Khovanskii-Rolle Continuation for Real Solutions

Algorithms in Real Algebraic Geometry and its Applications

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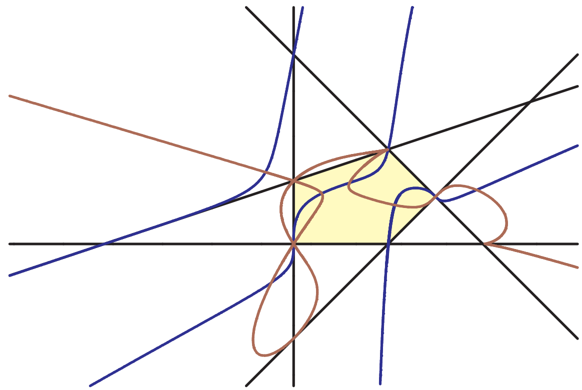

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Khovanskii-Rolle Continuation

This algorithm computes all positive solutions to a system of fewnomials using the real solutions to low degree polynomial systems and tracing arcs of real curves.

Its virtue is that the number of curves traced is controlled by the fewnomial bound, it takes advantage of any slack in that bound, and only real solutions of the fewnomial system are computed.

This talk will indicate portions of the algorithm, as well as some challenges that remain to make it effective.

Fewnomial Bounds

Khovanskii bounded the number, r, of positive solutions to a system of n real polynomials in n variables having $\ell + n + 1$ monomials, a *fewnomial system*.

Theorem (Khovanskii, 1980) $\quad r < 2^{\binom{\ell+n}{2}}(n+1)^{\ell+n}$.

This was improved using his main ideas in a novel way.

Theorem (Bihan, S., 2007) $\quad r < \frac{e^2+3}{4} 2^{\binom{n}{2}} n^{\ell}$.

The Khovanskii-Rolle continuation algorithm comes from the algorithmic proof of this last bound.

Steps in Algorithm

(i) *Gale duality* converts a fewnomial system in $\mathbb{R}^n_>$ to an equivalent system of functions on a polytope Δ in $\mathbb{R}^\ell$.

(ii) *Khovanskii-Rolle continuation* solves the system on Δ by path tracking arcs of curves in Δ from solutions to low degree polynomial systems on the faces of Δ.

(iii) These solutions are mapped to $\mathbb{R}^n_>$ and then refined and certified to give solutions to original fewnomial system.

Gale duality, via example

Suppose we have the system of polynomials,

$$\begin{aligned} v^2w^3 &= 1 - u^2v - uv^2w\,, \\ v^2w &= \tfrac{1}{2} - u^2v + uv^2w\,, \\ uvw^3 &= \tfrac{10}{11}(1 + u^2v - 3uv^2w)\,. \end{aligned} \qquad (*)$$

Observe that

$$\begin{aligned} (u^2v)^2\cdot(v^2w^3)^3 &= (uv^2w)^2\cdot(v^2w)\cdot(uvw^3)^2 \quad \text{and} \\ (uv^2w)^3\cdot(v^2w^3) &= (u^2v)\cdot(v^2w)^3\cdot(uvw^3)\,. \end{aligned}$$

Substituting $(*)$ into this, writing x for u^2v and y for uv^2w, and solving for 0, gives the polynomial form of the Gale dual system

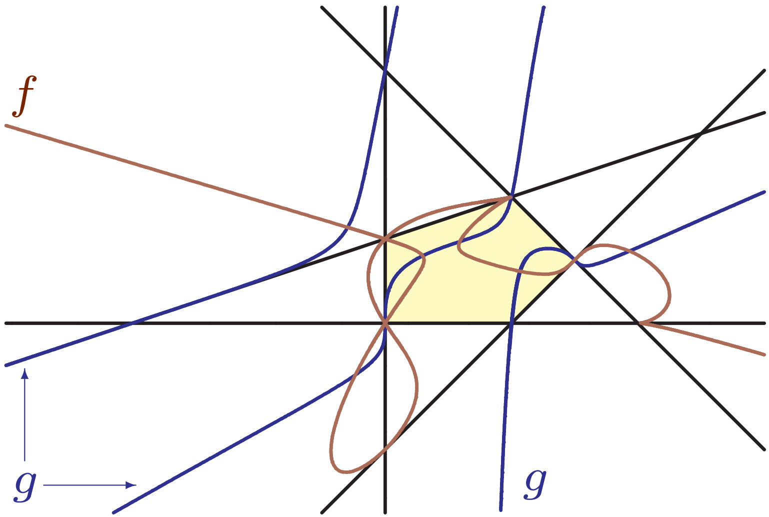
$$\begin{aligned} x^2(1-x-y)^3 - y^2(\tfrac{1}{2}-x+y)(\tfrac{10}{11}(1+x-3y))^2 &= 0\,, \\ y^3(1-x-y) - x(\tfrac{1}{2}-x+y)^3\tfrac{10}{11}(1+x-3y) &= 0\,. \end{aligned}$$

Gale duality, continued

The original system is equivalent to the Gale system

$$f := x^2(1-x-y)^3 / \left(y^2(\tfrac{1}{2}-x+y)(\tfrac{10}{11}(1+x-3y))^2\right) = 1\,,$$
$$g := y^3(1-x-y) / \left(x(\tfrac{1}{2}-x+y)^3\tfrac{10}{11}(1+x-3y)\right) = 1\,,$$

in the complement of the lines given by the linear factors.



