Semialgebraic Splines

SIAM Minisymposium on
Multivariate Splines and Algebraic Geometry
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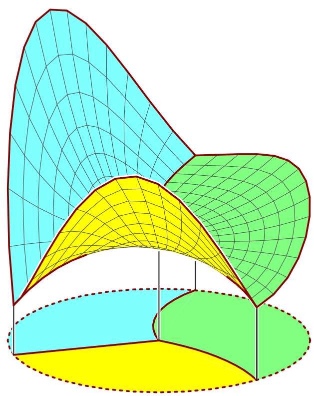
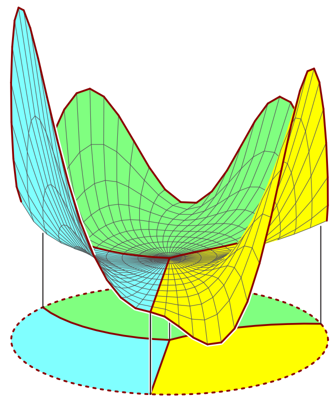




Work with Michael DiPasquale and Lanyin Sun

Motivating Goals (for Me)

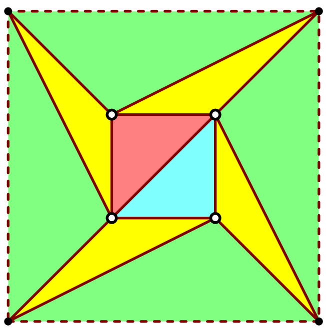
I: Compute dimensions of splines spaces on a semi-algebraic cell complex, to illustrate some phenomena not observed in traditional splines on simplicial or polyhedral complexes.

II: Learn something about splines and algebraic geometry.

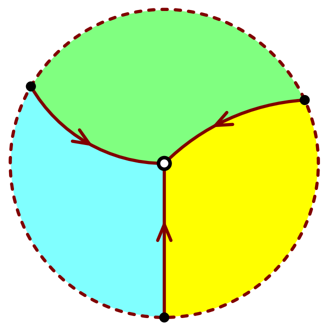
Definition: A (*basic*) *semialgebraic set* is one of the form

$$\{x \in \mathbb{R}^2 \mid h_i(x) \geq 0\,,\ \text{for } i = 1, \ldots, m\}\,,$$

where $h_1, \ldots, h_m$ are polynomials.



Simplicial Complex



Complex with Semialgebraic Cells

Semialgebraic Splines

A *semialgebraic spline* is a function that is piecewise a polynomial with respect to a complex Δ whose cells are semialgebraic sets.

$C^r_d(\Delta)$: vector space of splines on Δ of degree $\leq d$ and smoothness r.

Consider planar complexes Δ with a single interior vertex, υ, and whose edges are defined by polynomials $g_1, \ldots, g_N$ (with $g_i(\upsilon) = 0$), where $\deg(g_i) = n_i$ (In our examples here, $N = 3$ and $n_i = 2$).

Set $S := \mathbb{R}[x, y, z]$ and let $J(\upsilon) := \langle g_i \mid i = 1, \ldots, N \rangle$, a homogeneous ideal. The same homological algebra as for classical splines on a simplicial complex yields

$$\dim C^r_d(\Delta) \ = \ \sum_{i=1}^{N} \binom{d-(r+1)n_i+2}{2} \ + \ \dim(S/J(\upsilon))_d \,.$$

Pencils

Suppose that $g_1, \ldots, g_N$ all have degree n and form a pencil (dimension of linear span is 2). Suppose they define s distinct curves in $\mathbb{R}^2$. Set

$$t := \min\{s, r+2\}\,, \quad a := \lfloor \tfrac{r+1}{t-1} \rfloor\,,$$

$$s_1 := (t-1)a + t-r-2\,, \; s_2 := r+1-(t-1)a\,.$$

Theorem. $\dim C^r_d(\Delta)$ *equals* $\binom{d+2}{2}+$

$$(N-t)\binom{d-(r+1)n+2}{2} + s_1\binom{d-(r+1+a)n+2}{2} + s_2\binom{d-(r+2+a)n+2}{2}\,,$$

where $\binom{a}{b} = 0$ is zero if $a < b$. For $d > (r+2+a)n+1$ this is

$$N\binom{d-(r+1)n+2}{2} + n^2\left(\binom{a+r+2}{2} - t\binom{a+1}{2}\right)\,.$$

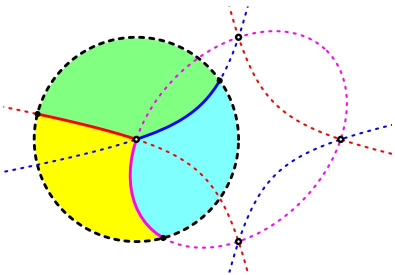
Consequently, the dimension of the spline space does not depend upon any real geometry of the curves underlying the edges. They define n^2 points in the complex projective plane, counted with multiplicity.

