Bounds for real solutions to polynomial equations

Solving Polynomial Equations Workshop

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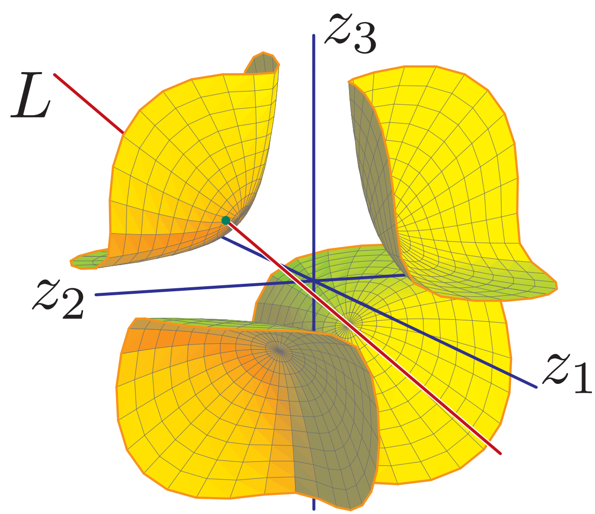


The problem of real solutions

Given a system of real polynomial equations

$$f_1(x_1,\ldots,x_n) \;=\; \cdots \;=\; f_n(x_1,\ldots,x_n) \;=\; 0\,,$$

with some number, say d, of complex solutions, what can we say about the number, r, of real solutions?

Besides $d \geq r \geq d \bmod 2$, typically not much.

We can often say much more for structured systems; there are many cases where either of these trivial bounds can be improved. That is, there are often nontrivial upper bounds or nontrivial lower bounds on the number of real soutions.

Fewnomial Upper Bounds

Descartes' (1636): *A univariate polynomial with $d+1$ terms*

$$c_0 x^{a_0} \;+\; c_1 x^{a_1} \;+\; \cdots \;+\; c_d x^{a_d}$$

has at most d positive real roots.

This bound for univariate polynomials was only recently extended.

Khovanskii (1980): *A system of n polynomial equations in n variables involving $k+n+1$ monomials will have at most $2^{\binom{k+n}{2}}(n+1)^{k+n}$ solutions with all coordinates positive.*

Enormous: When $n = k = 2$ the bound is $2^6 3^4 = 5184$.

What is the true bound, or *Khovanskii number*, $\chi(k,n)$?

Sharper Bounds

A smaller upper bound was found using Khovanskii's method and some geometry adapted to polynomials (Gale duality).

Bates, Bihan, S. $\chi(k,n) \leq \frac{e^2+3}{4} 2^{\binom{k}{2}} n^k$.

The bound for all real solutions is $\frac{e^4+3}{4} 2^{\binom{k}{2}} n^k$.

For $k = n = 2$, this bound is 20.778, but it can be lowered to 15.5, both of which are less than 5184.

Using an earlier construction of Bihan, this is sharp in the asymptotic sense, for k fixed and n large.

Bihan, Rojas, S. $\chi(k,n) \geq \lfloor \frac{n+k}{k} \rfloor^k$.

Open Problems

There are many open questions about fewnomial bounds.

- Bounds for other topological invariants of fewnomials.
- Systems of equations whose exponents exhibit more structure?
- Mixed systems (different exponents in each polynomial).
- What is the actual value of $\chi(k,n)$?
- Most lacking are constructions of fewnomial systems with many real solutions.

Tropical lower bounds

In 1990, Kontsevich gave a recursion for the number N_d of degree d rational curves interpolating $3d-1$ points in $\mathbb{P}^2$.

More recently, Welschinger proved that a particular signed (± 1) sum of the real rational curves was a constant, W_d.

Itenberg, Kharlamov, and Shustin used the tropical correspondence theorem of Mikhalkin to show that

$$W_d \geq \frac{d!}{3} \quad \text{and} \quad \lim_{d\to\infty} \frac{\log W_d}{\log N_d} = 1\,.$$

Thus W_d is a non-trivial lower bound for the number of real rational curves interpolating $3d-1$ points in $\mathbb{RP}^2$.

Partially inspired by this, Soprunova and I set out to develop a theory of lower bounds for systems of polynomial equations.

Lower bounds from topology

Topology provides a conceptually simple way to derive lower bounds on the number of real solutions to systems of polynomials.

Suppose that the real solutions are the fiber of a map

$$f^{-1}(x) \qquad \text{where} \qquad f\,:\,Y \longmapsto \mathbb{S}\,,$$

with Y and $\mathbb{S}$ oriented and $x \in \mathbb{S}$ is a regular value of f.

Then f has a well-defined *degree*

$$\deg(f) \;:=\; \sum_{y\in f^{-1}(x)} \operatorname{sign}\det df(y)\,.$$

(This sum is independent of x.)
Thus $|\deg(f)|$ is a lower bound on the number of solutions.

