A-Discriminant Coamoebas in Dimension Three

COALGA:

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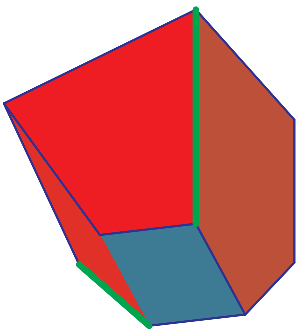
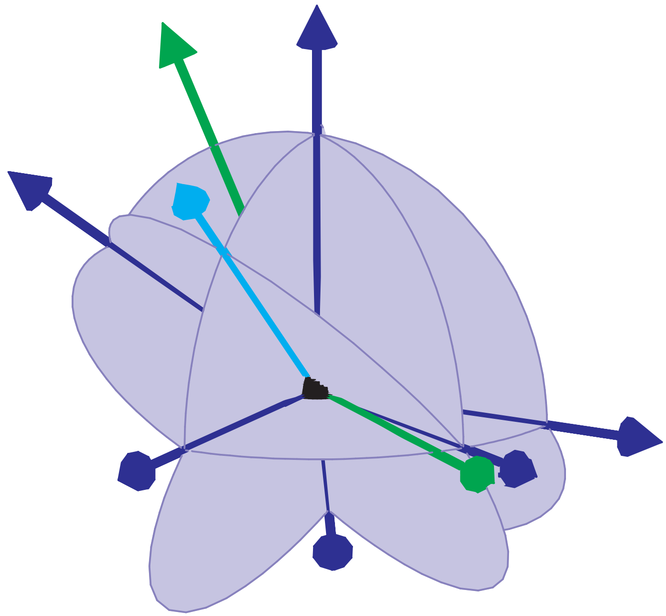




Joint work with Mounir Nisse

A-Discriminants

Let $A \subset \mathbb{Z}^n$ be homogeneous: $|a| \neq 0$ is constant for $a \in A$.

A polynomial f with support A

$$f \ := \ \sum_{a \in A} c_a x^a \qquad c_a \in \mathbb{C}\,,$$

defines a hypersurface $\mathcal{V}(f) \subset \mathbb{P}^{n-1}$.

A-discriminant : hypersurface in $\mathbb{P}^A$ of those f with $\mathcal{V}(f)$ singular.

This has many homogeneities, $(\mathbb{C}^\times)^n$ acts on $\mathbb{P}^A$ and on the A-discriminant via $t.[c_a \mid a \in A] = [t^a c_a \mid a \in A]$.

Taking the quotient by $(\mathbb{C}^\times)^n$ gives the *(reduced) A-discriminant*, or simply the A-discriminant.

(Reduced) A-discriminant

Taking the quotient by $(\mathbb{C}^\times)^n$ gives the *(reduced) A-discriminant*, which is a hypersurface in $\mathbb{P}^{d-1}$, where $|A| = n + d$. Henceforth, this is A-discriminant.

In algebraic geometry, quotients are typically nasty and the full A-discriminant was already complicated.
However, Kapranov generalized the classical Horn parametrization to A-discriminants

$$\mathbb{C}^d \ \ni \ z \ \longmapsto \ \prod_{b \in B} \langle b, z \rangle^b \ \in \ \mathbb{P}^{d-1} \,,$$

where B is Gale-dual to A, $\mathbb{Z}^d \xrightarrow{B} \mathbb{Z}^A \xrightarrow{A} \mathbb{Z}^n$ is exact.
He showed that the image is the (reduced) A-discriminant.

