1}$. Try these substitutions on the other integrals in Problem 4. (The method is not really recommended; it is easier to stick with trigonometric substitutions.)

***11.** The world's sneakiest substitution is undoubtedly

$$t = \tan \frac{x}{2}, \quad x = 2 \arctan t,$$
$$dx = \frac{2}{1 + t^2} \, dt.$$