Newton Polytopes via Witness Sets

Algebraic and Geometric Methods in Applied Discrete Mathematics

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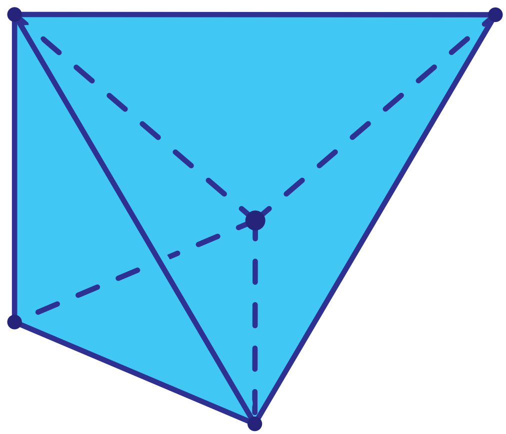


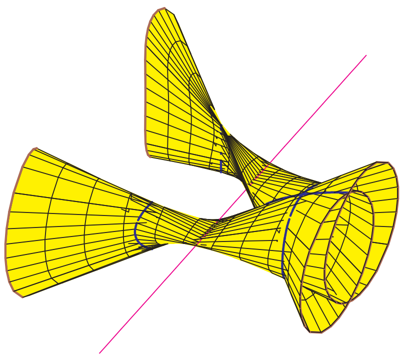
Work with Jon Hauenstein and Taylor Brysiewicz.

Fundamental Problem

By algebraic geometry, an irreducible hypersurface $\mathcal{H}$ in $\mathbb{C}^n$ is defined by the vanishing of a single irreducible polynomial, $f \in \mathbb{C}[x_1, \ldots, x_n]$,

$$\mathcal{H} \;=\; \mathcal{V}(f) \;:=\; \{x \in \mathbb{C}^n \mid f(x) = 0\}\,.$$

The problem I want to consider is: Suppose that we know the hypersurface, but not the polynomial?



$$= \; \mathcal{V}(f) \qquad f = ????$$

We would like to understand the polynomial f defining $\mathcal{H}$.

What Does *Understand* Mean?

Best: Complete knowledge. There are finite sets $\mathcal{A} \subset \mathbb{Z}^n$ and $\{c_a \mid a \in \mathcal{A}\} \subset \mathbb{C}$ such that

$$f \;=\; \sum_{a\in\mathcal{A}} c_a x^a \qquad (x^a := x_1^{a_1} \cdots x_n^{a_n})$$

Pretty Good: Knowing the support, $\mathcal{A}$.

We'll Settle For: Newton Polytope of $\mathcal{H}$,

$$N(\mathcal{H}) \;:=\; \text{convex hull of } \mathcal{A}\,.$$

Easier: The degree of $\mathcal{H}$.

How to Know $\mathcal{H}$ but not f

The hypersurface $\mathcal{H}$ might be the image of a map,

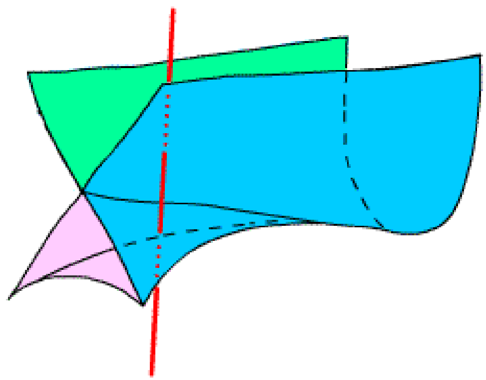
$$\varphi : X \longrightarrow \mathcal{H} \subset \mathbb{C}^n .$$

This is fairly common, for example

$$\Sigma := \{(p, f) \mid p \in \mathbb{P}^1 ,\ \deg(f) = d ,\ f_x(p) = f_y(p)\}$$

$$\mathbb{P}^1 \xleftarrow{\ pr\ } \Sigma \xrightarrow{\ \pi\ } \mathbb{P}^d$$

$pr\colon \Sigma \to \mathbb{P}^1$ is a projective bundle,
and $\pi(\Sigma)$ is the classical discriminant
of a d-form.

