Galois Groups of Schubert Problems

Georgia Tech Colloquium

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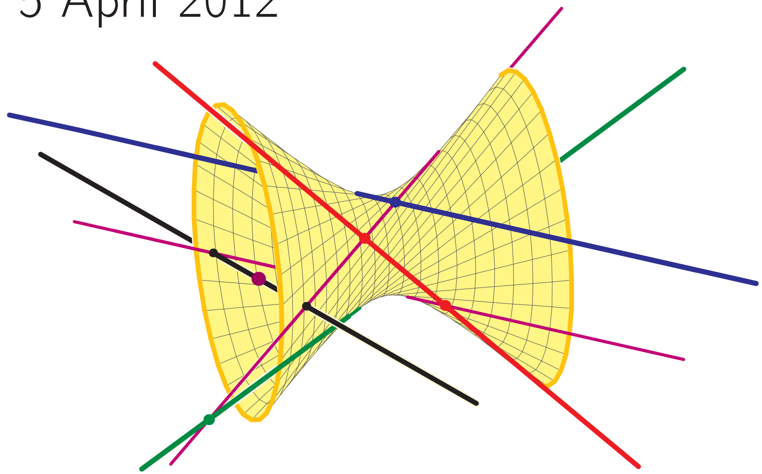

Frank Sottile

sottile@math.tamu.edu

Work with: Anton Leykin (Georgia Tech),

Christopher Brooks, Abraham Martín del Campo, and

Jon Hauenstein (Texas A&M)

Galois theory and the Schubert calculus

Galois theory had its origins in understanding the symmetries of polynomials. Later, Galois groups came to be understood as encoding all the symmetries of fields. Galois theory today is a pillar of number theory.

Galois groups also appear in geometry. The geometric study of Galois groups is not well-developed, because of its subtlety and because Galois groups are very hard to determine.

I will tell you of some ongoing work that is intended to shed more light on the topic of Galois groups in geometry. Specifically, I will describe a project to study Galois groups in the Schubert calculus, which is a well-studied class of geometric problems involving linear subspaces.

It is best to begin with examples.

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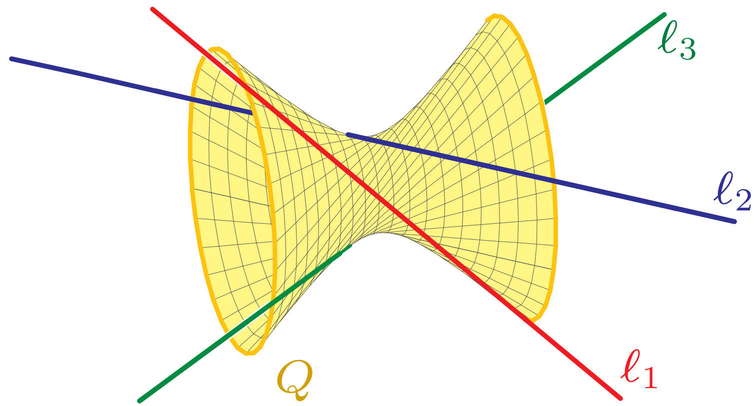

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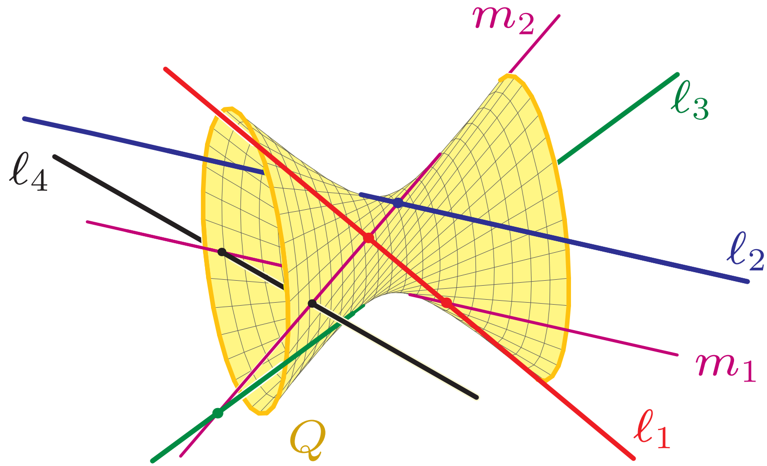

