Generalized Witness Sets

Software and Applications in Numerical Algebraic Geometry
SIAM Meeting on Applied Algebraic Geometry, 4 August 2015

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Based on discussions with: D. Bates, J. Hauenstein, and A. Leykin.

Numerical Algebraic Geometry

The origins of numerical algebraic geometry were in numerical homotopy continuation, a method to find all isolated complex solutions to a system of n equations in n variables

The subject rightly began around 2000, when Sommese, Verschelde, and Wampler developed the notion of a *witness set*

A witness set is a data structure for representing and manipulating an algebraic variety using numerical algorithms on a computer

Witness sets are the foundation for many geometrically appealing algorithms in numerical algebraic geometry

It is appropriate to consider the analogy:

$$\text{Witness set} \ : \ \text{Numerical Algebraic Geometry}$$
$$\Longleftrightarrow \text{Gröbner Basis} \ : \ \text{Symbolic Computation}$$

What is a Witness Set? I

Let $V \subset \mathbb{C}^n$ (or $\subset \mathbb{P}^n$) be a k-dimensional variety

A *witness set* for V is a triple (W, F, L) where

- $F = (f_1, \ldots, f_{n-k})$ are polynomials such that V is a component of $F^{-1}(0)$
- $L = (\ell_1, \ldots, \ell_k)$ are general affine forms defining a general $(n-k)$-plane $L^{-1}(0)$
- $W = V \cap L^{-1}(0)$

By Bertini's Theorem, W is a collection of $\deg V$ reduced points

Moving L enables us to sample points of V, test for membership in V, and perform many other geometric constructions

May view a witness set as a concrete manifestation of André Weil's notion of a general point

What is a Witness Set? II

$V \subset \mathbb{C}^n$ is a cycle whose class lies in the Chow group $A_k\mathbb{P}^n$
(Chow := algebraic cycles modulo rational equivalence, $\sim$)

L is a general representative of the distinguished generator of $A^k\mathbb{P}^n$

$W = V \cap L \in A_0 V$ (localized intersection product) is a reduced zero-cycle on V that witnesses the cap product $[V] \cap [L]$

By (Poincaré) duality, W represents V. Specifically, in $A_k\mathbb{P}^n$,

$$[V] = W \cdot [L_k]\,, \qquad L_k \text{ a } k\text{-plane}$$

There are acceptable variations, using images of algebraic cycles in cohomology, or numerical equivalence,...

Generalized Witness Sets I

Suppose that X is a smooth variety of dimension n with finitely generated Chow groups satisfying Poincaré duality

Let $\{L_{i,k} \mid k = 0, \ldots, n\,,\ \ i = 1, \ldots, \beta_k\}$ be cycles such that $\{[L_{i,k}] \mid i = 1, \ldots, \beta_k\}$ forms a basis for $A_k X$

We will also want that

- For every point x of X and i, k, there is a cycle Λ rationally equivalent to $L_{i,k}$ containing x
- For $Y \subset X$ of codimension k and any $i = 1, \ldots, \beta_k$, there is a cycle Λ rationally equivalent to $L_{i,k}$ with $Y \cap \Lambda$ is transverse

While apparently restrictive, projective spaces, Grassmannians, flag manifolds and products of these spaces all have these properties

Generalized Witness Sets II

Given such a variety X and representatives $L_{i,k}$

Let V be a subvariety of X of dimension $n-k$.
A *witness set* for V is a list of pairs

$$(W_1, \Lambda_1)\,,\ \ldots\ ,\ (W_{\beta_k}, \Lambda_{\beta_k})$$

where

- $\Lambda_i \sim L_{i,k}$ for $i = 1, \ldots, \beta_k$ with Λ_i general
- $W_i = V \cap \Lambda_i$ is a transverse intersection (and is a set of reduced points)

(This may be modified to be more computational by including $n-k$ hypersurfaces (equations) F whose intersection contains V as a component, and also equations for the Λ_i)

Rational Equivalence & Membership

Rational Equivalence. Suppose that $U \subset X \times \mathbb{C}$ is irreducible of dimension $k+1$ with k-dimensional fibers over $\mathbb{C}$ (the map f to $\mathbb{C}$ is flat). Then $f^{-1}(0) \sim f^{-1}(1)$. These elementary rational equivalences generate $\sim$ on algebraic cycles

$\longrightarrow$ Rational equivalence is just an algebraic homotopy

Membership. Given $x \in X$ and a nonempty witness set (W_i, Λ_i) for V. Let $\Lambda' \sim \Lambda_i$ contain x

The chain of elementary rational equivalences gives a homotopy between $W_i = V \cap \Lambda_i$ and $W' := V \cap \Lambda'$
Then $x \in V \Leftrightarrow x \in W'$

Other algorithms also extend to this setting

Examples

Grassmannians. The Grassmannian has distinguished Schubert varieties $X_\alpha F$ whose classes form a basis of its Chow ring, and satisfy duality

These cover the Grassmannian and satisfy a Bertini Theorem

Regeneration is also possible. The Picard group is $\mathbb{Z}$, so every hypersurface is a multiple of the Schubert divisor, D. The geometric Pieri rule (Schubert, 1884) gives an easy homotopy between

$$D \cap X_\alpha F \quad \text{and} \quad \sum_{\beta \lessdot \alpha} X_\beta F$$

Other varieties. These properties (except Pic, which is free abelian) hold for products of Grassmannians, including products of projective spaces: (See mss. of Hauenstein-Rodriguez on multiprojective varieties). Most are known to hold for flag manifolds

Final Comments and References

Challenge: Implement and refine these ideas

Oeding: Does there exist a reasonable notion of an equivariant witness set?

References.

- J. Hauenstein and J. Rodriguez, *Numerical irreducible decomposition of multiprojective varieties*, `arXiv:1507.07069`
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