Trace Test

Special Session on Theory and Applications of Numerical Algebraic Geometry
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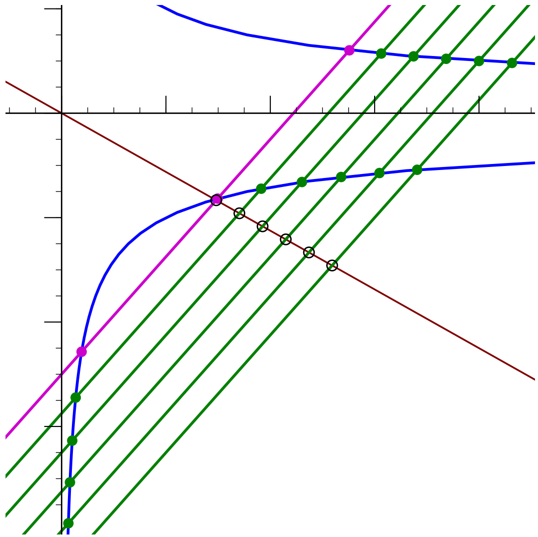

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Work with Anton Leykin and Jose Israel Rodriguez.

Witness sets in Numerical Algebraic Geometry

Numerical algebraic geometry uses the ability to solve systems of polynomial equations to study algebraic varieties on a computer.

It represents a variety $V \subset \mathbb{C}^n$ as a *witness set* $W = V \cap L$, where L is a general affine linear space of codimension $m = \dim V$.

Basic problem: Given a subset $W' \subset W$ how to certify that $W' = W$?

Trace Test: *Suppose that $L(t)$ for $t \in \mathbb{C}$ is a general pencil of affine-linear spaces with $L(0) = L$. Use continuation to follow points of W' along t, obtaining sets $W'(t)$. Then the trace of points in $W'(t)$,*

$$Tr(W'(t)) \;:=\; \sum \{w \mid w \in W'(t)\}\,,$$

is an affine function of t if and only if $W' = W$.

Proof of Trace Test

A general irreducible curve in $\mathbb{C}^2$ is defined by a dense irreducible polynomial $f \in \mathbb{C}[x,t]$ of degree d. Normalize f so that $1 =$ coefficient of x^d.

$f \in \mathbb{C}(t)[x]$ is irreducible and monic. The negative sum of its roots is its coefficient of x^{d-1}. Thus

$$\text{trace}(K/\mathbb{C}(t))(x) \;=\; c_0 t + c_1 \qquad c_0, c_1 \in \mathbb{C}\,, \tag{1}$$

where K contains the roots of f.

A general pencil $L(t)$ spans a codimension $m-1$ plane M with $M \cap V$ a curve, and M has coordinates $(\underline{x}, t)$. By (1), $\text{Tr}(W(t))$ is an affine function when W is a witness set.

This does not hold for $\text{Tr}(W'(t))$ if $W' \subsetneq W$, as the monodromy in t is the full symmetric group.

Explain

Multihomogeneous Witness Sets

A subvariety $V \subset \mathbb{P}^A \times \mathbb{P}^B$ of dimension m has *multidegrees* $d_{a,b}$ for $a+b = m$: For a general codimension a plane $L \subset \mathbb{P}^A$ and a general codimension b plane $M \subset \mathbb{P}^B$,

$$d_{a,b}(V) = \#V \cap (L \times M)\,.$$

Definition (Hauenstein-Rodriguez) An intersection $W_{a,b} = V \cap (L \times M)$ is a *multihomogeneous witness set* of bidimension (a, b) for V.

Advantages:

(1) Reflects the structure of V in $\mathbb{P}^A \times \mathbb{P}^B$.

(2) Smaller than alternatives. Embedding V into $\mathbb{P}^{AB+A+B}$ via Segre σ,

$$\deg(\sigma(V)) = \sum_{a+b=m} \binom{m}{a} d_{a,b}\,.$$

This is huge.

Using Multihomogeneous Witness Sets

Hauenstein and Rodriguez showed that many algorithms in numerical algebraic geometry work well with multihomogeneous witness sets. These include regeneration, membership, and using a multihomogeneous witness set in one bidimension to populate another.

What does not work well is the trace test.

Fact. If $L(t) \subset \mathbb{P}^A$ and $M(s) \subset \mathbb{P}^B$ are pencils of affine spaces of codimensions a and b, respectively, then $\mathrm{Tr}(V \cap (L(t) \times M(s)))$ is not a bilinear function in s and t.

