Semialgebraic Splines

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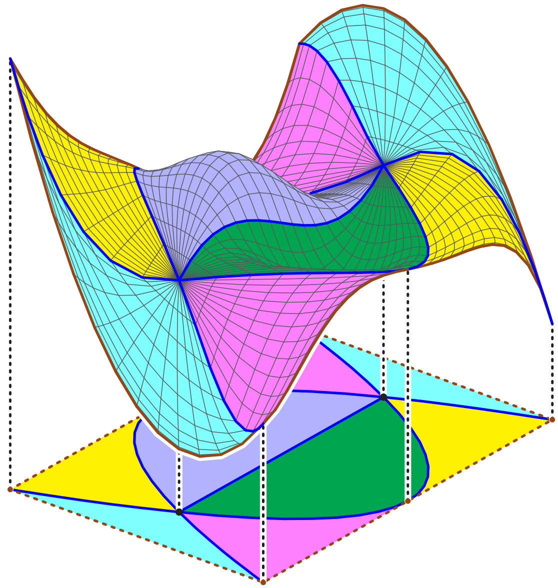
Frank Sottile
sottile@math.tamu.edu



Work with Michael DiPasquale

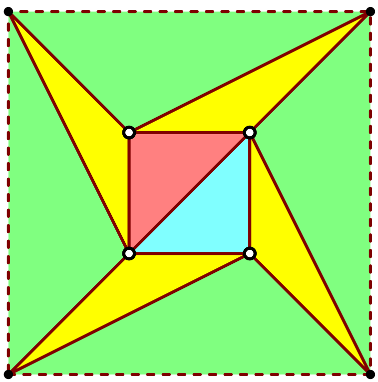
Motivating Goals

Compute dimensions of spline spaces on a complex of semialgebraic cells, to illustrate some phenomena not observed in traditional splines on simplicial or polyhedral complexes, and also to compare to traditional splines.

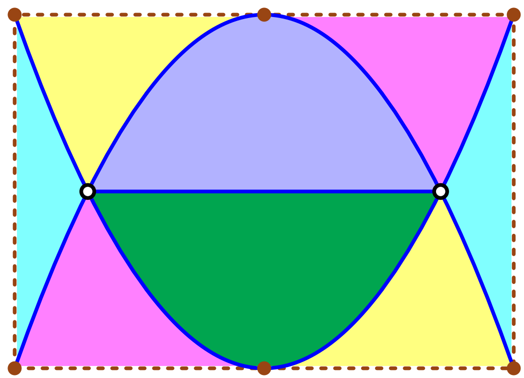
Definition: A (*basic*) *semialgebraic set* is one of the form

$$\{x \in \mathbb{R}^2 \mid h_i(x) \geq 0\,,\ \text{for } i = 1, \ldots, m\}\,,$$

where $h_1, \ldots, h_m$ are polynomials.



Simplicial Complex



Semialgebraic Cell Complex

Semialgebraic Splines

A *semialgebraic spline* is a function that is piecewise a polynomial with respect to a complex Δ whose cells are semialgebraic sets.

$C^r_d(\Delta)$: vector space of splines on Δ of degree $\leq d$ and smoothness r.

Let Δ be a planar complex with edges defined by real polynomials.
We previously showed the homological approach of Billera-Rose-Schenck-Stillman computes the spline module $C^r_d(\Delta)$ and thus $\dim C^r_d(\Delta)$.

When Δ has a single interior vertex, v, we determined $\dim C^r_d(\Delta)$ in two extreme cases:

- The curves incident to v form a *pencil* (as lines incident to v do)
- The curves incident to v have distinct tangents at v and are sufficiently *generic* (a classical case for rectilinear splines).

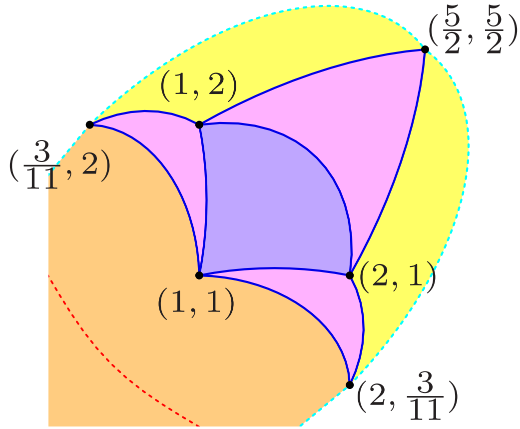
We continue these two cases for more involved complexes Δ.

Nets

Suppose that the edge forms span a two-dimensional space of forms (a *net*). Then the forms at a vertex form a pencil, and if the vertices are in general position, there is a unique edge between two vertices.

At right is the net spanned by $\{x^2 - yz, y^2 - xz, z^2 + xy\}$ with the indicated vertices.

