The Optimal Littlewood-Richardson Homotopy

Algorithms and Complexity in Polynomial System Solving
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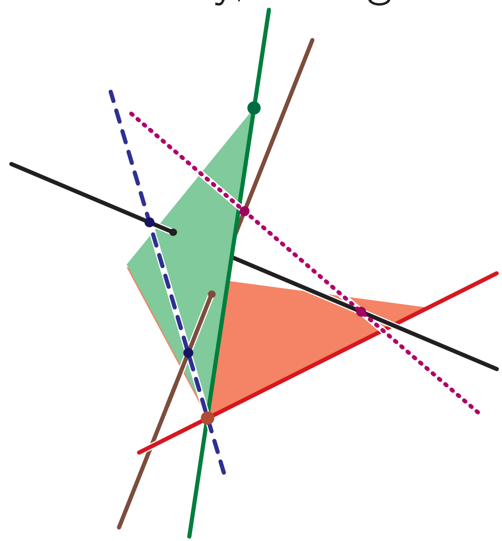


Work with: Abraham Martín del Campo,
Anton Leykin, Ravi Vakil, and Jan Verschelde

Homotopy Continuation Algorithms

Numerical homotopy continuation computes all solutions to a system of polynomial equations. There are several approaches

→ Bézout homotopy $F\colon (f_1, \ldots, f_n \mid \deg(f_i) = d_i)$

$$H(t;x) = tF(x) + (1-t)(x_i^{d_i} - 1 \mid i = 1, \ldots, n)$$

- Optimal (no extraneous paths) for generic systems F
- Poorly behaved for non-generic systems with structure

→ Polyhedral homotopy. Optimal for sparse systems w/ BKK bound

→ Equation by equation/regeneration. Default method for `Bertini`

Very general and very flexible

Equations in Geometry

In algebraic geometry, varieties do not have natural *square* formulations (number of equations=number of variables)

This is even more true in enumerative geometry, which is concerned with zero-dimensional transverse intersections of varieties

Even when square, the number of solutions is far less than BKK bound

Typically, all methods are non-optimal

Point de départ:
Classical 19^{th}c enumerative geometry is based on the principle of continuity and the method of specialization—this is just a homotopy continuation algorithm in reverse

Schubert Problems

Scubert problems are a fundamental class of problems in enumerative geometry

The set of linear spaces having position α with respect to a flag of subspaces $F: F_1 \subset \cdots \subset F_n = \mathbb{C}^n$ is a Schubert variety, $X_\alpha F$

Schubert problems are formulated as intersections of Schubert varieties

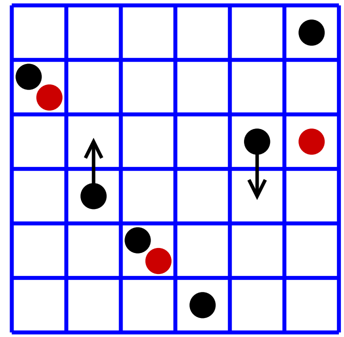
$$(*) \qquad X_{\alpha^1} F^1 \cap \cdots \cap X_{\alpha^s} F^s ,$$

where the flags $F^1, \ldots, F^s$ are general

We want to compute the points in $(*)$

Geometric Littlewood-Richardson Rule

This transforms $Y(F,M) := X_\alpha F \cap X_\beta M$ $(F, M$ general) into a union of Schubert varieties



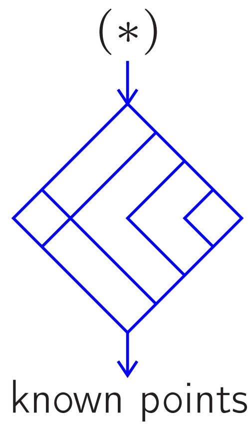
The flag M is moved to coincide with F in $\binom{n}{2}$ steps, deforming $Y(F,M)$ in the process

Components $Y_{\bullet\bullet}(F,M)$ are encoded by checkerboard patterns. Their deformations are recorded by checkerboard games

Iterating $s-1$ times resolves our Schubert problem

$$(*) \qquad X_{\alpha^1}F^1 \cap \cdots \cap X_{\alpha^s}F^s$$

This sequence of deformations is organized combinatorially by a directed acyclic graph



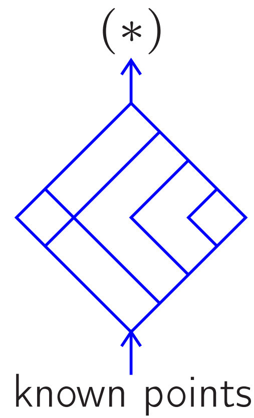
Animations

.html

Homotopy Steps

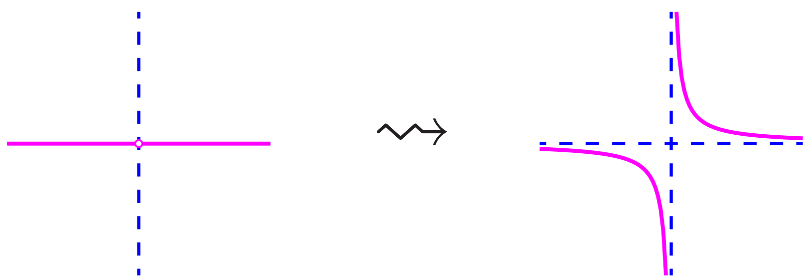
Reversing the directed acyclic graph & putting deformations into coordinates gives the Littlewood-Richardson Homotopy

The action is in changing the parametrizations of the checkerboard varieties $Y_{\bullet\bullet}(F, M)$



Each of the $\binom{n}{2}$ steps has one of three geometries:

- Geometrically constant (Just a coordinate change)
- Simple homotopy (Subspace rotates with flag)
- Subtle homotopy (Read the paper/code)



References

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