Periodic Operators for Algebraic Geometry

Minisymposium on Nonlinera
Algebra in Applications

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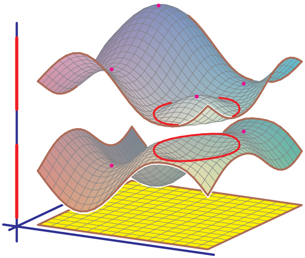

I thank my co-authors: Do, Faust, Kuchment, Lopez-García, Regan, Robinson, Shipman, and Telen, as well as colleagues Berkolaiko, Fillman, and Liu.

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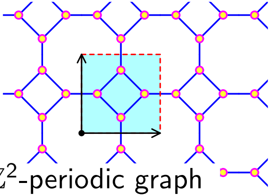
Periodic graph operators to matrices



a $\mathbb{Z}^2$-periodic graph

A $\mathbb{Z}^d$-periodic graph $\Gamma = (\mathcal{V}, \mathcal{E})$ is a discretization of a crystal.
Schrödinger operator (on $\ell_2(\mathcal{V})$) is

$$H \;=\; V \;+\; \Delta\,,$$

where $V \colon \mathcal{V} \to \mathbb{R}$ is a potential and Δ is a weighted graph Laplacian (V and Δ are both periodic).

As H is self-adjoint, its spectrum $\sigma(H) := \{\lambda \mid H - \lambda \text{ not invertible}\}$ is a subset of $\mathbb{R}$.
It has finitely many intervals, representing electron energy bands.

After Fourier (Floquet) transform, H becomes multiplication by the *Floquet matrix*, $L(z) \in \mathrm{Mat}_{n\times n}\,\mathbb{R}[z_1^{\pm}, \ldots, z_d^{\pm}]$.

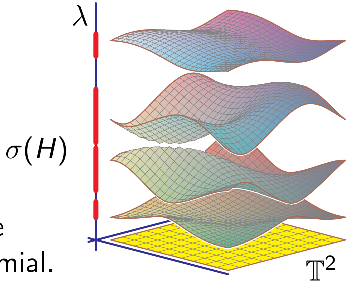
Here, n is the number of $\mathbb{Z}^d$-orbits on $\mathcal{V}$.

As Γ is undirected, $L(Z)^T = L(z^{-1})$.

Twisted reality of Bloch varieties

Let $\mathbb{T} := \{z \in \mathbb{C}^\times \mid \overline{z} = z^{-1}\}$, the unit complex numbers.
Then $\mathbb{T}^d$ = unitary characters of $\mathbb{Z}^d$, and
$\sigma(H) = \{\lambda \in \mathbb{R} \mid \exists z \in \mathbb{T}^d \text{ such that } \det(L(z) - \lambda I_n) = 0\}$.

The *dispersion polynomial* is
$D(z, \lambda) := \det(L(z) - \lambda I_n) \in \mathbb{R}[z^\pm]$.
It defines the *(real) Bloch variety* $\mathrm{BV}_\mathbb{R}$ in $\mathbb{T}^d \times \mathbb{R}$, and the spectrum $\sigma(H)$ is its image under the coordinate function λ.



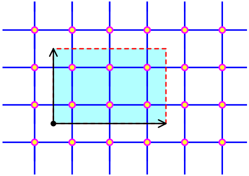
The *Bloch variety* $\mathrm{BV} \subset (\mathbb{C}^\times)^d \times \mathbb{C}$ is the hypersurface defined by dispersion polynomial.

As $L(z)^T = L(z^{-1})$, $D(z, \lambda) = D(z^{-1}, \lambda)$, and $\mathrm{BV}_\mathbb{R}$ is the subset of BV fixed by the *twisted complex conjugation*
$(z, \lambda) \longmapsto (\overline{z^{-1}}, \overline{\lambda})$ on $(\mathbb{C}^\times)^d \times \mathbb{C}$ and BV.

Everything old is new again

1979: van Moerbeke and Mumford considered $\mathbb{Z}$-periodic *directed graphs*, showing an equivalence between the operators and curves with certain divisors. (The curves are the Bloch varieties).

