Webs and Maximally Inflected Curves?

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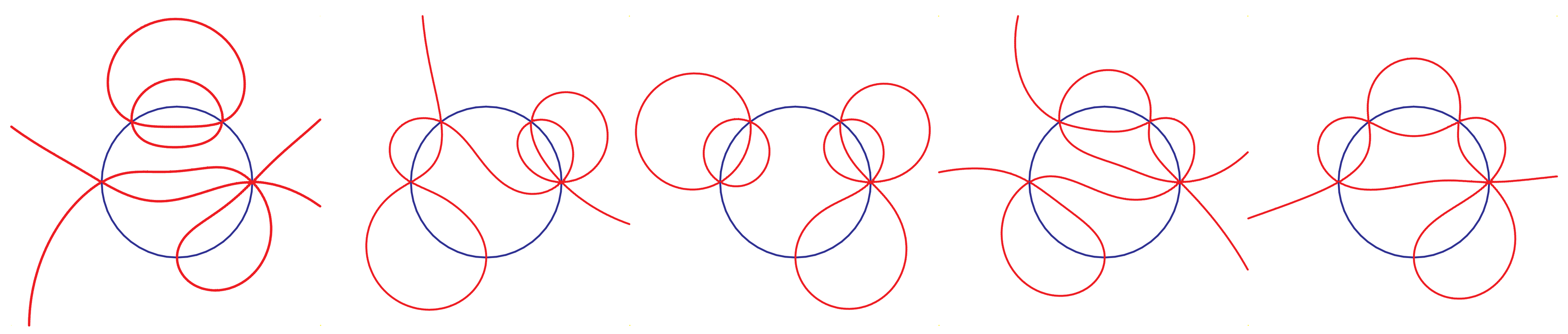
Maximally Inflected Curves

A real rational curve $\varphi\colon \mathbb{P}^1 \to \mathbb{P}^n$ of degree $d{+}n$ is *maximally inflected* if its $N = (n{+}1)d$ ramification points (generalized flexes) lie in $\mathbb{RP}^1$.

Purbhoo: For given ramification, these are in bijection with tableaux of shape $(n{+}1) \times d$ (a consequence of the Shapiro Conjecture).

Purbhoo: The geometry of these curves encode most tableaux combinatorics.

$n = 1$**:** The map $\varphi \mapsto \varphi^{-1}(\mathbb{RP}^1) \cap \mathcal{H}$ ($\mathcal{H}$ is upper-half plane of $\mathbb{CP}^1$) is a bijection between curves and nets preserving tableaux, and this persists when ramification points collide:



1	1	1	2	2
3	3	4	4	5

1	1	1	3	4
2	2	3	4	5

1	1	1	3	3
2	2	4	4	5

1	1	1	2	4
2	3	3	4	5

1	1	1	2	3
2	3	4	4	5

$\varphi^{-1}(\mathbb{RP}^1)$ and tableaux of ramified maximally inflected sextics $\varphi\colon \mathbb{P}^1 \to \mathbb{P}^1$

