Alternating Monodromy From an Involution

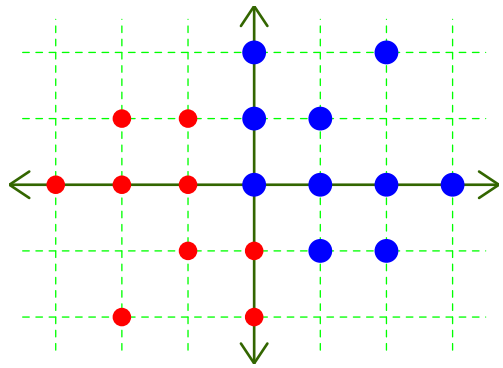
Special Session on Applied Algebra . . .

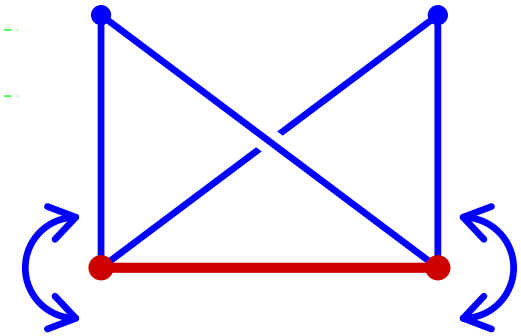
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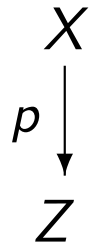


Work with Timothy Duff of U of Missouri
and Josef Schicho of RICAM-Linz

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Galois/Monodromy

Branched covers (maps $p\colon X \to Z$ of irreducible equidimensional varieties) are ubiquitous in enumerative geometry and applications of algebraic geometry.



The *degree* (number points in a typical fiber $p^{-1}(z)$) is the *enumerative* in enumerative geometry. It is *a* measure of the complexity of solving.

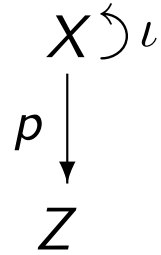
A finer invariant, reflecting subtle internal structure in $p\colon X \to Z$, is its Galois/monodromy group $\mathcal{G}_p$, which also controls algorithms for finding/computing $p^{-1}(z)$.

Today, I will discuss the consequences for the Galois group when X admits an involution ι which preserves the fibers.

This is surprisingly common and has the elementary consequence that the degree, $\#p^{-1}(z)$, is even, say $2d$.

Involution $\Longrightarrow$ Alternating

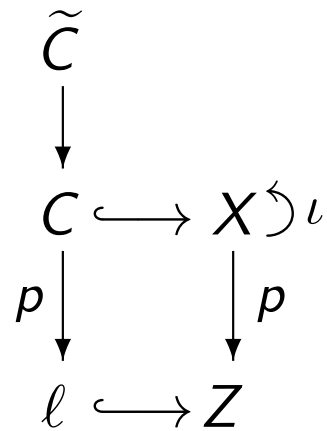
When X admits an involution ι commuting with p, ι acts freely on a general fiber. This shows $\deg(p) = 2d$ is even *and* that the monodromy group is imprimitive and is a subgroup of $(\mathbb{Z}/2\mathbb{Z})^d \rtimes S_d =: B_d$.



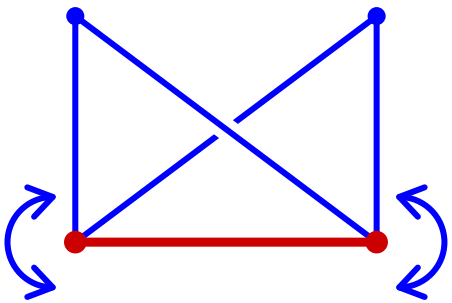
Theorem. *If p is a proper map, the image $p(X^\iota)$ of the fixed point locus of ι has codimension more than 1 in Z, and Z is an affine space, then $\mathcal{G}_p \subset B_d \cap Alt_{2d} =: D_d$.*

- The *same* hypotheses imply that the number of real points in any two regular fibers are congruent modulo 4.

Idea of Proof. Let $\mathbb{C}^1 \simeq \ell \hookrightarrow Z$ be a line meeting the branch locus transversally and missing $p(X^\iota)$. Let $C \hookrightarrow X$ be its pullback and $\widetilde{C} \to C$ the normalization. By Lefschetz, $\mathcal{G}_p = \mathcal{G}_{p|_\ell}$. The involution ι on $\widetilde{C}$ shows the ramification $\widetilde{C} \to \ell$ comes in pairs.



Monodromy of Laman Graphs



A *Laman graph* $G = (V \cup \{v_1, v_2\}, E \cup \{(v_1, v_2)\})$ is a graph with finitely many, δ, realizations in $\mathbb{R}^2$ for a general choice of edge distances (modulo translations and rotations).

$\mathcal{G} = \mathbb{Z}_2 \times \mathbb{Z}_2$
$= D_4 \cap A_4$

Setting $\mathbb{E} := \mathbb{C}^2$ and fixing the vertices $v_1, v_2 \in \mathbb{E}$, the vertex-to-distance map is a branched cover of some degree δ.