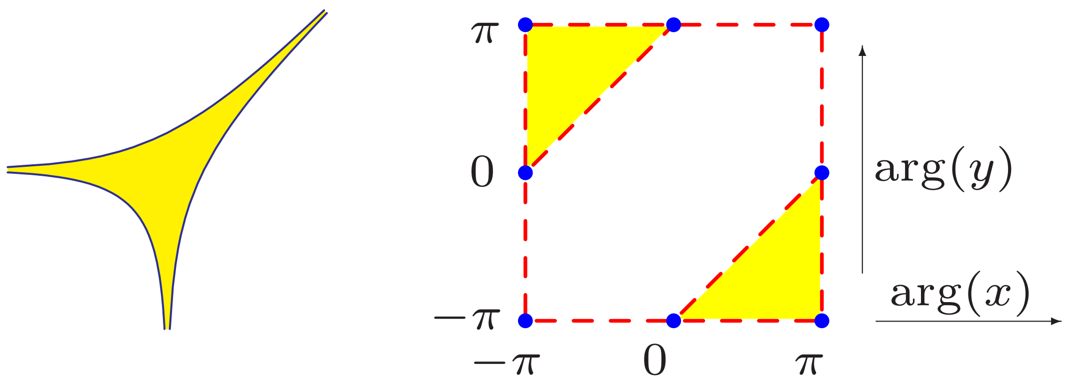
This map is central to the study of A-discriminants.

(Pre-) Tropical Objects

Let $V \subset (\mathbb{C}^\times)^n$ be a variety. Its *amoeba* $\mathcal{A}(V)$ is the set of lengths in V and its *coamoeba* $co\mathcal{A}(V)$ is the set of arguments in V. Gel'fand, Kapranov, and Zelevinsky inroduced amoebas and Passare introduced coamoebas.

Formally, identify $\mathbb{C}^\times$ with $\mathbb{R} \times \mathbb{T}$, where $\mathbb{T} = S^1$ is the unit complex numbers, and $e^r\theta \mapsto (r, \theta)$. Then $\mathcal{A}(V)$ is the projection of V to $\mathbb{R}^n$ and $co\mathcal{A}(V)$ is its projection to $\mathbb{T}^n$.

Example : The amoeba and coamoeba of $\mathcal{V}(x + y + 1)$ are

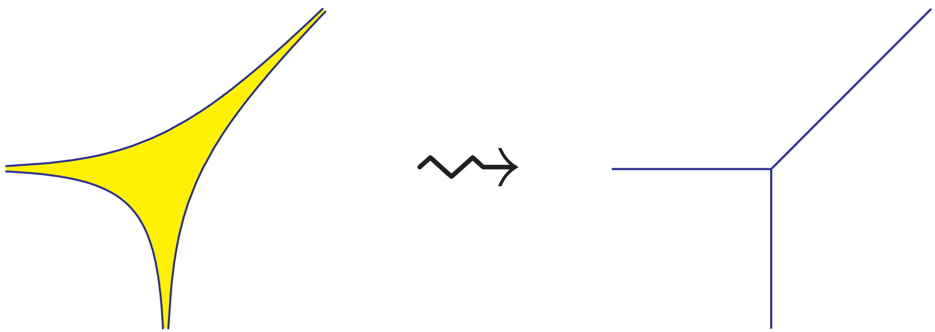


Tropical Variety

Tropical variety $\mathcal{T}(V)$ of V : cone over the limiting directions of $\mathcal{A}(V)$.

By results of Bergman and Bieri-Groves, $\mathcal{T}(V)$ is a rational polyhedral fan of the same dimension (in $\mathbb{R}^n$) as V (as a complex variety).

For our line, this is the tripod,



Tropical Objects for Discriminants

Discriminants have relatively simple and understandable tropical objects.

Theorem (Passare, Sadykov, Tsikh)
Principal A-determinants have soid amoebas. (No bounded components of complement).

We will see that the tropical variety and coamoeba of a discriminant are also special.

Tropical Discriminants

Alicia w/ Feichtner and Sturmfels found a beautiful structure theorem for tropical discriminants, using the Horn-Kapranov parametrization.

The Horn-Kapranov parametrization is a composition of two simple maps

$$\lambda_B \;:\; \mathbb{C}^d \;\ni\; z \;\longmapsto\; (\langle b, z\rangle \mid b \in B) \;\in\; \mathbb{C}^B\,,$$

$$\pi_B \;:\; \mathbb{C}^B \;\ni\; x \;\longmapsto\; \prod_{b\in B} x_b^b \;\in\; \mathbb{P}^{d-1}\,,$$

with λ_B linear and π_B a homomorphism on dense tori.

Their work involved two steps.

(a) Tropical variety of a linear space (Bergman fan: Sturmfels, Ardila-Klivans, and Sturmfels-Feichtner).