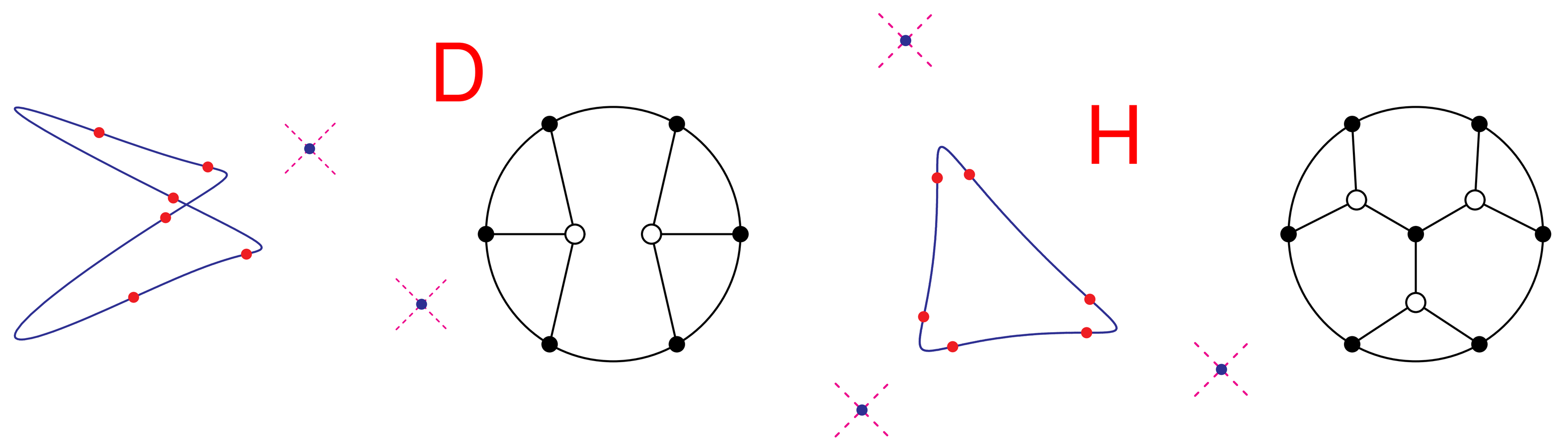
Plane curves

A rational plane curve of degree $d{+}2$ has $\frac{1}{2}d(d{+}1)$ double points. For real curves, these include nodes $\times$ and *solitary points* $\times$.

The *Welschinger invariant* of a curve φ is the parity of the number of its solitary points.

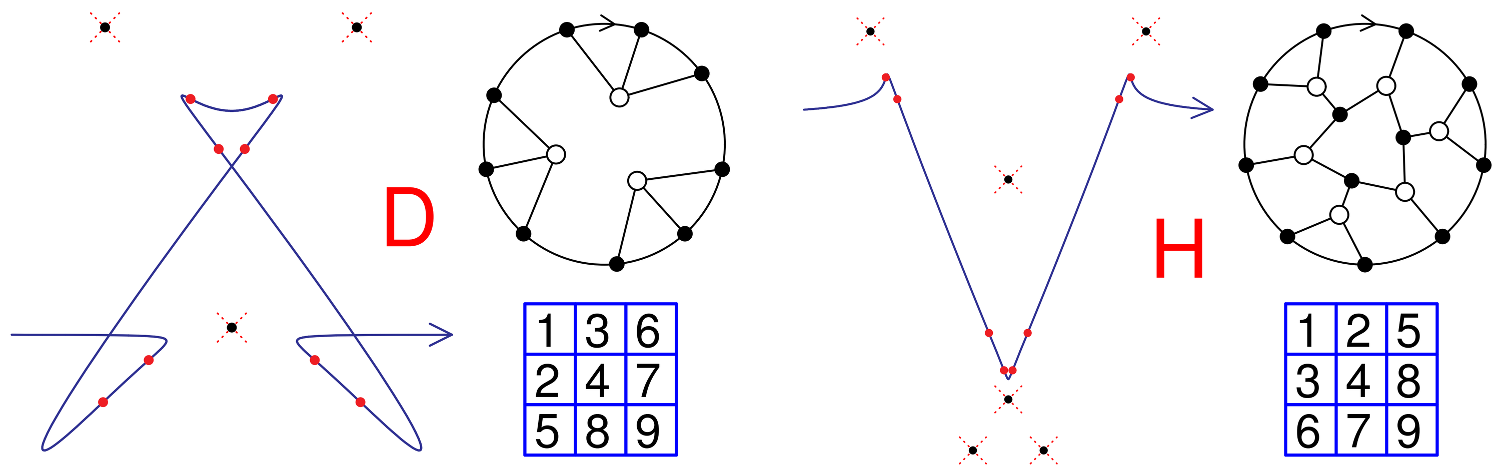
We have a non-rigorous method associating a tableau to a curve.

Maximally inflected quartics and their webs:



Conjecture. The number of solitary points equals the number of regions $(-d-1)$.