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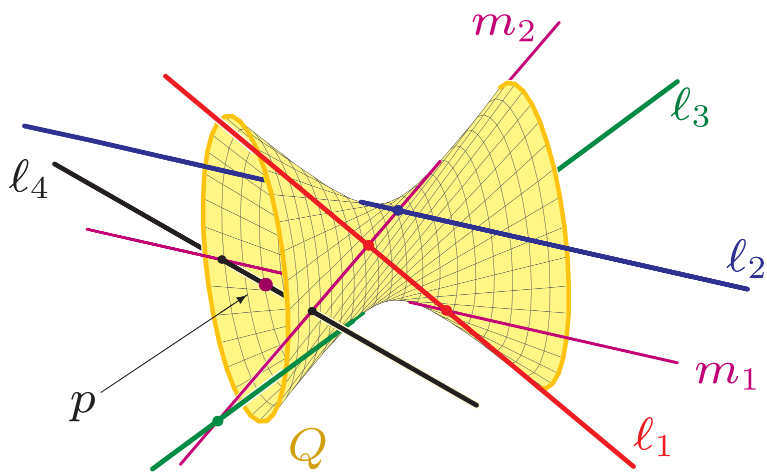

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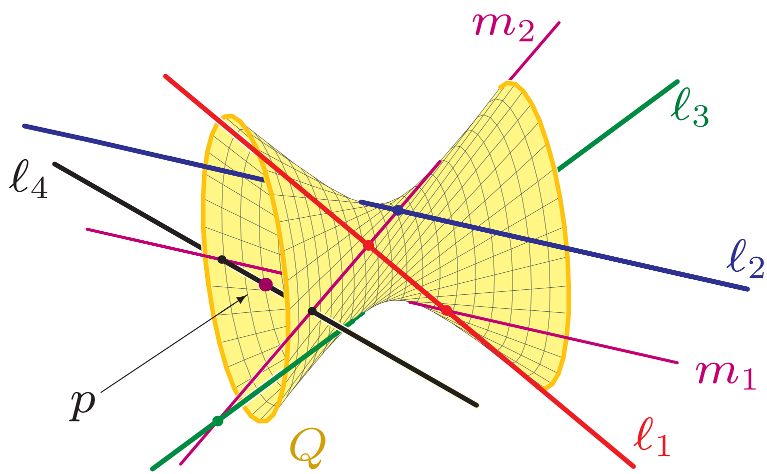


This shows that

The Galois group of the problem of four lines is the symmetric group $\mathcal{S}_2$.

A Problem with Exceptional Geometry

Q: What 4-planes H in $\mathbb{C}^8$ that meet each of general 4-planes K_1, K_2, K_3, K_4 in a 2-dimensional subspace?

Auxiliary problem: There are four (h_1, h_2, h_3, h_4) 2-planes in $\mathbb{C}^8$ meeting each of K_1, K_2, K_3, K_4. Schematically, $\square\square\square^{\,4} = 4$.

Fact: All solutions H to our problem have the form $H_{i,j} = \langle h_i, h_j \rangle$ for $1 \leq i < j \leq 4$. Schematically, $\boxplus^{\,4} = 6$.

It follows that the Galois group of $\boxplus^{\,4} = 6$ is equal to the Galois group of $\square\square\square^{\,4} = 4$, which is known to be the symmetric group $\mathcal{S}_4$.

This problem $\boxplus^{\,4} = 6$ also has exceptional reality: If K_1, K_2, K_3, K_4 are real, then either two or six of the $H_{i,j}$ are real, and never four or zero.

