Critical Points of Discrete Periodic Operators

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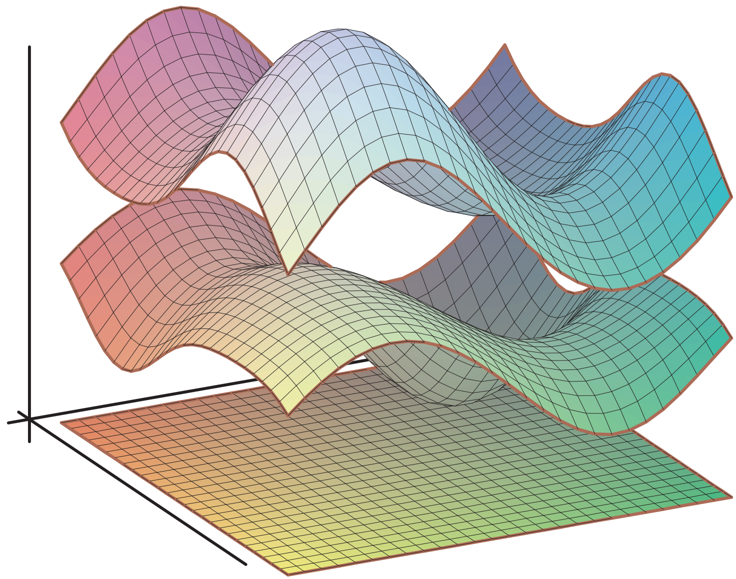
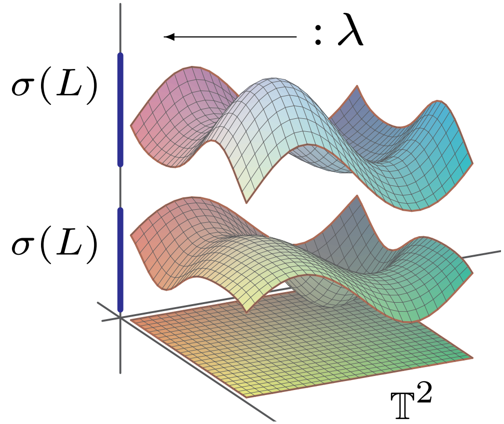


Work with Ngoc Do,
Matthew Faust, and
Peter Kuchment.

Critical Points of Discrete Periodic Operators

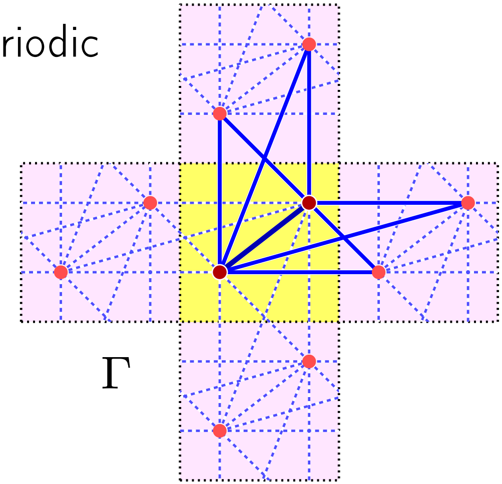
It is a widely held belief in physics that the dispersion relation of a sufficiently general periodic operator is non-degenerate near the edges of its spectrum.
This is needed, for example, for effective masses in solid state physics, and many other properties (ask Peter Kuchment).



This belief holds if the critical points of the function λ on the dispersion relation are Morse non-degenerate.

For a discrete operator on a periodic graph, the dispersion relation is an algebraic variety and degeneracy is an algebraic condition. Thus (non)degeneracy may be studied through the lens of algebraic geometry. Today, I discuss some early results in this program.

Discrete Periodic Laplace-Beltrami Operators



Let $\Gamma \subset \mathbb{R}^n$ be a $\mathbb{Z}^n$-periodic graph with periodic labels $c_e \in \mathbb{C}$ for each edge e. The discrete *Laplace-Beltrami* operator $L = L_c$ is

$$L_c f(u) := \sum c_e(f(u) - f(v)),$$

the sum over edges $e = (u, v)$ of Γ.

