Bloch Discriminants

Minisymposium on Discrete and
Continuous Schrödinger Operators

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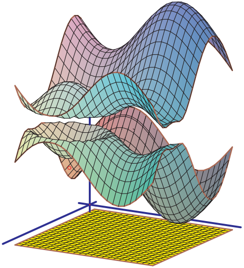

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Bloch Discriminants

Goal: Study how the Bloch variety of a discrete Schrödinger operator depends upon parameters.

Why: This is a special case of the *Geography of Parameter Space* problem in real algebraic geometry: how do features of an algebraic object depend upon the parameters.

Discriminant = parameters where the features change.

Why Bloch: The Bloch variety has a nonstandard real algebraic structure. (Induced by $(\mathbb{C}^\times)^d \ni z \longmapsto 1/\overline{z} \in (\mathbb{C}^\times)$.)

Existing computational methods only treat the standard real structure $\mathbb{R}^d \subset \mathbb{C}^d$. Studying Bloch varieties will yield insight and computational tools for nonstandard real structures.

Bloch Varieties

A *Schrödinger operator* $H = \Delta + P$ on a graph $\Gamma = (V, E)$ acts on $\mathbb{C}^V := \{f\colon V \to \mathbb{R}\}$ with difference operator Δ and potential P. For $f \in \mathbb{C}^V$ and $v \in V$, we have

$$(Hf)(v) = \sum_{(u,v)\in E} c_{u,v}(f(v) - f(u)) \;\; + \;\; P(v)f(v)\,.$$

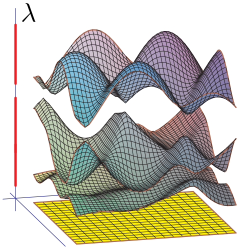
Suppose $\mathbb{Z}^d$ acts on Γ (P & $\{c_{u,v}\}$ invariant) with finitely many orbits. For a character $z\colon \mathbb{Z}^d \to S^1$ ($z \in (S^1)^d$) let

$$\ell_z(\Gamma) := \{f \in \mathbb{C}^V \mid f(v+\alpha) = z^\alpha f(v), v \in V, \alpha \in \mathbb{Z}^d\}\,,$$

a finite-dimensional vector space.

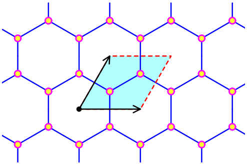
Then H restricts to H_z on $\ell_z(\Gamma)$ as a matrix $L(z)$ of Laurent polynomials. The *Bloch variety* is

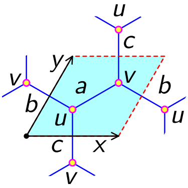
$$\{(z,\lambda) \mid \exists f\in\ell_z(\Gamma) \text{ s.t. } H_z f = \lambda f\} \;=\; \mathcal{V}(\det(L(z) - \lambda I))\,.$$

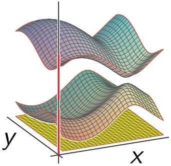


Warm up Exercise: Hexagonal Lattice

We study the critical points of the function λ on the Bloch variety for the hexagonal lattice with given potential and edge labels:







We have

$$L(x,y) = \begin{pmatrix} u+a+b+c & -a-bx^{-1}-cy^{-1} \\ -a-bx-cy & v+a+b+c \end{pmatrix} .$$

The dispersion polynomial is $D(x,y,\lambda) = \det(L-\lambda I)$ and the critical points are given by $D = \partial D/\partial x = \partial D/\partial y = 0$.

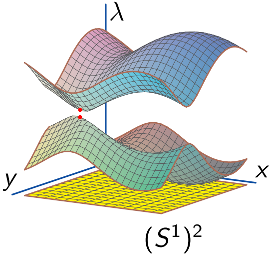
Eliminating x & y gives a degree 10 polynomial in λ with 4 quadratic factors and 2 linear factors.
Its roots are the *critical energies*.

General potential $u \neq v$ $\quad a, b, c > 0$

The quadratic factors give two critical points

$$\frac{u+v}{2} + a + b + c \pm \sqrt{(a \pm b \pm c)^2 + (\frac{u-v}{2})^2}$$

above each corner point $(x, y) = (\pm 1, \pm 1)$.



$(a, b, c) = (5, 3, 2)$

At each linear factor

$$(a + b + c + u - \lambda)\ (a + b + c + v - \lambda)$$

the level set (Fermi curve) is defined by

$$(a + bx + cy)\ (a + bx^{-1} + cy^{-1})\ =\ 0\,.$$

The critical points are above the 2 common zeroes of both factors

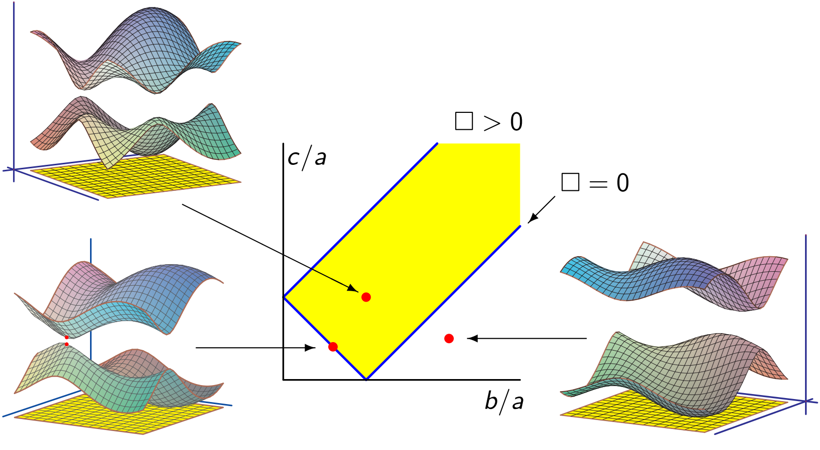
$$x = \frac{c^2 - a^2 - b^2 \pm \Box^{1/2}}{2ab} \quad y = \frac{b^2 - a^2 - c^2 \pm \Box^{1/2}}{2ac}\,,$$

where $\Box = (a + b + c)(a - b + c)(a + b - c)(a - b - c)$.