We cannot even fix t and let s vary for irreducible decomposition, for $V \cap L$ could be reducible even if V is irreducible.

Dimension Reduction

Let $V \subset \mathbb{P}^A \times \mathbb{P}^B$ be irreducible of dimension $m \geq 2$, $a+b = m$ with $d_{a,b}(V) \neq 0$, $L' \subset \mathbb{P}^A$ a general linear space of codimension $a-1$, and $M' \subset \mathbb{P}^B$ a general linear space of codimension $b-1$.

$U := V \cap (L' \times M')$ is irreducible of dimension 2 with multidegrees

$$d_{0,2} = d_{a-1,b+1}(V)\,, \quad d_{1,1} = d_{a,b}(V)\,, \quad d_{2,0} = d_{a+1,b-1}(V)\,.$$

Either (1) $d_{0,2} = d_{2,0} = 0 \;\Rightarrow\; U$ is a product of curves. Then V is also a product and we may treat each factor separately.

Or (2) a further linear slice is possible, reducing V to a curve in a product of projective spaces.

The cases are detected from the tangent spaces at general points of V or of U.

A Multihomogeneous Trace Test

Assume that V is not a product. Given nonzero adjacent multidegrees $d_{\alpha+1,\beta}$ and $d_{\alpha,\beta+1}$, $L' \subset \mathbb{P}^A$ and $M' \subset \mathbb{P}^B$ of codimensions α and β containing hyperplanes $L \subset L'$ and $M \subset M'$, then

$$W_{10} := V \cap (L \times M') \text{ and } W_{01} := V \cap (L' \times M)$$

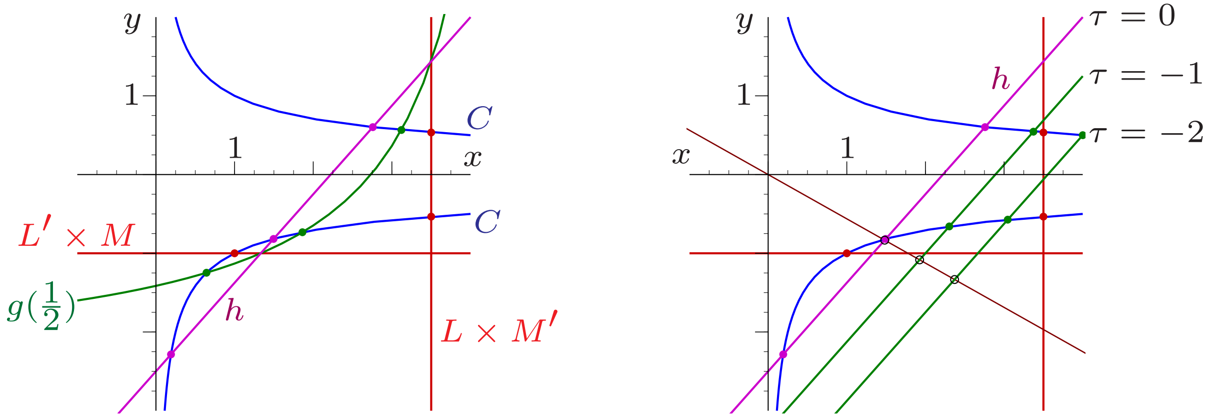
are the corresponding multihomogeneous witness sets.

Then $C := V \cap (L' \times M')$ is an irreducible curve with multidegrees $d_{10} = d_{\alpha+1,\beta}$ and $d_{01} = d_{\alpha,\beta+1}$ having witness sets W_{10} and W_{01}.

Working in an affine patch $\mathbb{C}^n \oplus \mathbb{C}^m$ on $L' \times M'$, C has degree $d_{10} + d_{01}$ and $W_{01} \cup W_{10}$ can be used to get a witness set $W = C \cap H$, which we may use for a trace test in the affine space $\mathbb{C}^n \oplus \mathbb{C}^m$.

Example

Suppose that $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ is defined locally by $y^2 x = 1$.



Left: Linear spaces $x = x_0$ and $y = y_0$, line $H : h = 0$, and the curve $g(\frac{1}{2})$, where $g(t) := (x - x_0)(y - y_0)(1 - t) + th$. These are $g(t)$ at $t = 0,\ \frac{1}{2},\ 1$.

Right: the parallel slices $h = \tau$ are in green, and the averages of witness points ($\frac{1}{3}$ of the trace) lies on the brown line.