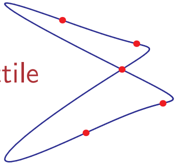
Reality, perhaps non-standard

Real Solutions in Numerical Algebraic Geometry

SIAM conference on Applied Algebra and Geometry
UW Madison, 8 July 2025



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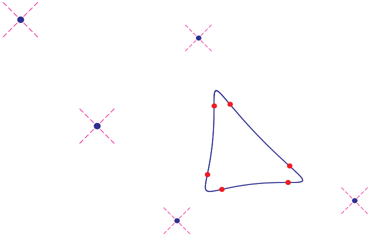


With Tom Brazelton, Maggie Regan, Simon Telen.

Alternate Reality in Nature

Reality is not always what you expect.

Let X be a complex space. A *real structure* on X is an anti-holomorphic involution, $\tau\colon X \to X$ with $\tau^2 = \mathrm{id}$ and $d_x(\sqrt{-1}\xi) = -\sqrt{-1}d_x(\xi)$, for $\xi \in T_xX$.

The real points $X_{\mathbb{R}}$ of (X, τ) are the fixed points X^τ of τ.

Standard. $X = \mathbb{C}^n$ and $\tau(x) := \overline{x}$. Then $X_{\mathbb{R}} = \mathbb{R}^n$.

Twisted diagonal. Any X. On $X \times X$, set $\tau(x, y) := (\overline{y}, \overline{x})$. Then $(X \times X)_{\mathbb{R}} = \{(x, \overline{x}) \mid x \in X\}$, the *twisted diagonal*.

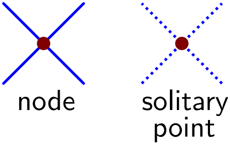
Hermitian matrices. On $X = M_n(\mathbb{C})$, define $\tau(M) := \overline{M}^T$. Then $X_{\mathbb{R}}$ is the set of Hermitian matrices. It combines the two previous structures.

Unitary group. On $X = \mathrm{GL}_n(\mathbb{C})$, define $\tau(M) := \overline{M}^{-T}$. Then $X_{\mathbb{R}}$ is the unitary group U_n.

The reality in our world is often non-standard

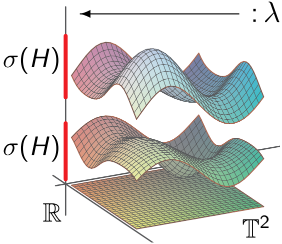
In applications, we have to take reality as nature presents itself.

A real rational normal curve $\gamma\colon \mathbb{P}^1 \to \mathbb{P}^2$ has three types of nodes:



The first two correspond to different real structures on a double point scheme. They give important invariants of real plane curves.

Bloch varieties encode the spectrum of a periodic operator H. They are real with respect to the real structure on $(\mathbb{C}^\times)^d \times \mathbb{C}$ where $\tau(z, \lambda) = (1/\overline{z}, \overline{\lambda})$. We want to understand their features w.r.t. parameters for H.



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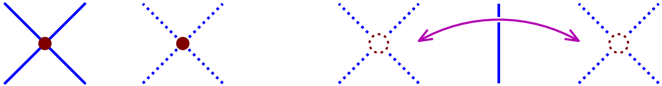


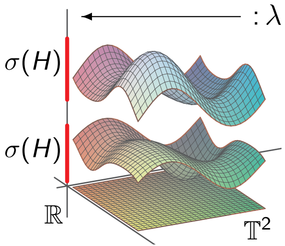
node

solitary point

conjugate pair of nodes

The first two correspond to different real structures on a double point scheme. They give important invariants of real plane curves.

Bloch varieties encode the spectrum of an operator H on a periodic medium. They are real with respect to the real structure on $(\mathbb{C}^\times)^d \times \mathbb{C}$ where $\tau(z,\lambda) = (1/\overline{z}, \overline{\lambda})$. We want to understand their features w.r.t. parameters for H.



How to study reality on a computer

Standard (off the shelf) software and algorithms study standard reality. E.g. Sturm sequences & scores of packages.

Studying non-standard reality includes, e.g. understanding real roots of reciprocal polynomials (f with $t^{\deg(f)}f(t^{-1}) = f(t)$). While some of this is known, it is not well-studied.

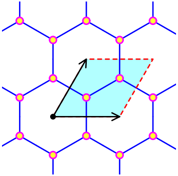
An approach: Any non-standard affine real algebraic variety admits a standard re-embedding (this is just invariant theory).