Galois Groups of Enumerative Problems

In 1870, Jordan explained how *algebraic* Galois groups arise naturally from problems in enumerative geometry, and in 1979 Harris showed that such an algebraic Galois group coincides with a geometric monodromy group.

This Galois group of a geometric problem is a subtle invariant. When it is *deficient* (i.e. not the full symmetric group), the geometric problem has some exceptional, intrinsic structure.

Work of Harris and of Vakil give several methods to determine Galois groups, at least experimentally.

I will describe a project to study Galois groups for problems coming from the Schubert calculus using numerical algebraic geometry, symbolic computation, and combinatorics. We expect more questions than answers.

Some Theory

A degree e dominant map $E \xrightarrow{\pi} B$ of equidimensional irreducible varieties (up to codimension one, $E \to B$ is a covering space of degree e)
$\rightsquigarrow$ degree e extension of fields of rational functions $\pi^* K(B) \subset K(E)$.
Define the Galois group Gal$(E/B) \subset \mathcal{S}_e$ to be the Galois group of the Galois closure of this extension.

Harris's Theorem. (Work over $\mathbb{C}$.) Restricting $E \to B$ to open subsets over which π is a covering space, $E' \to B'$, the Galois group is equal to the monodromy group of deck transformations.

This is the group of permutations of a fixed fiber induced by analytically continuing the fiber over loops in the base.

Point de départ: Such monodromy permutations are readily and reliably computed using methods of numerical algebraic geometry.

Enumerative Geometry

"Enumerative Geometry is the art of determining the number e of geometric figures x having specified positions with respect to other, fixed figures b."
— Hermann Cäser Hannibal Schubert, 1879.

B := configuration space of the fixed figures, and X := the space of the figures x we count. Then $E \subset X \times B$ consists of pairs (x, b) where $x \in X$ has given position with respect to $b \in B$.

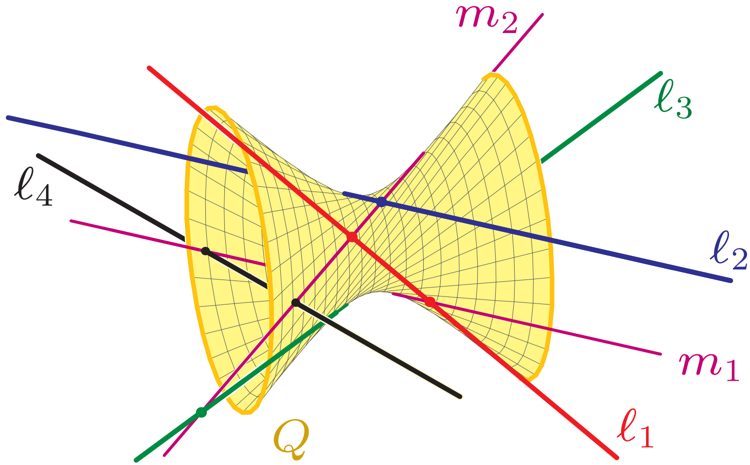
The projection $E \to B$ is a degree e cover outside of some discriminant locus, and the *Galois group of the enumerative problem* is Gal(E/B).

In the problem of four lines, B = four-tuples of lines, X = lines, and E consists of 5-tuples $(m, \ell_1, \ell_2, \ell_3, \ell_4)$ with m meeting each ℓ_i. We showed that this has Galois group the symmetric group $\mathcal{S}_2$.

Schubert Problems

The Schubert calculus is an algorithmic method promulgated by Schubert to solve a wide class of problems in enumerative geometry.

Schubert problems are problems from enumerative geometry involving linear subspaces of a vector space incident upon other linear spaces, such as the problem of four lines, and the problems $\square\square\square^4 = 4$ and $\boxplus^4 = 6$.



As there are many millions of computable Schubert problems, many with their own unique geometry, they provide a rich and convenient laboratory for studying Galois groups of geometric problems.