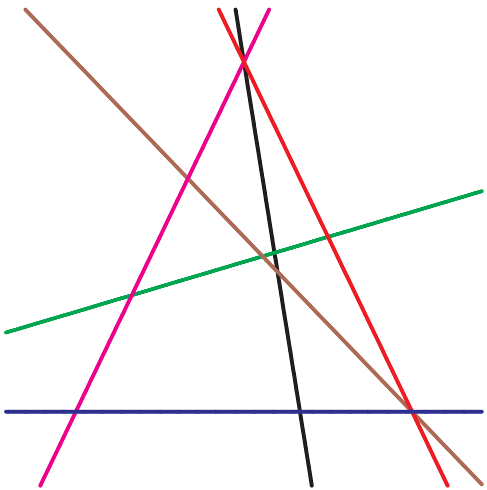
(b) Its image under the linear map induced by the homomorphism π_B.

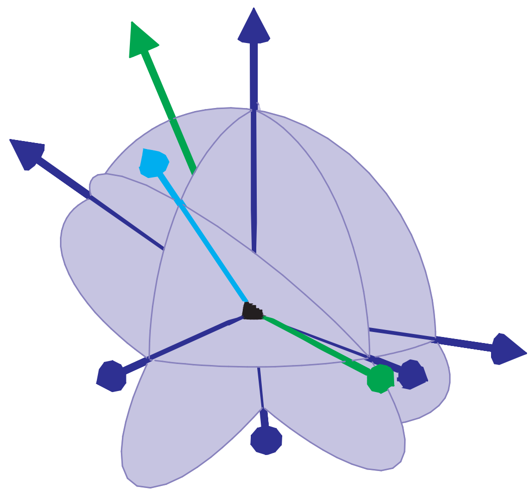
$\rightsquigarrow$ Description of tropical discriminant *and* Newton polytope of discriminant.

Example

Let B be the column vectors in $\begin{pmatrix} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{pmatrix}$

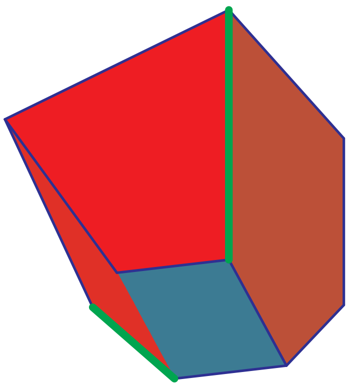
This defines a line arrangement in $\mathbb{P}^2$ and a tropical fan.

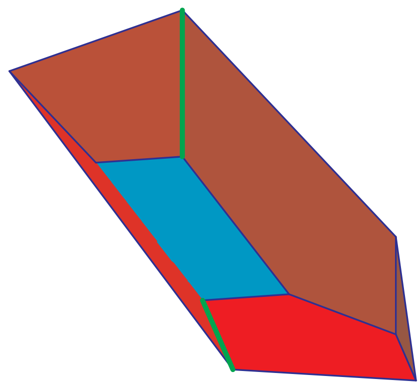




Discriminant and Its Newton Polytope

$$3125q^4r^4 - 1024p^5q^2 + 1280p^4q^2r - 40p^3q^2r^2 + 4000p^2q^3r^2 - 40p^2q^2r^3 + 500pq^3r^3 + 1280pq^2r^4 + 4000q^3r^4 - 1024q^2r^5 - 432p^6 - 1152p^5q + 768p^4q^2 + 864p^5r + 1584p^4qr + 512p^3q^2r - 432p^4r^2 + 1584p^3qr^2 + 5038p^2q^2r^2 - 200pq^3r^2 - 1152p^2qr^3 + 512pq^2r^3 - 200q^3r^3 + 768q^2r^4 + 216p^5 + 832p^4q - 192p^3q^2 + 216p^4r + 532p^3qr - 208p^2q^2r + 832p^2qr^2 - 208pq^2r^2 + 16q^3r^2 - 192q^2r^3 - 27p^4 - 200p^3q + 16p^2q^2 - 200p^2qr + 16q^2r^2 + 16p^2q$$





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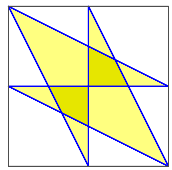
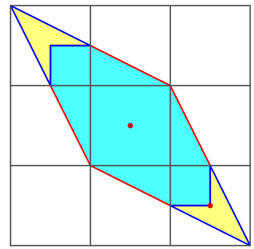
Discriminant Coamoebas

Nilsson and Passare described the *discriminant coamoeba* when $B \subset \mathbb{Z}^2$.