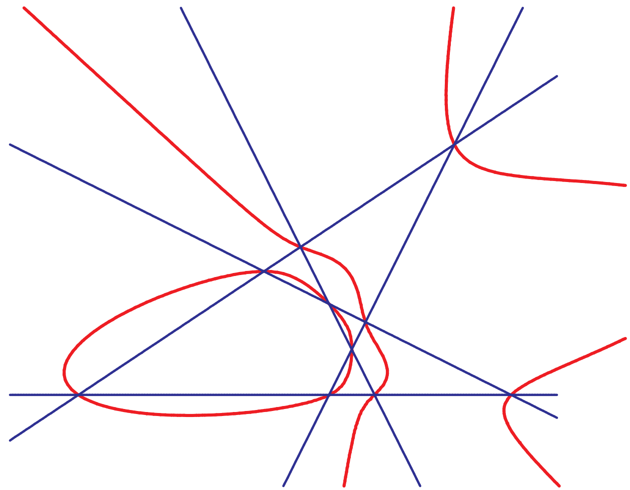


Example: Lüroth Quartics

Lüroth, 1869: If $\ell_1, \ldots, \ell_5$ are equations for lines, then

$$q \;:=\; \ell_1\ell_2\ell_3\ell_4\ell_5\Bigl(\tfrac{1}{\ell_1}+\tfrac{1}{\ell_2}+\tfrac{1}{\ell_3}+\tfrac{1}{\ell_4}+\tfrac{1}{\ell_5}\Bigr)$$

defines a quartic that inscribes the great pentagon, $\mathcal{V}(\ell_1\ell_2\ell_3\ell_4\ell_5)$.



This set of quartics is the image of a map $(\mathbb{C}^3)^5 - \to \mathbb{P}^{14}$ ($\mathbb{P}^{14}$ = plane quartics), and it forms a hypersurface, $\mathcal{L}$.

Morley, 1919: $\mathcal{L}$ has degree 54.

The defining equation of $\mathcal{L}$ is the *Lüroth invariant*, which could have as many as $\binom{54+14}{14} = 12323427976816 0$ monomials.

How to Represent a Polytope?

$P =$ convex hull of a finite subset of $\mathbb{R}^n$.

$$P = \bigcap_{\omega \text{ in a finite set}} \{x \mid \omega \cdot x \leq b_\omega\},$$

the intersection of finitely many half-spaces.

Oracle Representation:

For $\omega \in \mathbb{R}^n$, set $h(\omega) = \max\{\omega \cdot x \mid x \in P\}$.

The face P_ω of P *exposed* by ω is

$$P_\omega := \{x \in P \mid \omega \cdot x = h(\omega)\}$$

The *oracle representation* of P is a function that given $\omega \in \mathbb{R}^n$ returns P_ω, if it is a vertex.

We propose a method, based on numerical algebraic geometry to compute an oracle representation of the Newton polytope of a hypersurface.

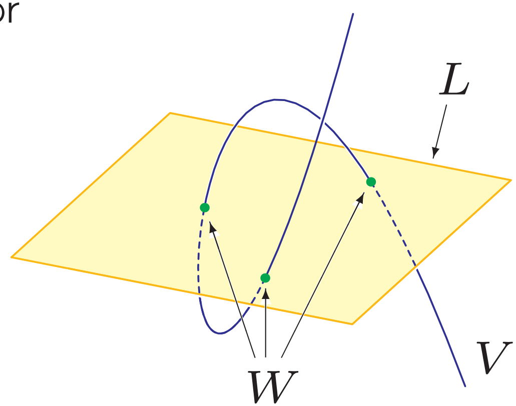
Witness Sets

Numerical Algebraic Geometry uses numerical analysis to represent and manipulate varieties on a computer.

