Solving Sparse Decomposable Systems

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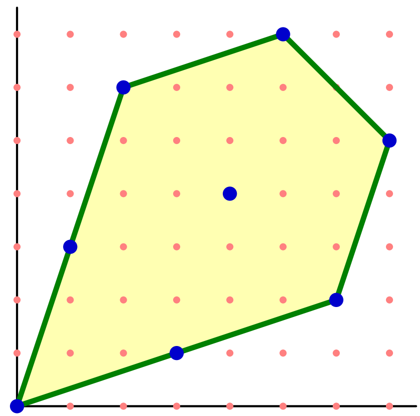


Work with Taylor Brysiewicz, Jose Rodriguez, and Thomas Yahl

Solving Structured Systems

Goal: Develop numerical methods to solve systems of equations that exploit natural structures of the equations.

My current favourite structure:
A family of systems of equations $F(x) = 0$ on $\mathbb{C}^n$ ($x \in \mathbb{C}^n$) parameterized by $\mathbb{C}^N$ ($F \in \mathbb{C}^N$) has an incidence variety

$$\mathcal{X} := \{(x, F) \in \mathbb{C}^n \times \mathbb{C}^N \mid F(x) = 0\}\,.$$

The projection $\pi\colon \mathcal{X} \to \mathbb{C}^N$ has fibre $\pi^{-1}(F) = \{x \mid F(x) = 0\}$.

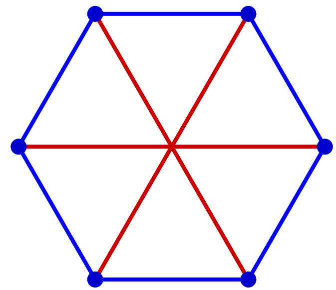
This is a *branched cover*, over an open set $U \subset \mathbb{C}^N$, $\mathcal{X}|_U \to U$ is a covering space with monodromy group G_π, the *Galois group* of this family $\pi\colon \mathcal{X} \to \mathbb{C}^N$ of equations.
(G_π is a Galois group in the usual sense.)

Imprimitivity

Recall: $\pi : \mathcal{X} \to \mathbb{C}^N$ with $\pi^{-1}(F) = \{x \mid F(x) = 0\}$.

The monodromy group G_π of this branched cover acts on fibers. This action is *imprimitive* if G_π preserves a nontrivial partition.

Example. The dihedral group D_6 acts imprimitively on the vertices of the hexagon, preserving opposite pairs of vertices.
This gives: $\mathbb{Z}/2\mathbb{Z} \hookrightarrow D_6 \twoheadrightarrow S_3$.



Proposition. G_π is imprimitive if and only if π factors

$$\pi : \mathcal{X} \to \mathcal{Y} \to \mathbb{C}^N \qquad (*)$$

as a composition of nontrivial branched covers.

Améndola and Rodriguez explained how to exploit such a *decomposable branched cover* $(*)$ in numerical algebraic geometry.

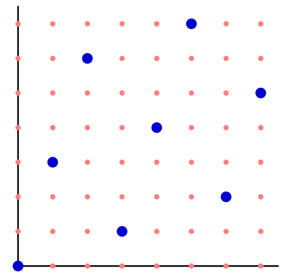
Obstruction: How to compute such a decomposition.

Sparse Polynomial Systems

A point $a \in \mathbb{Z}^n$ corresponds to a monomial $x^a := x_1^{a_1} \cdots x_n^{a_n}$.

Let $\mathcal{A} \subset \mathbb{Z}^n$ be finite with $0 \in \mathcal{A}$. Then $f = \sum_{a \in \mathcal{A}} c_a x^a$ for $c_a \in \mathbb{C}$ is a *sparse polynomial* with *support* $\mathcal{A}$. Write $f \in \mathbb{C}^{\mathcal{A}}$.

Example. The support of $f = 1 + 2x^3y + 3x^6y^2 + 4xy^3 + 5x^4y^4 + 6x^7y^5 + 7x^2y^6 + 8x^5y^7$ is at right.



Let $\mathcal{A}_\bullet = \mathcal{A}_1, \ldots, \mathcal{A}_n$ with $0 \in \mathcal{A}_i \subset \mathbb{Z}^n$. $F = (f_1, \ldots, f_n) \in \mathbb{C}^{\mathcal{A}_\bullet} = \mathbb{C}^{\mathcal{A}_1} \times \cdots \times \mathbb{C}^{\mathcal{A}_n}$ is a *system of polynomials with support* $\mathcal{A}_\bullet$.