Example: $\varphi\colon \mathbb{C}^\times \ni z \longmapsto (\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2\sqrt{-1}}) \in \mathbb{C}^2$ satisfies $\varphi(1/\overline{z}) = \overline{\varphi(z)}$. We have $\varphi(\mathbb{C}^\times) = \mathcal{V}(x^2+y^2-1)$, which has codimension 1 and degree 2.

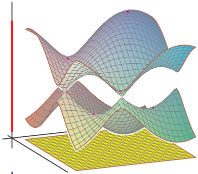
For $(\mathbb{C}^\times)^d$, we get a variety of codimension d and degree 2^d in $\mathbb{C}^d$. This results in unacceptable computational complexity.

Numerical methods can sidestep this complexity, working in situ, just looking for fixed points of the involution τ.

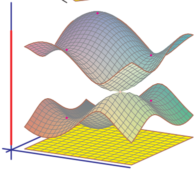
Bloch varieties (with Regan & Telen)



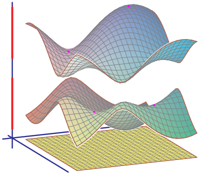
Bloch varieties encode the energy (spectrum) of a Schrödinger operator H on a periodic medium, with respect to the representation theory of the translation group $\mathbb{Z}^d$ (its unitary characters $\mathbb{T}^d$).

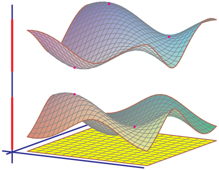


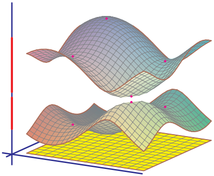
For an operator on a discrete graph, the Bloch variety is a hypersurface in $\mathbb{T}^d \times \mathbb{R}$, whose geometry reflects physical properties of the medium.

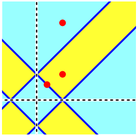


Understanding how its features vary with respect to parameters is a non-standard version of the problem of real root classification, *i.e.* understanding discriminant chambers.



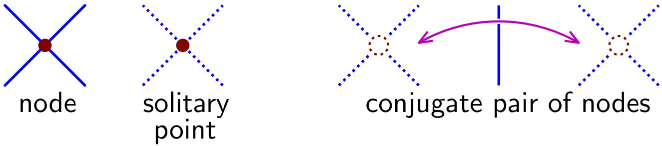






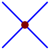
Real rational curves I (w/ Brazelton)

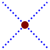
A (standard) real rational plane curve $\varphi\colon \mathbb{P}^1 \to \mathbb{P}^2$ of degree d has three types of nodes:



The *double point scheme* is

$$\mathrm{DP} \;:=\; \{(s,t) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid s \neq t \text{ and } \varphi(s) = \varphi(t)\}\,.$$

 $\Longleftrightarrow$ $\mathrm{DP} \cap (\mathbb{P}^1 \times \mathbb{P}^1)_{\mathbb{R}}$ (standard)

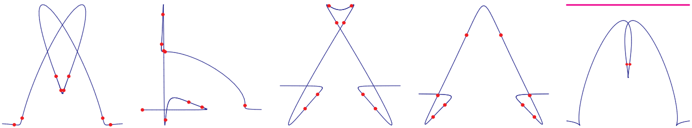
 $\Longleftrightarrow$ $\mathrm{DP} \cap (\mathbb{P}^1 \times \mathbb{P}^1)_{\mathbb{R}}$ (twisted diagonal)

Given the points of DP numerically, one may sort them by which are real with respect to which real structures, by simply checking if they are (nearly) fixed under the corresponding involution.

Real rational curves II: A cautionary tale

(Skipping details......)

Circa 2003, computing $\sim 10^5$ sets of the 42 quintics φ with 9 given real flexes, always found the same vector $(c_0, \ldots, c_6) = (0, 0, 0, 12, 18, 9, 3)$.
(Here, $c_i = \#$curves with i solitary points.)



C. 2023, tried to upgrade this to sextics. Computed sets of 462 sextics φ with 12 given real flexes, and typically found $(c_0, \ldots, c_{10}) = (0, 0, 0, 0, 55, 132, 132, 88, 39, 12, 4)$, but this was only for 80% of instances.

Tried to compute the 6006 septics, but failed most of the time.

The issue appears to be numerical instability. This is an issue of the extreme numerics in this computation.

Last words

Nature, by way of applications, may present us with a different version of reality than what we expect.

Several approaches, including an appealing one advantaged by numerical computation (e.g. inspection), may be bedeviled by issues of complexity.

(Joos Heintz: You may not care about complexity, but complexity cares about you!)

Better, new tools are needed.