Let $V \subset \mathbb{C}^n$ be a variety of codimension k, given as a component of $F(x) = 0$. A *witness set* for V is a pair (W, L), where

– L is a general affine plane of dimension k, and

– $W = V \cap L$.



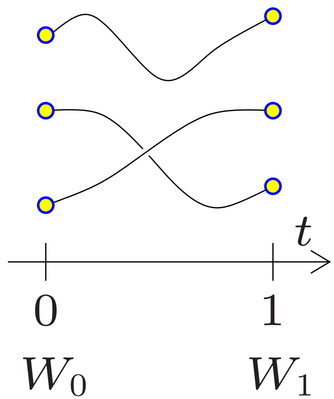
L is either parameterized, or cut out by $n-k$ affine forms, and W consists of numerical approximations to $V \cap L$.

Continuation

In numerical algebraic geometry, the basic operation is continuation, which traces points along implicitly-defined paths.

Suppose that $L(t)$ for $t \in \mathbb{C}$ is a family of k-planes, and we have a witness set $(W_1, L(1))$ for V.

We numerically continue these points in $V \cap L(t)$ from $t = 1$ to $t = 0$ to get another witness set $(W_0, L(0))$ for V.



This allows us to sample points from V.

Witness Sets of Projections

Our hypersurfaces come as the image under a projection.

Suppose that $X \subset \mathbb{C}^n \oplus \mathbb{C}^m$ and $\mathcal{H} = \pi(X) \subset \mathbb{C}^n$ is a hypersurface.

Hauenstein, Sommese, Wampler: Given a witness set $(X \cap L, L)$ for X, compute a witness set $(\mathcal{H} \cap \ell, \ell)$ for $\mathcal{H}$:
— Choose a general lines $\ell \subset \mathbb{C}^n$
— Move L to a non-general plane Λ with $\pi(\Lambda) = \ell$.
Set $W' := X \cap \Lambda$. Then $(\pi(W'), \ell)$ is a witness set for $\mathcal{H}$.

Delicate: Λ is not in general position.

Witness set of a Hypersurface

Suppose $f = \sum_{a\in\mathcal{A}} c_a x^a$ is a polynomial, $\mathcal{H} := \mathcal{V}(f)$, and $P = \text{conv}(\mathcal{A})$.

Let $p, q \in \mathbb{C}^n$ be general, and define

$$\ell_{p,q}(s) = \ell(s) \;:=\; \{sp - q \mid s \in \mathbb{C}\}\,.$$

Then $f(\ell(s)) = 0$ defines the witness set $\mathcal{H} \cap \ell$.

Thus a witness set gives roots of $f(\ell(s))$.

For $\omega \in \mathbb{R}^n$ and $t > 0$, set $t^\omega := (t^{\omega_1}, \ldots, t^{\omega_n})$. Then,

$$f(t^\omega.\ell(s)) \;=\; \sum_{a\in\mathcal{A}} c_a(sp_1 - q_1)^{a_1}\cdots(sp_n - q_n)^{a_n}\; t^{\omega\cdot a}$$

$$\stackrel{!}{=}\; t^{h(\omega)}\Big(\sum_{\mathcal{A}\cap P_\omega} c_a(sp-q)^a \;+\; \sum_{\mathcal{A}\smallsetminus P_w} c_a(sp-q)^a t^{\omega\cdot a - h(\omega)}\Big)$$

$\Rightarrow\;\; \exists d_\omega > 0$ such that $\omega\cdot a - h(\omega) < -d_\omega$ for $a \in \mathcal{A}\smallsetminus P_\omega$.