A net defines a map $\varphi\colon \mathbb{P}^2 \to \mathbb{P}^2$, and $\varphi(\Delta)$ is a rectilinear complex on $\mathbb{R}^2$.

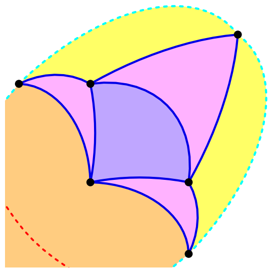


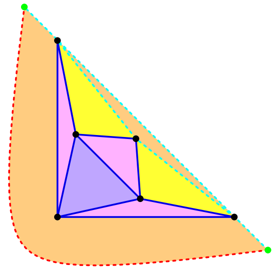
The spline module for Δ is a flat base-change along φ^* of the spline module for $\varphi(\Delta)$.

This gives simple formulas for $\dim C^r_d(\Delta)$ in terms of $\dim C^r_j(\varphi(\Delta))$.

Morgan-Scott for Nets

In the example from the previous page, here are Δ and $\varphi(\Delta)$:





Morgan-Scott for Nets

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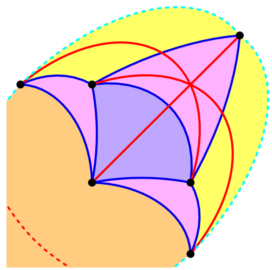
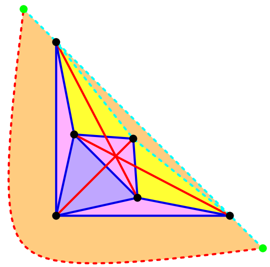




These exhibit the Morgan-Scott phenomena. Let Δ' be a generic complex from this net with the same topology as Δ. Then we have

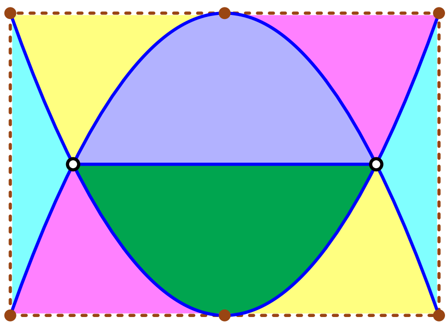
d	0	1	2	3	4	5	6	7	8	9
$\dim C^1_d(\Delta')$	1	3	6	10	15	21	34	54	81	115
$\dim C^1_d(\Delta)$	1	3	6	10	16	24	37	55	81	115

The difference 1,3,3,1 is $\dim \mathbb{R}[x,y,z]/\langle x^2-yz,\, y^2-xz,\, z^2+xy\rangle$

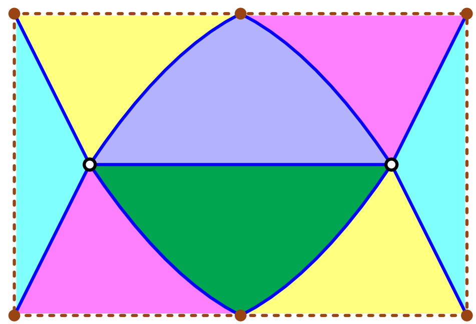
Generic Complexes Δ

When Δ has a single interior vertex v and the edge forms at v have distinct tangents, we gave a combinatorial formula for the dimension of $C^r_d(\Delta)$ for d sufficiently large.

For more general Δ, a more subtle acyclicity condition from local cohomology, which appeared in work of Schenck and Stillman, also gives a combinatorial formula for $\dim C^r_d(\Delta)$ for d sufficiently large.



Not Generic



Generic

What is the point of Generic?

Formulas for $\dim C^r_d(\Delta)$ for d large have two pieces:

- A regular, combinatorial part, and
- A possible difficult homology module.

For a generic complex, the regular part is even more regular, (this is a consequence of our earlier paper), and the difficult homology module has finite length, so in the long run it does not contribute, and we get a formula for $\dim C^r_d(\Delta)$ for d large:

$$(\phi_2-\phi_1)\binom{d+2}{2}+\sum_{\tau\in\Delta_1^\circ}\binom{d-(r+1)n_\tau+2}{2}+\sum_{v\in\Delta_0^\circ}\left(\binom{r+a_v+2}{2}-t_v\binom{a_v+1}{2}\right)$$

Here, $\phi_2, \phi_1, n_\tau, a_v, t_v$ are combinatorial data from the complex Δ.

The point of this work is not the formulae, but rather that methods for splines on rectilinear complexes mostly also work for semialgebraic complexes.

References

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- H. Schenck and M. Stillman, *A family of ideals of minimal regularity and the Hilbert series of $C^r(\hat{\Delta})$*, Adv. Appl. Math. **19** (1997), 169–182.
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