Theorem. (Kushnirenko-Bernstein) The number of solutions in $(\mathbb{C}^\times)^n$ to a general system with support $\mathcal{A}_\bullet$ is the mixed volume $MV(\mathcal{A}_\bullet)$ of the convex hulls of the $\mathcal{A}_i$.

Esterov's Theorem

As before, the incidence variety

$$\mathcal{X}_{\mathcal{A}_\bullet} := \{(x, F) \in (\mathbb{C}^\times)^n \times \mathbb{C}^{\mathcal{A}_\bullet} \mid F(x) = 0\}$$

is a branched cover over $\mathbb{C}^{\mathcal{A}_\bullet}$ with Galois group $G_{\mathcal{A}_\bullet}$.

This has two sources of imprimitivity

(1) Lacunary. For example, $f(x) = g(x^3)$.
(2) Triangular. For example, $f(x, y) = g(x) = 0$.

For both, the solutions of f given a solution of g are the preserved partition.

Theorem. (Esterov) $G_{\mathcal{A}_\bullet}$ is the symmetric group if neither (1) nor (2) occurs. Otherwise, $G_{\mathcal{A}_\bullet}$ is imprimitive (besides trivial cases).

We now explain these two cases of lacunary and triangular supports.

Lacunary

Suppose that $\mathcal{A}_\bullet = \mathcal{A}_1, \ldots, \mathcal{A}_n$ are supports with $0 \in \mathcal{A}_i$, and the span $\mathbb{Z}\mathcal{A}_\bullet \subset \mathbb{Z}^n$ has rank n.

Smith normal form of the matrix whose columns are $\mathcal{A}_\bullet \rightsquigarrow d_1, \ldots, d_n \in \mathbb{N}$ and coordinate changes such that $\mathcal{A}_i \subset d_1\mathbb{Z} \oplus d_2\mathbb{Z} \oplus \cdots \oplus d_n\mathbb{Z}$.

Then $f_i(x) = g_i(x_1^{d_1}, \ldots, x_n^{d_n})$, where support of g_i is $\mathcal{B}_i = \mathrm{diag}(\frac{1}{d_1}, \ldots, \frac{1}{d_n})\mathcal{A}_i$.

To solve $F = 0$:

(1) Solve $g_1 = \cdots = g_n = 0$.

(2) For each solution y, get solutions x of F with coordinates

$$x_j := \exp\left(\frac{2\pi \arg(y_j)\sqrt{-1}}{d_j}\right)|y_j|^{\frac{1}{d_j}},$$ up to d_j-th roots of unity.

The Galois group is imprimitive if $MV(\mathcal{B}_\bullet) > 1$ and $d_1 \cdots d_n > 1$.

Triangular

After permuting and changing coordinates using Smith normal form, $\mathbb{Z}\{\mathcal{A}_1, \ldots, \mathcal{A}_k\} \subset \mathbb{Z}^k \oplus 0^{n-k}$ and has rank k.

This gives a projection $p\colon \mathbb{Z}^k \oplus \mathbb{Z}^{n-k} \twoheadrightarrow \mathbb{Z}^{n-k}$ and corresponding coordinates $(x, z) \in (\mathbb{C}^\times)^k \times (\mathbb{C}^\times)^{n-k}$.

To solve $F = 0$:

(1) Solve $f_1(x) = \cdots = f_k(x) = 0$ in $(\mathbb{C}^\times)^k$.

(2) For each solution y, solve the new system

$$G \;:\; f_{k+1}(y, z) = \cdots = f_n(y, z) = 0,$$

which has support $p(\mathcal{A}_{k+1}), \ldots, p(\mathcal{A}_n)$.

The Galois group is imprimitive when $1 \leq k < n$ and $MV(\mathcal{A}_1, \ldots, \mathcal{A}_k) > 1$ and $MV(p(\mathcal{A}_{k+1}), \ldots, p(\mathcal{A}_n)) > 1$.

(Recursive) Algorithm

Given a polynomial system F with support $\mathcal{A}_\bullet$,

If neither lacunary nor triangular, call PHCpack to solve,
otherwise:

If lacunary follow the algorithm given two pages ago.

If triangular follow the algorithm given on last page.

On (admittedly) manufactured examples of systems that are lacunary and/or triangular, perhaps with several levels of structure, this algorithm outperforms PHCpack.

Moral: Exploit structure. Understand Galois groups.

Thanks! Paper to come.....