Main Lemma

$$f(t^\omega.\ell(s)) = t^{h(\omega)}\Big(\sum_{\mathcal{A}\cap P_\omega} c_a(sp-q)^a + \sum_{\mathcal{A}\smallsetminus P_w} c_a(sp-q)^a t^{\omega\cdot a-h(\omega)}\Big)$$

Set f_ω to be the sum of terms in f from P_ω

Lemma. *In the limit as $t \to \infty$, $t^{-h(\omega)} f(t^\omega.\ell(s)) \to f_\omega(\ell(s))$. $\deg(f)-\deg(f_\omega)$ zeroes will diverge to ∞, while the remaining $\deg(f_\omega)$ remaining bounded.*

If P_ω is a vertex, say a (which holds when ω is generic), then

$$f_\omega(\ell(s)) \;=\; c_a(sp_1 - q_1)^{a_1}\cdots(sp_n - q_n)^{a_n}\,.$$

Thus a_i zeroes of $f(t^\omega.\ell(s))$ coalesce to q_i/p_i as $t \to \infty$.

Our paper describes how to turn this idea into an algorithm.

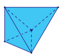
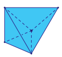
Lüroth quartics, again

We created a test implementation and used it to compute a few vertices of the *Lüroth polytope* (Newton polytope of the Lüroth hypersurface).

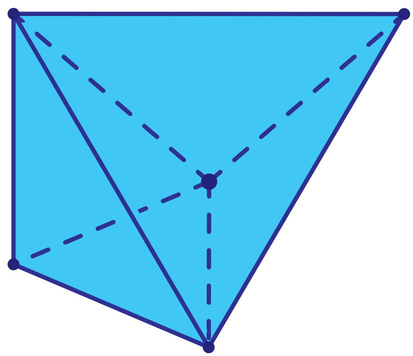
Ciani quartics: The Lüroth quartics whose monomials are squares,

$$\alpha x^4 + \beta y^4 + \gamma z^4 + 2(\delta x^2 y^2 + \rho x^2 z^2 + \sigma y^2 z^2)\,,$$

form a face of the Lüroth polytope, which we computed.

It is 14  $+\,\alpha^4\beta^4\gamma^4$,
where  is equivalent to the bipyramid,

$$\text{conv}\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} =$$

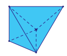
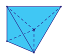


Ciani Face of Lüroth Invariant

We used a numerical factorization algorithm and an LLL-based interpolation method to compute the Lüroth invariant, restricted to this face of Ciani quartics, $f(\mathcal{L})_\omega$.

$$f(\mathcal{L})_\omega \;=\; \alpha^4\beta^4\gamma^4 f_1^4 f_2^2 f_3^2 f_4^2 \cdot f_5\,,$$

where $f_1, \ldots, f_5$ have integer coefficients, between 1 and $2401 = 7^4$.

f_1, f_2, f_3, f_4 have the same Newton polytope, , but
f_5 has Newton polytope 4 .

Subsequently, Basson, Lercier, Ritzenthaler, and Sijsling found an expression for the Lüroth invariant in terms of the fundamental and secondary invariants of $GL(3)$ acting on quartics.