1993: Gieseker, Knörrer, Trubowitz (GKT) studied pure the Schrödinger operator on the grid graph $\mathbb{Z}^2$ where $\mathbb{Z}^2$ acts via $a\mathbb{Z} \oplus b\mathbb{Z}$, with $\gcd(a, b) = 1$. We show this with $a = 3$ and $b = 2$.

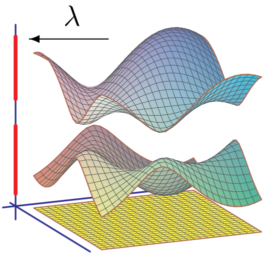


They studied/determined:

- Density of states (gave a formula).
- Irreducibility of Bloch and Fermi ($\lambda =$ const. on BV) varieties.
- Smoothness of Bloch and Fermi varieties.
- Used a toric compactification and the Torelli Theorem.

Presented in a Bourbaki Lecture by Peters in 1992.

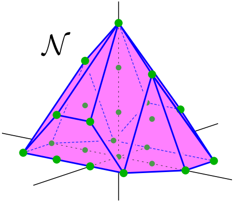
Critical Points



Kuchment: Physicists assume that the critical points on $BV_{\mathbb{R}}$ lying over edges of the spectrum are non-degenerate, which implies many important physical properties. This *Spectral Edges Nondegeneracy Conjecture* is largely open.

$\rightsquigarrow$ First step: study complex critical points of λ on BV.

Faust-S.: Number of critical points is at most the volume of the Newton polytope $\mathcal{N}$ of $D(z, \lambda)$.
It is less than the volume only if $\mathcal{N}$ has vertical faces, or if the Bloch variety is singular along the boundary divisors.
Used the toric compactification given by $\mathcal{N}$.

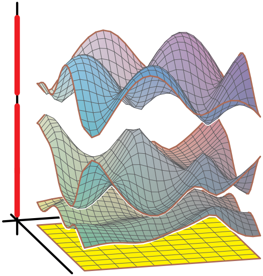


10:45 Sunday: Jonah Robinson will explain how to quantify this asymptotic contribution to the number of critical points.

Irreducibility

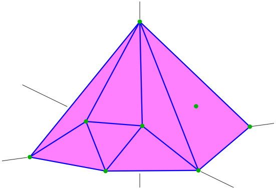
A *Fermi variety* is a λ-level set of Bloch variety.

(Ir)reducibility has long been studied (GKT).
Kuchment-Vainberg: Irreducible Fermi variety implies no embedded eigenvalues, and
Liu: Irreducibility implies quantum ergodicity.



Fillman-Liu-Matos: Used "top homogeneous component" of $D(z, \lambda)$ to show irreducibility for operators on certain graphs.

Faust-Lopez: Generalized this to study irreducibility using *facial forms* of $D(z, \lambda)$. These are restrictions of $D(z, \lambda)$ to faces of Newton polytope.



Sheaves and Compactification

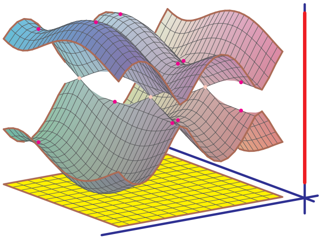
$L(z)\colon \mathbb{C}[z^{\pm}]^n \to \mathbb{C}[z^{\pm}]^n$ is a map of free modules.

Kravaris: Used a free resolution of L to study density of states.

Better: $L(z)$ is an endomorphism of $\left(\mathcal{O}_{(\mathbb{C}^\times)^d}\right)^n$, and the Bloch variety in $(\mathbb{C}^\times)^d \times \mathbb{C}$ is the support of the kernel sheaf to $L(z) - \lambda I_n$ consisting of solutions to $L\psi = \lambda\psi$.

Faust-Lopez-Shipman-S.: Study toric compactfications of

- $\left(\mathcal{O}_{(\mathbb{C}^\times)^d}\right)^n$ (in any toric variety).
- solution sheaf in toric variety of $\mathcal{N}$.
- the operator H (for certain graphs).



15:55 Saturday: Jordy Lopez-García will sketch this toric compactification.

Bestiary of Bloch varieties