Proof-of-concept computation

Leykin and I used off-the-shelf numerical homotopy continuation software, and an implementation of (an easy version of) the Pieri homotopy algorithm to compute Galois groups of some Schubert problems formulated as the intersection of a skew Schubert variety with Schubert hypersurfaces.

In every case, we found enough monodromy permutations to generate the full symmetric group (determined by `Gap`). This included one Schubert problem with $e = 17,589$ solutions.

We conjecture that problems of this type will always have the full symmetric group as Galois group.

To investigate this question for more general Schubert problems, both on Grassmannians and on other flag manifolds, we need more algorithms and implementations.

Numerical Project

Recent work, including certified continuation (Beltrán and Leykin), Littlewood-Richardson homotopies (Vakil, Verschelde, and S.), regeneration (Hauenstein), implementation of Pieri and of Littlewood-Richardson homotopies (Martín del Campo and Leykin) will enable the reliable numerical computation of Galois groups of more general Schubert problems.

We plan to use a supercomputer whose day job is calculus instruction to investigate many of the millions of accessible and computable Schubert problems. Our intention is to build a library of Schubert problems (expected to be very few) whose Galois groups are deficient.

These data would be used to generate conjectures, leading to proofs about Galois groups of Schubert problems, as well as showcase the possibilities of numerical computation.

Vakil's Alternating Lemma

Suppose $S \subset B$ has a dense set of regular values of $E \to B$. Then

$$\text{Gal}(E|_S/S) \hookrightarrow \text{Gal}(E/B).$$

This occurs often in enumerative geometry. Also common are geometric degenerations

$$X \cap Y \rightsquigarrow W \cup Z$$

which give natural families $S \subset B$ such that

$$E|_S \simeq F \bigcup G \quad \text{where } F \to S \text{ and } G \to S$$

are the sub-enumerative problems for W & Z of degrees f, g, where $f + g = e$.

Vakil's Alternating Criterion. If $f \neq g$ and both $\text{Gal}(F/S)$ and $\text{Gal}(G/S)$ contain the alternating groups A_f and A_g, then $\text{Gal}(E/B)$ contains the alternating group A_e.

Application of Vakil's Criterion

Vakil's geometric Littlewood-Richardson rule allows the use of his criterion.

Christopher Brooks has an efficient `python` script implementing Vakil's geometric Littlewood-Richardson rule and criterion. We intend to use it in our study of Galois groups. Managing mountains of data is a challenge.

We found that all Schubert problems involving lines in $\mathbb{P}^n$ for $n \leq 40$ have at least alternating Galois groups, which inspired us prove the following.

Theorem. (Brooks, Martín del Campo, S.) *The Galois group of any Schubert problem involving lines in* $\mathbb{P}^n$ *is at least alternating.*

This proof reduces to an inequality of Kostka numbers, part of which has an easy combinatorial proof. For the other cases, we use Fourier analysis on the representation ring of $\mathfrak{sl}_2$ to convert the combinatorial inequality into an integral inequality, which we establish by estimation. This Fourier analysis gives a new interpretation of the cohomology of Grassmannians, and this is all part of the Ph.D. thesis of Martín del Campo.

Specialization Lemma

Given $\pi\colon E \to B$ with B rational, the fibre $\pi^{-1}(y)$ above a $\mathbb{Q}$-rational point $y \in B(\mathbb{Q})$ has a minimal polynomial $p_y(t) \in \mathbb{Q}[t]$. In this situation, the algebraic Galois group of $p_y(t)$ is a subgroup of Gal(E/B).

This can be applied to Schubert problems (and many other geometric problems). The minimal polynomial of such fibers are easy to compute in many cases when $e \lesssim 200$, which should enable this method to be used to study Galois groups of many thousands to millions of Schubert problems.

Using reduction modulo many primes, it is not hard to determine the Galois group when it is the full symmetric group.

However, when it is not the full symmetric group, we do not know of software that can compute Galois groups of the huge polynomials we generate. The software that we know of cannot handle even moderate-sized polynomials (degree ~ 20, with integer coefficients $\sim 10^{200}$).

Thank You!