f_5

$$2401\alpha^4\beta^4\gamma^4 - 196\alpha^4\beta^3\gamma^3\sigma^2 + 102\alpha^4\beta^2\gamma^2\sigma^4 - 4\alpha^4\beta\gamma\sigma^6 + \alpha^4\sigma^8 - 196\alpha^3\beta^4\gamma^3\rho^2$$
$$-196\alpha^3\beta^3\gamma^4\delta^2 + 840\alpha^3\beta^3\gamma^3\delta\rho\sigma - 820\alpha^3\beta^3\gamma^2\rho^2\sigma^2 - 820\alpha^3\beta^2\gamma^3\delta^2\sigma^2 + 232\alpha^3\beta^2\gamma^2\delta\rho\sigma^3$$
$$-12\alpha^3\beta^2\gamma\rho^2\sigma^4 - 12\alpha^3\beta\gamma^2\delta^2\sigma^4 - 40\alpha^3\beta\gamma\delta\rho\sigma^5 + 4\alpha^3\beta\rho^2\sigma^6 + 4\alpha^3\gamma\delta^2\sigma^6 - 8\alpha^3\delta\rho\sigma^7$$
$$+102\alpha^2\beta^4\gamma^2\rho^4 - 820\alpha^2\beta^3\gamma^3\delta^2\rho^2 + 232\alpha^2\beta^3\gamma^2\delta\rho^3\sigma - 12\alpha^2\beta^3\gamma\rho^4\sigma^2 + 102\alpha^2\beta^2\gamma^4\delta^4$$
$$+232\alpha^2\beta^2\gamma^3\delta^3\rho\sigma + 128\alpha^2\beta^2\gamma^2\delta^2\rho^2\sigma^2 - 80\alpha^2\beta^2\gamma\delta\rho^3\sigma^3 + 6\alpha^2\beta^2\rho^4\sigma^4 - 12\alpha^2\beta\gamma^3\delta^4\sigma^2$$
$$-80\alpha^2\beta\gamma^2\delta^3\rho\sigma^3 + 220\alpha^2\beta\gamma\delta^2\rho^2\sigma^4 - 24\alpha^2\beta\delta\rho^3\sigma^5 + 6\alpha^2\gamma^2\delta^4\sigma^4 - 24\alpha^2\gamma\delta^3\rho\sigma^5$$
$$+24\alpha^2\delta^2\rho^2\sigma^6 - 4\alpha\beta^4\gamma\rho^6 - 12\alpha\beta^3\gamma^2\delta^2\rho^4 - 40\alpha\beta^3\gamma\delta\rho^5\sigma + 4\alpha\beta^3\rho^6\sigma^2 - 12\alpha\beta^2\gamma^3\delta^4\rho^2$$
$$-80\alpha\beta^2\gamma^2\delta^3\rho^3\sigma + 220\alpha\beta^2\gamma\delta^2\rho^4\sigma^2 - 24\alpha\beta^2\delta\rho^5\sigma^3 - 4\alpha\beta\gamma^4\delta^6 - 40\alpha\beta\gamma^3\delta^5\rho\sigma$$
$$+220\alpha\beta\gamma^2\delta^4\rho^2\sigma^2 - 272\alpha\beta\gamma\delta^3\rho^3\sigma^3 + 48\alpha\beta\delta^2\rho^4\sigma^4 + 4\alpha\gamma^3\delta^6\sigma^2 - 24\alpha\gamma^2\delta^5\rho\sigma^3$$
$$+48\alpha\gamma\delta^4\rho^2\sigma^4 - 32\alpha\delta^3\rho^3\sigma^5 + \beta^4\rho^8 + 4\beta^3\gamma\delta^2\rho^6 - 8\beta^3\delta\rho^7\sigma + 6\beta^2\gamma^2\delta^4\rho^4 - 24\beta^2\gamma\delta^3\rho^5\sigma$$
$$+24\beta^2\delta^2\rho^6\sigma^2 + 4\beta\gamma^3\delta^6\rho^2 - 24\beta\gamma^2\delta^5\rho^3\sigma + 48\beta\gamma\delta^4\rho^4\sigma^2 - 32\beta\delta^3\rho^5\sigma^3 + \gamma^4\delta^8 - 8\gamma^3\delta^7\rho\sigma$$
$$+24\gamma^2\delta^6\rho^2\sigma^2 - 32\gamma\delta^5\rho^3\sigma^3 + 16\delta^4\rho^4\sigma^4 .$$

The Future

This approach to finding Newton polytopes of hypersurface, and possibly using that information with interpolation to find a defining polynomial appears feasible, and would have many applications, were a proper implementation made.

This is a current project of Taylor Brysiewicz, a graduate student at TAMU.

References

[1] R. Basson, R. Lercier, C. Ritzenthaler, and J. Sijsling, An explicit expression of the Lüroth invariant, ISSAC 13, ACM, New York, 2013, pp. 31–36.

[2] J. Hauenstein and F. Sottile, Newton Polytopes and Witness Sets,] Mathematics in Computer Science, 8 (2014), pp. 235–251.