Pencils, continued

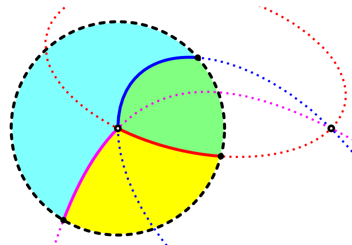
Theorem. *For* $d > (r+2+a)n+1$, $\dim C^r_d(\Delta)$ *is*

$$N\binom{d-(r+1)n+2}{2} + n^2\left(\binom{a+r+2}{2} - t\binom{a+1}{2}\right).$$

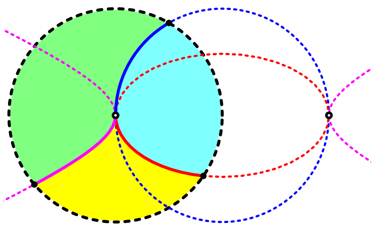
$\dim C^r_d(\Delta)$ is independent of the real geometry of the edge curves.



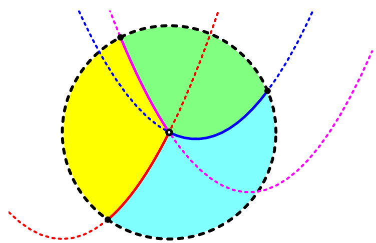
Four real points



Two real and two complex points



Two double points



A real point and a triple point at infinity

Distinct Tangents

Another extreme is when $g_1, \ldots, g_N$ are smooth at v with distinct tangents, given by $L_1, \ldots, L_N$, respectively.

Theorem. *For d sufficiently large, $\dim C_d^r(\Delta)$ equals*

$$\sum_{i=1}^{N} \binom{d-(r+1)n_i+2}{2} \;+\; \binom{a+r+2}{2} - t\binom{a+2}{2}\,,$$

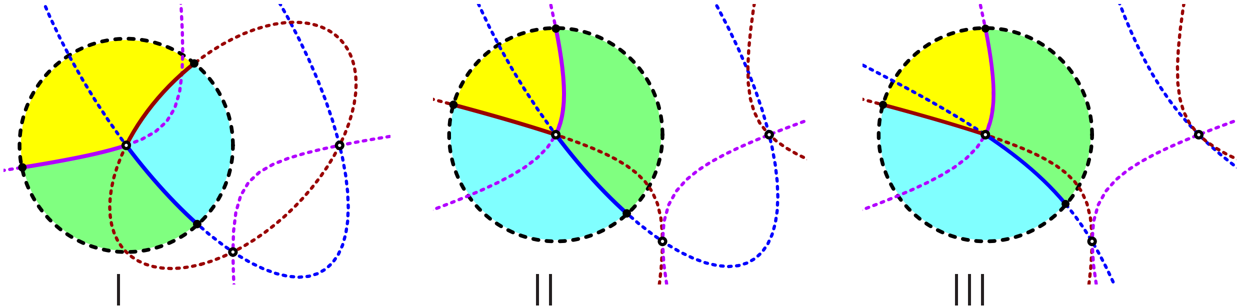
where $t := \min\{N, r+1\}$ and $a := \lfloor \frac{r+1}{t-1} \rfloor$.

For this, we prove that the ideals $J(v) = \langle g_i^{r+1} \mid i = 1, \ldots, N\rangle$ and $\langle L_i^{r+1} \mid i = 1, \ldots, N\rangle$ define schemes (supported at v) with the same multiplicity, even though they are not isomorphic when r is large.

$\rightsquigarrow$ Similar to the classical case of edges having distinct slopes.

Possible Extensions

If $g_1, \ldots, g_N$ do not form a pencil, but vanish at another point, there are some subtleties.



As $\dim C^r_d(\Delta) = \binom{d-(r+1)2+2}{2} + \dim(S/J(v))_d$, we consider $\dim(S/J(v))_d$ for d large and $r = 0, 1, 2, 3, 4$.

	0	1	2	3	4
I	3	9	21	36	57
II	3	10	22	38	60
III	3	11	23	40	63