We consider the equation

$$L_c f = \lambda f \text{ for } \lambda \in \mathbb{C}.$$

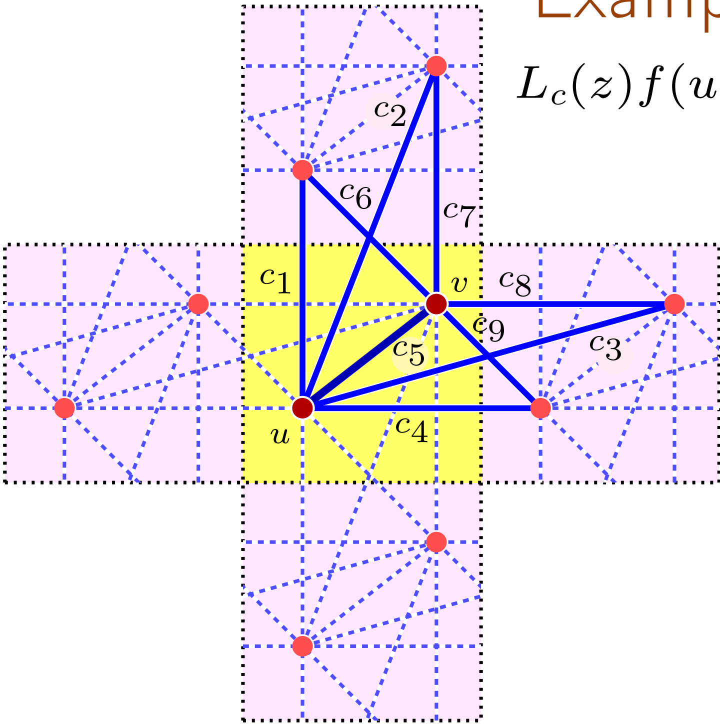
Floquet theory seeks quasi-periodic solutions f for $z \in (\mathbb{C}^\times)^n$:

$$f(v + \alpha) = z^\alpha f(v) \quad \text{for} \quad \alpha \in \mathbb{Z}^n$$

Such a function f is determined by its values at the vertices of a fundamental domain G, so that $f \in \mathbb{C}^{V(G)} = \mathbb{C}^d$.

In the example, $n = 2$, $d = 2$ ($G = K_2$ is diatomic), and $c \in \mathbb{C}^9$.

Example



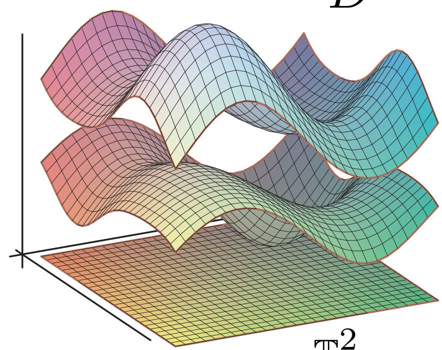
$$
\begin{aligned}
L_c(z)f(u) \;=\;& c_1(f(u) - z_2^{-1}f(u)) \\
&+c_1(f(u) - z_2 f(u)) \\
&+c_2(f(u) - z_2 f(v)) \\
&+c_3(f(u) - z_1 f(v)) \\
&+c_4(f(u) - z_1^{-1} f(v)) \\
&+c_4(f(u) - z_1 f(v)) \\
&+c_5(f(u) - f(v)) \\
&+c_6(f(u) - z_2^{-1} f(v)) \\
&+c_9(f(u) - z_1^{-1} f(v)).
\end{aligned}
$$

$$
\begin{aligned}
L_c(z)f(u) \;=\;& f(u)(2c_1 + c_2 + c_3 + 2c_4 + c_5 + c_6 + c_9 \\
&\qquad - c_1(z_2 + z_2^{-1}) - c_4(z_1 + z_1^{-1})) \\
&-f(v)(c_2 z_2 + c_3 z_1 + c_5 + c_6 z_2^{-1} + c_9 z_1^{-1})\,.
\end{aligned}
$$

Generic (Non)Degeneracy

Set $d = \#V(G)$ and let N be the number of orbits of edges in Γ.