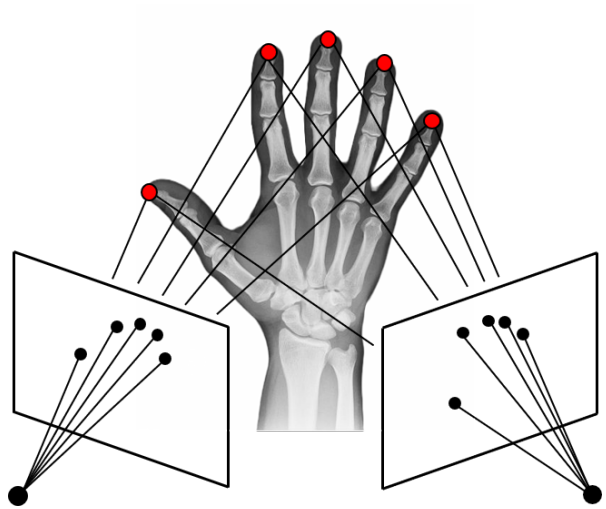
$$\begin{array}{c} \mathbb{E}^V \\ \Big\downarrow p \\ \mathbb{C}^E \end{array}$$

The map $\mathbb{E} \ni (x, y) \mapsto (x, -y)$ induces an involution of the branched cover, and the map p may be compactified so that it is proper.

$\rightsquigarrow$: The monodromy of the branched cover for Laman graphs is a subgroup of $D_{\delta/2}$.

Five-point Reconstruction Problem:

Given images of five points in two views of a camera, find the relative pose of the cameras (and the world points).



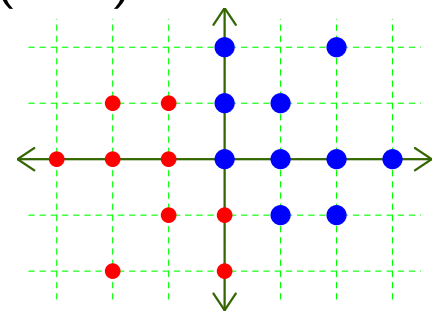
This problem possesses a hidden symmetry, called a *twisted pair*: Rotating the second camera 180° about the line connecting the two camera centres (and a reflection of the world points) gives a second solution.

This involution implies that Galois group is $D_{10} = (\mathbb{Z}/2\mathbb{Z})^{10} \rtimes S_{10} \cap \mathrm{Alt}_{20}$, which is exploited in solvers.

There are many other examples from applications. To paraphrase Joos Heintz: "You may not care about Galois, but Galois cares about you(r problems)".

Unmixed Reciprocal Systems

$f \in \mathbb{C}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]$ is *reciprocal* if $f(z) = f(z^{-1})$.



The support, $\mathcal{A} = \{a \in \mathbb{Z}^n \mid z^a \text{ appears in } f\}$, of f is symmetric in that $\mathcal{A} = -\mathcal{A}$.
Let $0 \in B \subset \mathcal{A}$ be a transversal to the ± 1 orbits.

A reciprocal polynomial f with support B is a linear combination $f = \sum_{b \in B} f_b \frac{z^b + z^{-b}}{2}$.
The coefficients identify $f \in \mathbb{C}^B$. We have the incidence variety

$$X := \{(f_1, \ldots, f_n, z) \in (\mathbb{C}^B)^n \times (\mathbb{C}^\times)^n \mid f_i(z) = 0 \text{ for } i = 1, \ldots, n\}$$

$$\Big\downarrow p$$

$$(\mathbb{C}^B)^n$$

whose fibers $p^{-1}(f_1, \ldots, f_n)$ are the solutions to $f_i(z) = 0$.

Theorem. *$X \xrightarrow{p} (\mathbb{C}^B)^n$ is a branched cover of even degree $2d = vol(conv(\mathcal{A}))$, and if $n > 1$, $\mathcal{G}_p = D_d = B_d \cap Alt_{2d}$.*

Reciprocal Systems are General Enough

$$X := \{(f_1, \ldots, f_n, z) \in (\mathbb{C}^B)^n \times (\mathbb{C}^\times)^n \mid f_i(z) = 0 \ i = 1, \ldots, n\}$$

$$p \downarrow$$

$$(\mathbb{C}^B)^n \ \ni \ f = (f_1, \ldots, f_n)$$

By Kushnirenko's Theorem, $\mathrm{vol}(\mathrm{conv}(\mathcal{A}))$ is an upper bound for the number of solutions $p^{-1}(f)$, counted with multiplicity.

(This is because the solutions $p^{-1}(f)$ are a subset of a linear section of the toric variety $X_{\mathcal{A}}$ of $\mathcal{A}$.)

We have equality, as facial systems of systems of reciprocal polynomials are general. (No solutions at toric infinity.)

To remove multiplicities, we consider the *Chebyshev Variety* $Ch(B)$ defined by Bel-Afiä, Meroni, and Telen.
This is the closure of $\psi_B((\mathbb{C}^\times)^n$, where

$$\psi_B(z) \ := \ [\tfrac{z^b + z^{-b}}{2} \mid b \in B] \ \in \ \mathbb{P}^B .$$

This map is two-to-one on $(\mathbb{C}^\times)^n \setminus \{\pm 1\}^n$.....

Galois Groups of Reciprocal Systems

The Chebyshev Variety $Ch(B)$ is the closure of $\psi_B((\mathbb{C}^\times)^n$, where

$$\psi_B(z) \;:=\; [\tfrac{z^b+z^{-b}}{2} \mid b \in B] \;\in\; \mathbb{P}^B\,.$$

Bertini's Theorem removes multiplicity:
The map $\psi_B\colon (\mathbb{C}^\times)^n \to Ch(B)$ is generically two-to-one, and

$$p^{-1}(f) \;=\; \psi_B^{-1}(\text{ a linear section of } Ch(B)\ )\,.$$

Monodromy of linear sections of $Ch(B)$ is S_d.

A local computation near a singular linear section (inducing a simple transposition in S_d), shows that $\mathcal{G}_p$ contains an involution of the form $(z,\zeta)(z^{-1},\zeta^{-1})$.

This implies $D_d \subset \mathcal{G}_p$. This completes the proof, as the

Alternating Theorem implies $\mathcal{G}_p \subset D_d$.