This is an explicit polyhedral object that is the complement of a zonotope generated by B in vol(A) times a fundamental domain.

For $B = \left(\begin{smallmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{smallmatrix}\right)$, here is the coamoeba in its fundamental domain and as the complement of the zonotope.





Passare-S. Gave a new proof using the Horn-Kapranov parametrization.

$$\lambda_B \;:\; \mathbb{C}^d \;\ni\; z \;\longmapsto\; (\langle b, z\rangle \mid b \in B) \;\in\; \mathbb{C}^B\,,$$

$$\pi_B \;:\; \mathbb{C}^B \;\ni\; x \;\longmapsto\; \prod_{b\in B} x_b^b \;\in\; \mathbb{P}^{d-1}\,.$$

Extending this to higher dimension is now work with Mounir Nisse.

Phase Limit Set

The *phase limit set* $\mathcal{P}^\infty(V)$ of a subvariety $V \subset (\mathbb{C}^\times)^n$ is the set of accumulation points of arguments of unbounded sequences $\{x_i \mid i \in \mathbb{N}\}$ in V. (Definition similar to the logarithmic limit set.)

Each cone σ of the tropical variety $\mathcal{T}(V)$ of V gives an initial scheme $\text{in}_\sigma V$, on which the subgroup of the torus $(\mathbb{C}^\times)^n$ corresponding to the linear span of σ acts freely. The quotient is the part of V at the infinity corresponding to σ.

Theorem (Nisse-S.)
The closure of $coA(V)$ is $co\mathcal{A}(V) \cup \mathcal{P}^\infty(V)$, and

$$\mathcal{P}^\infty(V) \;=\; \bigcup_{\sigma\neq 0} co\mathcal{A}(\text{in}_\sigma V)\,.$$

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Phase Limit Set of Linear Space

A *flat* L of the hyperplane arrangement given by B is any intersection of hyperplanes. The set of hyperplanes $B_L \subset B$ containing L gives an arrangement on $\mathbb{C}^d/L$ and its complement B^L gives an arrangement in L. There are corresponding linear spaces $\lambda_{B_L}(\mathbb{C}^d/L) \subset \mathbb{C}^{B_L}$ and $\lambda_{B^L}(L) \subset \mathbb{C}^{B^L}$. (Deletion-restriction).

Using the definition of phase limit set, we deduce:

Theorem (Nisse-S.)
The closure of the coamoeba of $\lambda_B(\mathbb{C}^d)$ is the union of sets, one for each flat L of B. The set corresponding to L is

$$\overline{co\mathcal{A}(\lambda_{B_L}(\mathbb{C}^d/L))} \times co\mathcal{A}(\lambda_{B^L}(L))\,.$$

The shape of this recovers what we would get from the Bergman fan using the structure theorem for phase limit sets.

Discriminant Coamoebas in Dimension Three

The phase limit set of the discriminant is simply $\pi_B(\mathcal{P}^\infty(\lambda_B(\mathbb{C}^d)))$.

Lemma If $\mathbf{dim} > \mathbf{2}$, the coamoeba discriminant equals its phase limit set.

We need only to understand parts of the phase limit set coming from the rays of the tropical discriminant. There are three overlapping types.

(a) Irreducible hyperplanes, given by $b \in B$. Such a component equals $\mathbb{T}_b \times \overline{co\mathcal{A}(\mathcal{V}(b))}$, which we understand recursively.

(b) Other irreducible flats. If one-dimensional, the component is collapsed by π_B, and does not contribute.

(c) Rays arising from the projection π_B of the Bergman fan. These do not contribute.

In dimension three, only the first type contributes, and we obtain a description of the coamoeba as a union of polyhedra, and as lying outside of the zonotope of B, but there is more work to be done...