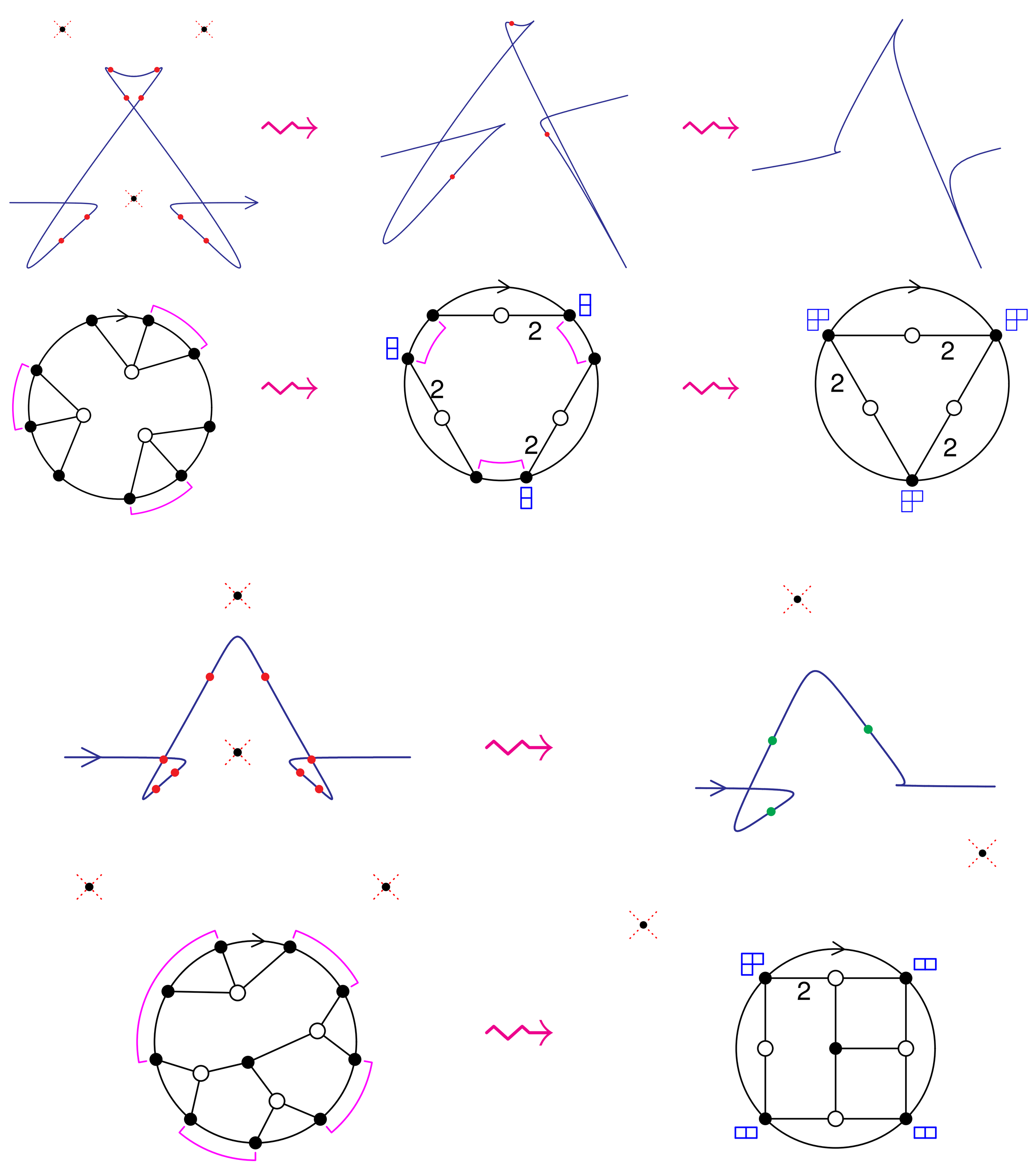
This holds for quintics with simple flexes,