When $a \pm b \pm c = 0$, there is a degenerate critical point at $(\pm 1, \pm 1)$, and when $\Box < 0$, they are both in $(S^1)^2$. ($\Box > 0$ in $\mathbb{R}^2_>$)

Geography when for $u \neq v \quad a, b, c > 0$

The location and type of critical points in terms of $\Box$:



Equipotential $u = v$ with $a, b, c > 0$

When $u = v$ the linear factors become

$$(a + b + c + u - \lambda)^2$$

and the corresponding critical points are singular.

Above $(x, y) = (\pm 1, \pm 1)$ we have critical points

$$v + a + b + c \pm (a \pm b \pm c)\,.$$

There are singular critical points (Dirac points) above the points

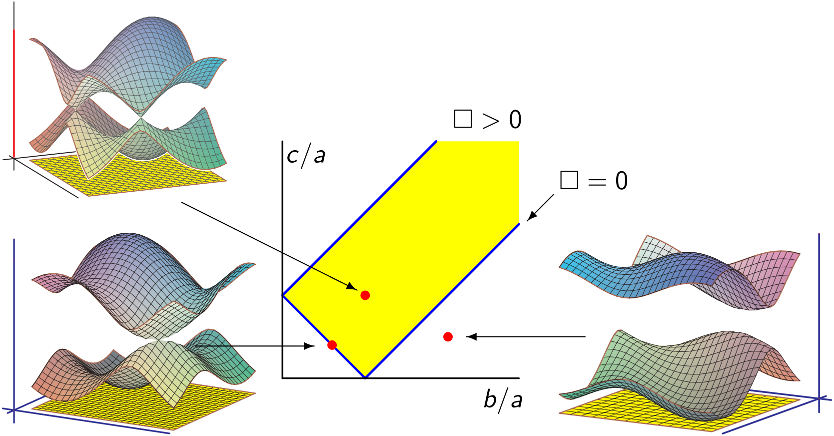
$$x = \frac{c^2 - a^2 - b^2 \pm \Box^{1/2}}{2ab} \quad y = \frac{b^2 - a^2 - c^2 \pm \Box^{1/2}}{2ac}\,,$$

where $\Box = (a + b + c)(a - b + c)(a + b - c)(a - b - c)$.

When $a \pm b \pm c = 0$, there is a degenerate critical point at $(\pm 1, \pm 1)$, and when $\Box < 0$, they are both in $(S^1)^2$. $(\Box > 0 \text{ in } \mathbb{R}^2_>)$

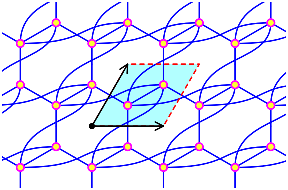
Geography when for $u = v$ $\quad a, b, c > 0$

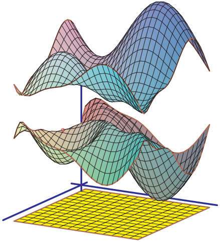
The location and type of critical points in terms of □:



A more serious bipartite graph

Consider the bipartite graph:





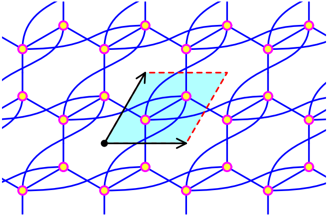
Here, $L(x, y)$ is

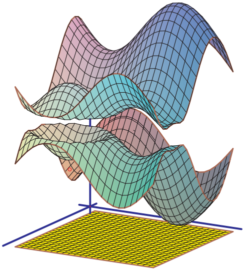
$$\begin{pmatrix} u+a+b+c+d+e & -a-bx^{-1}-cy^{-1}-dx-ey \\ -a-bx-cy-dx^{-1}-ey^{-1} & v+a+b+c+d+e \end{pmatrix} .$$

Eliminating x and y from the critical point equation gives a degree 18 polynomial in λ which factors into two linear, four quadratic, and one degree 8 polynomial.

As before, the quadratic factors give critical points above each of $(x, y) = (\pm 1, \pm 1)$.

More





The linear factors each have reducible Fermi curve

$$(a + bx^{-1} + cy^{-1} + dx + ey)(a + bx + cy + dx^{-1} + ey^{-1}) = 0$$

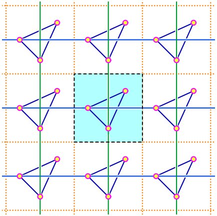
This typically has four singularities, giving four critical points.
(We understand them, their realities, and when they coincide)

When $b = d$ and $c = e$, the whole curve is singular, which is the example of Filonov and Kachkovskiy above.

We partially understand reality for the degree 8 polynomial and all this for equal potentials.

Robinson's Graph

The graph at right has an extremely fascinating Bloch variety. It has singularities, reality issues, critical points at infinity, etc. It is a deep challenge to study this, in part because of the lack of tools for treating non-standard real structures.



We display two views of its Bloch variety; it has two singular points and the apparent curve of self-intersection is not what it appears.