The *dispersion relation* $D = D_c$ is $\{(z,\lambda) \mid \exists f \in \mathbb{C}^d\ L_c(z)f = \lambda f\}$, which is defined in $(\mathbb{C}^\times_z)^n \times \mathbb{C}_\lambda$ by $\Phi := \det(L_c(z) - \lambda I_d) = 0$.



Set $U_{dg} := \{c \in \mathbb{C}^N \mid D_c$ has a degenerate λ-critical point$\}$

Theorem. (DKS) *Exactly one of U_{dg} or $\mathbb{C}^N \smallsetminus U_{dg}$ is dense in $\mathbb{C}^N$, and the other lies in a proper algebraic hypersurface.*

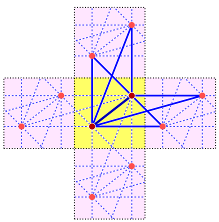
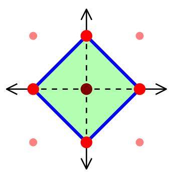
This is because the set of points (c,p) with $p \in D_c$ a degenerate critical point for the function λ is an algebraic subset of $\mathbb{C}^N \times (\mathbb{C}^\times)^n \times \mathbb{C}$.

Dense Periodic Graphs

A *dense* periodic graph Γ is one with as many edges as possible.
(1) The fundamental domain G of Γ is a complete graph K_d.
(2) If there is an edge between G and $G+\alpha$ for some $\alpha \in \mathbb{Z}^n$,
then Γ has all possible edges between G and $G+\alpha$.

Let $\mathcal{A} := \{\alpha \in \mathbb{Z}^n \mid \exists$ an edge between G and $G+\alpha\}$.
Then for $u \in V(G)$, $\qquad L_c(z)f(u) \;=\; \sum_{v \in V(G)} g_{u,v}(c,z)f(v)\,,$

where the matrix entry $g_{u,v}(c,z)$ is a Laurent polynomial in z whose coefficients are linear in c and its monomials are $\{z^\alpha \mid \alpha \in \mathcal{A}\}$.

$\Gamma =$  $\qquad \mathcal{A} =$  $\qquad \left(\begin{array}{c}\mathcal{A} = -\mathcal{A} \text{ is always} \\ \text{symmetric}\end{array}\right)$

Critical Point Equations

$L_c(z)$ is a $d \times d$ matrix of Laurent polynomials, each with support $\mathcal{A}$.

$\Phi = \det(L_c(z) - \lambda I_d)$ is a Laurent polynomial with support $d \cdot (\mathcal{A} \cup \{(\mathbf{0}^n, 1)\})$, a pyramid P (the *Newton polytope* of Φ)

Critical points are defined by the equations $0 = \frac{\partial \lambda}{\partial z_i}, \quad i = 1, \ldots, n$.

As $\;0 = \frac{\partial \Phi}{\partial z_i} + \frac{\partial \Phi}{\partial \lambda}\frac{\partial \lambda}{\partial z_i}$ and $z_i \neq 0$, the *Critical Point Equations* are

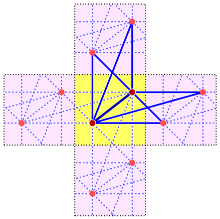
$$\Phi\,,\;\; z_i \frac{\partial \Phi}{\partial z_i} \qquad i = 1, \ldots, n.$$

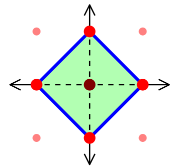
Let P_i be the Newton polytope of $z_i \frac{\partial \Phi}{\partial z_i}$.
We have the following estimate on the number of critical points.

