Cohomological Consequences of the Pattern Map

Geometry and combinatorics on homogeneous spaces

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Schubert Polynomials

Lascoux and Schützenberger (1982) defined *Schubert Polynomials*, $\mathfrak{S}_w(x)$, for w a permutation. These form a basis for all polynomials in x and include all Schur polynomials.

For $w \in S_n$, they form a system of polynomial representatives of Schubert classes in the cohomology ring the flag variety $\mathbb{F}\ell(n)$.

In "*Structure de Hopf...*" ('82) L-S show that in the expansion

$$\mathfrak{S}_w(y_1, \ldots, y_n\,,\, z_1, \ldots, z_m) \;=\; \sum_{u,v} d_w^{u,v}\, \mathfrak{S}_u(y)\mathfrak{S}_v(z)\,,$$

the coefficients $d_w^{u,v}$ are nonnegative.

Arbitrary Substitutions

Arbitrarily substituting y's and z's in some $\mathfrak{S}_w(x)$, e.g.,

$$\mathfrak{S}_w(y_1, y_2, z_1, y_3, y_4, z_2, z_3 \ldots) = \sum_{u,v} d_w^{u,v} \, \mathfrak{S}_u(y)\mathfrak{S}_v(z),$$

defines $d_w^{u,v} \in \mathbb{Z}$ depending on the positions of the y's and z's.

Bergeron–S. ('98): These coefficients $d_w^{u,v}$ are naturally Schubert structure constants $c^{\alpha}_{\beta,\gamma}$ for multiplication in Schubert basis.

Used projection formula along the map $\mathbb{F}\ell(n) \times \mathbb{F}\ell(m) \hookrightarrow \mathbb{F}\ell(n{+}m)$ whose pullback gives the substitution.

Lenart, Robinson, S. ('06): Generalized to the Grothendieck ring of $\mathbb{F}\ell(n{+}m)$.

Geometry of Permutation Patterns

Billey-Braden ('03): G: Semisimple linear algebraic group.
Let $\mathcal{F}$ be the flag variety of G, parametrizing Borel subgroups.
Let $\eta \in G$ be semisimple. Set $G_\eta := Z_G(\eta)$.

$B \mapsto B_\eta := B \cap G_\eta$ defines the *geometric pattern map*, π_η,

$$\mathcal{F}^\eta := \{B \in \mathcal{F} \mid \eta \in B\} \xrightarrow{\pi_\eta} G_\eta/B_\eta\,.$$

Let W, W_η be the Weyl groups of G, G_η. If $\pi_\eta \colon W \to W_\eta$ is the Billey-Postnikov generalised pattern map, then we have

Theorem [BB]. $\pi_\eta(X_w \cap \mathcal{F}^\eta) \;=\; X_{\pi_\eta(w)}$.

In type A, the map $\mathbb{F}\ell(n) \times \mathbb{F}\ell(m) \hookrightarrow \mathbb{F}\ell(n+m)$ is a section of π_η, where $\eta = \left(\begin{smallmatrix} \alpha I_n & 0 \\ 0 & \beta I_m \end{smallmatrix}\right)$.

Sections of the Pattern Map

Sections of the pattern map

$$G_\eta/B_\eta \xrightarrow{\text{Ш}} \mathcal{F}^\eta$$

correspond to right cosets Ш of W_η in W.

On cohomology rings, this is just a substitution of the canonical generators $\mathfrak{h}^*$ of $H^*(\mathcal{F})$ for the canonical generators $\mathfrak{h}^*$ (!) of $H^*(G_\eta/B_\eta)$. (Here η lies in a maximal torus for G, which is also a maximal torus for G_η.)

In type A, this is just renaming and shuffling the variables in a Schubert polynomial. E.g.,

$$\text{Ш}^*(\mathfrak{S}_w) \;=\; \mathfrak{S}_w(y_1, y_2, z_1, y_3, y_4, z_2, z_3, \ldots)\,.$$

Sections in Homology

For $u \in W_\eta$, $X_u B_\eta = X_u B_\eta \cap X_{\omega_\eta} B'_\eta$, where B'_η is the Borel opposite to B_η and ω_η is the longest element in W_η. Thus

$$Ш(X_u B_\eta) \;\subset\; X_{Ш(u)} B \cap X_{Ш'(\omega_\eta)} B' \,.$$

A dimension computation shows equality, so

$$Ш_*[X_u B_\eta] \;=\; [X_{Ш(u)} B \cap X_{Ш'(\omega_\eta)} B'] \,.$$

For $w \in W$, we have $Ш^*(\mathfrak{S}_w) = \sum_{u \in W_\eta} d^u_w \mathfrak{S}_u$, so d^u_w is

$$p_*(Ш^*(\mathfrak{S}_w) \cap [X_u]) \;=\; p_*(\mathfrak{S}_w \cap [X_{Ш(u)} \cap X'_{Ш'(\omega_\eta)}]) \,,$$

where p is the map to a point.
$\rightsquigarrow$ d^u_w is a particular Schubert structure constant.

The Actual Formula

The section Ш corresponds to a right coset of W_η. Let $\varsigma \in W$ be the minimal length coset representative. Then $d_u^w = c_{w,\varsigma}^{u\varsigma}$.

Algorithm:

Expand the product $\mathfrak{S}_w \cdot \mathfrak{S}_\varsigma$ in Schubert basis for $H^*(\mathcal{F})$.

Restrict to terms of the form $\mathfrak{S}_{u\varsigma}$ for $u \in W_\eta$.

Replace $\mathfrak{S}_{u\varsigma}$ by $\mathfrak{S}_u$ to obtain formula for $\text{Ш}^*(\mathfrak{S}_w)$.

Example: $G = C_4$, $G_\eta = A_3$, and $\varsigma = \overline{2}\,\overline{1}\,3\,4$

$$\mathfrak{C}_{3\,\overline{1}\,4\,2} \cdot \mathfrak{C}_{\overline{2}\,\overline{1}\,3\,4} \;=\; \mathfrak{C}_{\overline{3}\,\overline{2}\,4\,\overline{1}} + 2\mathfrak{C}_{2\,\overline{3}\,4\,\overline{1}} + 2\mathfrak{C}_{\overline{4}\,\overline{3}\,1\,2} + 2\mathfrak{C}_{\overline{2}\,\overline{3}\,4\,1} + 2\mathfrak{C}_{\overline{1}\,\overline{4}\,3\,2} + 2\mathfrak{C}_{\overline{4}\,\overline{2}\,3\,1}\,.$$

so $\text{Ш}^*\left(\mathfrak{C}_{3\,\overline{1}\,4\,2}\right) \;=\; 2\mathfrak{S}_{3412} + 2\mathfrak{S}_{3241} + 2\mathfrak{S}_{4132} + 2\mathfrak{S}_{2431}\,.$