1	3	6
2	4	7
5	8	9

1	2	5
3	4	8
6	7	9

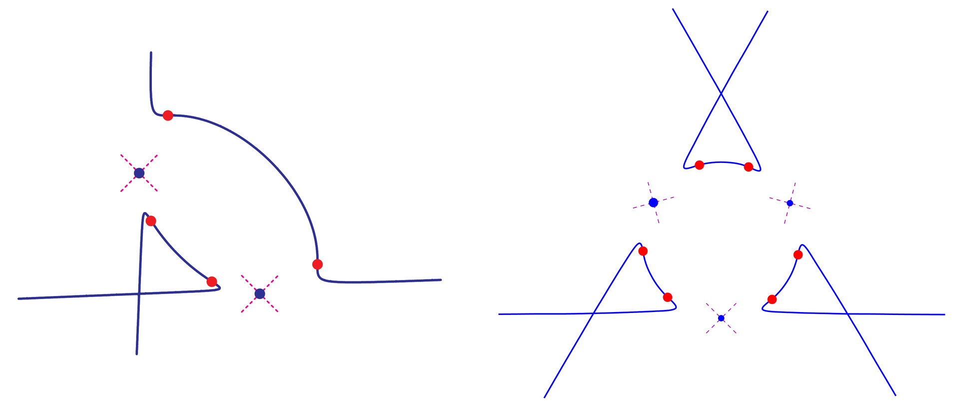
and when ramification points collide:



Constructions

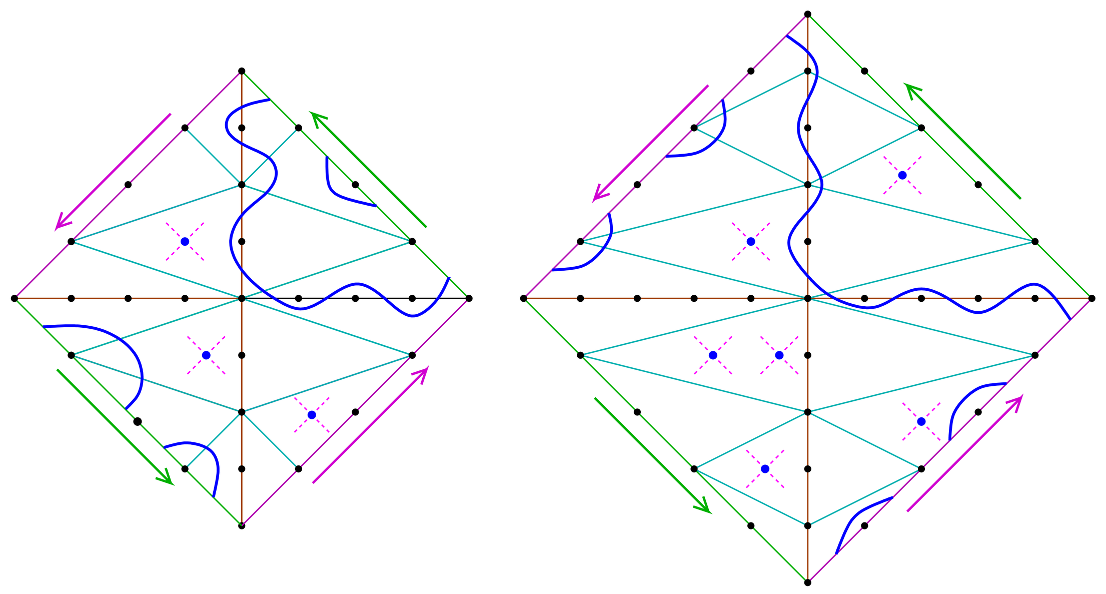
[Kh-S] Construct maximally inflected plane curves of degrees $d{+}2$ generalizing D and H.

Deforming d lines tangent to a conic:



1	4	$\cdots$	$3d{-}2$
2	5	$\cdots$	$3d{-}1$
3	6	$\cdots$	$3d$

Patchworking rational Harnack curves:



1	2	$\cdots$	d
$d{+}1$	$\cdots\cdots$		$2d$
$2d{+}1$	$\cdots\cdots$		$3d$

For both constructions, the conjecture on solitary points and regions of the web holds.

Colliding ramification on curves gives a well-defined notion of degeneration and ramification of webs.

Bibliography

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