Bernstein's Theorem. *For $c \in \mathbb{C}^N$, the number of λ-critical points on D_c is bounded above by the mixed volume of $P, P_1, \ldots, P_n$.*

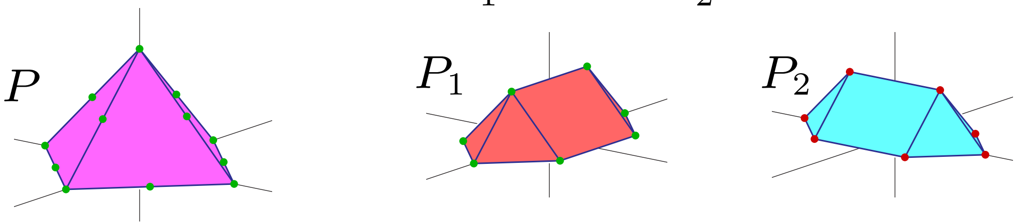
Example

In our running example

$\Gamma =$ 

$\mathcal{A} =$ 

The Newton polytopes of Φ, $z_1 \frac{\partial \Phi}{\partial z_1}$, and $z_2 \frac{\partial \Phi}{\partial z_2}$ are



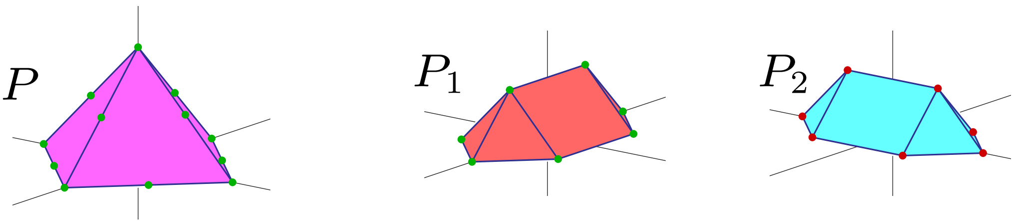
Bernstein's Theorem. *For $c \in \mathbb{C}^9$, the number of critical points on D_c is bounded above by the mixed volume of P, P_1, P_2.*

Monotonicity

Since $P_i \subset P$, we have $MV(P, P_1, \ldots, P_n) \leq (n+1)!\,\text{vol}(P)$.

Theorem. (Rojas) *We have equality if for every k-face F of P, k of the P_i have a k-face along F.*

This holds in our example,



Theorem. (Faust-S.) *For any dense graph, $MV = (n+1)!\,vol(P)$.*

The key to Rojas's result is a lemma of Bernstein:

Bernstein's Lemma. *The number of solutions to the critical point equations is less than the mixed volume if and only if there is a face F of P such that the critical point equations restricted to F have a solution.*

Toric Varieties

For $c \in \mathbb{C}^N$, $D_c = \{\Phi(c, z) = 0\} \subset (\mathbb{C}^\times)^n \times \mathbb{C}$.

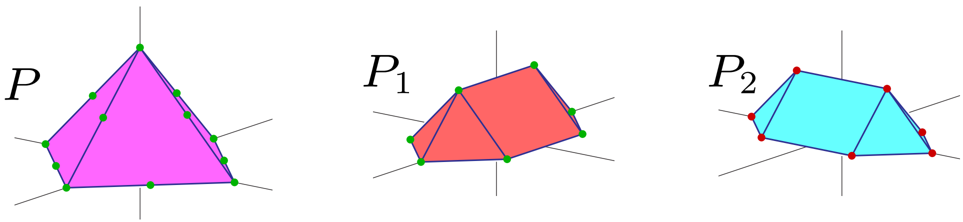
Its closure is a hypersurface in the toric variety X_P corresponding to P.

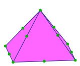
This has the property that $X_P \supset (\mathbb{C}^\times)^n \times \mathbb{C}$, and the difference $X_P \smallsetminus (\mathbb{C}^\times)^n \times \mathbb{C}$ is a union of toric varieties X_F for a face F of P.

Also, $\overline{D_c} \cap X_F$ is defined by the restriction of $\Phi(c, z)$ to F.

Lemma. (Faust-S.) *If $\overline{D_c} \cap X_F$ is smooth for all faces F of P, then the number of critical points is $(n+1)!\,\mathrm{vol}(P)$.*

In our example,



we have $3!\mathrm{vol}($$) = 32$.