Gale Duality

A system of polynomials w/ monomials $\{x^\alpha \mid \alpha \in \mathcal{A}\}$,

$$f_1(x_1,\ldots,x_n) \;=\; \cdots \;=\; f_n(x_1,\ldots,x_n) \;=\; 0\,,$$

is the pullback of a linear section of the toric variety $X_{\mathcal{A}}$

$$\varphi_{\mathcal{A}}^{-1}(X_{\mathcal{A}} \cap L) \qquad (L \simeq \mathbb{C}^\ell)\,,$$

where $X_{\mathcal{A}} \subset \mathbb{C}^{\mathcal{A}}$ is parametrized by $\varphi_{\mathcal{A}}\colon x \mapsto (x^\alpha \mid \alpha \in \mathcal{A})$.

If $p\colon \mathbb{C}^\ell \xrightarrow{\sim} L \subset \mathbb{C}^{\mathcal{A}}$ parametrizes ℓ, the *Gale Dual System* on $\mathbb{C}^\ell$ is

$$p^{-1}(\text{equations for } X_{\mathcal{A}})\,.$$

(These equations are ℓ binomials in the components of p, which give ℓ rational functions whose logarithms are linear combinations of the logarithms of the components of p.)

Khovanskii-Rolle continuation

Given a logarithmic Gale system,

$$\psi_j := \sum_{i=1}^{\ell+n} a_{i,j} \log(p_i(y)) = 0 \qquad j = 1, \ldots, \ell\,, \qquad (*)$$

($p_i(y)$ linear), we find solutions in the polyhedron

$$\Delta := \{y \in \mathbb{R}^\ell \mid p_i(y) > 0\}\,.$$

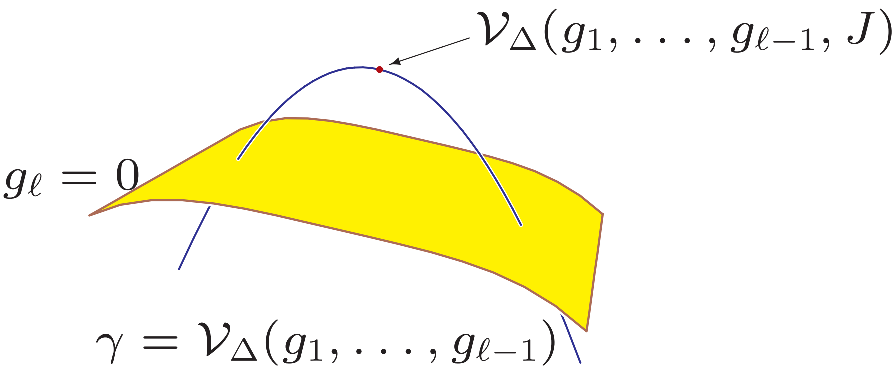
(Δ corresponds to the positive orthant $\mathbb{R}^n_>$.)

By Khovanskii-Rolle Theorem (next slide), solutions of $(*)$ come from solutions of low degree polynomial systems via path continuation.

Khovanskii-Rolle Theorem

Let $g_1, \ldots, g_\ell$ be functions on Δ, $\gamma := \mathcal{V}_\Delta(g_1, \ldots, g_{\ell-1})$ a smooth curve with $\mathrm{ubc}_\Delta(\gamma)$ unbounded components. Then

$$|\mathcal{V}_\Delta(g_1, \ldots, g_\ell)| \;\leq\; \mathrm{ubc}_\Delta(\gamma) \;+\; |\mathcal{V}_\Delta(g_1, \ldots, g_{\ell-1},\, J)|\,.$$



Starting at points where γ meets $\partial\Delta$ and $J = 0$, tracing arcs of γ in both directions finds all solutions $\mathcal{V}_\Delta(g_1, \ldots, g_{\ell-1}, g_\ell)$.

Degree Reduction & Solutions

Given $\psi_j := \displaystyle\sum_{i=1}^{\ell+n} a_{i,j} \log(p_i(y)) \;=\; 0 \quad j = 1, \ldots, \ell$, set

$J_j :=$ numerator of Jacobian of $\psi_1, \ldots, \psi_j, J_{j+1}, \ldots, J_\ell$,

$\gamma_j := \mathcal{V}_\Delta(\psi_1, \ldots, \psi_{j-1}, J_{j+1}, \ldots, J_\ell)$.

Then $\deg J_j = 2^{\ell-j}n$ and $T_j := \gamma_j \cap \partial\Delta$ is $J_{j+1} = \cdots = J_\ell = 0$ on the $\ell-j$ skeleton of Δ.

In computing $S_0 := \mathcal{V}_\Delta(J_1, \ldots, J_\ell)$ we compute the endpoints T_j by regeneration, and recursively compute $S_j := \mathcal{V}_\Delta(\psi_1, \ldots, \psi_j, J_{j+1}, \ldots, J_\ell)$ by Khovanskii-Roll continuation. Our solutions are S_ℓ.

Features and Status

— Solution of Gale system that we just saw proposed by Bates-S., with a Maple/Bertini implementation for $\ell = 2$.

— Algorithmic issues in setting up Gale system, passing solutions back to original fewnomial system, and using regeneration to compute T_j is being written up.

— The curves γ_j are mildly singular at endpoints $\gamma_j \cap \partial\Delta$. Have a method to overcome this. Not written.

— Genericity of exponents $\mathcal{A}$ and polynomials f_i is assumed. Need to remove this.

— Needs a proper implementation.

Thank You !