Sparse polynomials, geometrically

A polynomial with support $\mathcal{A} \subset \mathbb{Z}^n$ is

$$f \;=\; \sum_{\alpha\in\mathcal{A}} c_\alpha x^\alpha\,,$$

where $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.
This is the pullback of a linear form along the map

$$\varphi\ :\ (\mathbb{C}^*)^n \ni x \longmapsto [x^\alpha \mid \alpha \in \mathcal{A}] \in \mathbb{P}^{\mathcal{A}}\,.$$

If $X_\mathcal{A}$ is the closure of the image, then a system of polynomials with support $\mathcal{A}$ corresponds to a linear section of $X_\mathcal{A}$,

$$f_1 \;=\; \cdots \;=\; f_n \;=\; 0 \quad\longleftrightarrow\quad X_\mathcal{A} \cap L\,,$$

and real solutions are real points in the section.

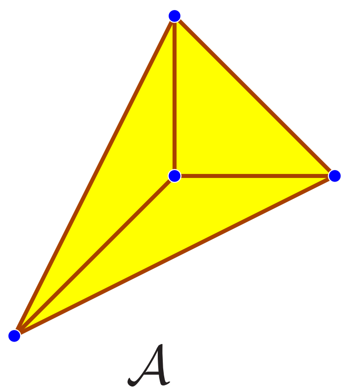
An example

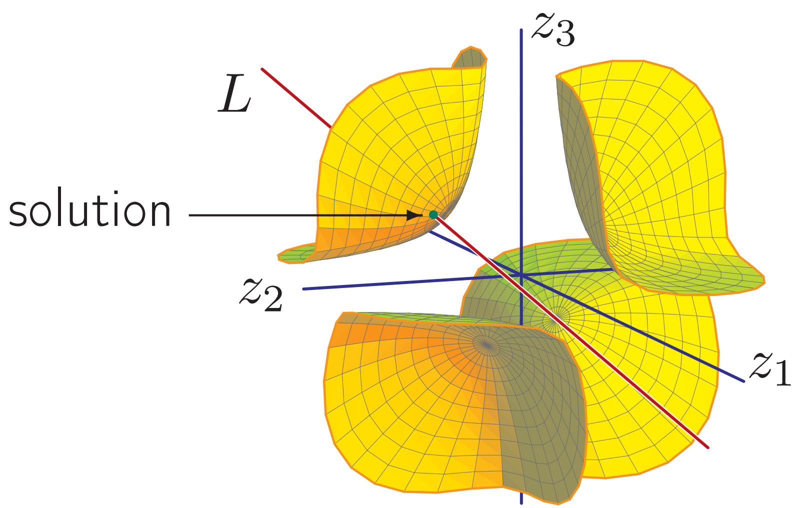
The system of polynomials

$$x^2y + 2xy^2 + xy - 1 \;=\; x^2y - xy^2 - xy + 2 \;=\; 0\,,$$

corresponds to a linear section of the toric variety

$$X_{\mathcal{A}} \;:=\; [xy : x^2y : xy^2 : 1] \;=\; \mathcal{V}(z_1z_2z_3 - z_0^3)$$



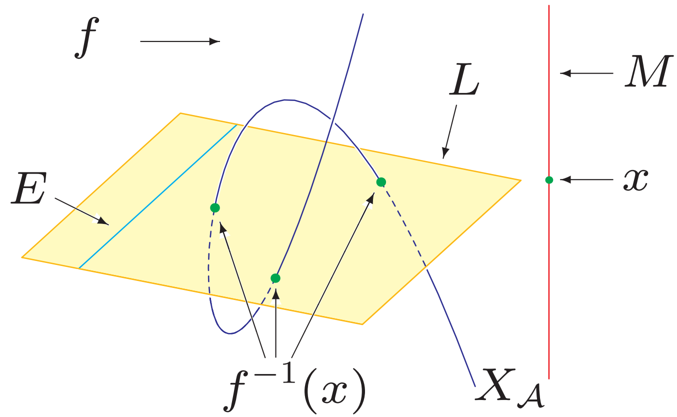


Polynomial systems as fibers

We realize $X_{\mathcal{A}} \cap L$ as the fiber of a map.

Let $E \subset L$ be a codimension one linear subspace and $M \simeq \mathbb{P}^n$ a complementary linear space.

The projection f from E sends $X_{\mathcal{A}}$ to M with $X_{\mathcal{A}} \cap L$ the fiber above $x = L \cap M$.



Restricting to $Y_{\mathcal{A}} := X_{\mathcal{A}} \cap \mathbb{RP}^{\mathcal{A}}$, the real solutions are fibers of

$$f \ : \ Y_{\mathcal{A}} \ \rightarrow \ M \cap \mathbb{RP}^{\mathcal{A}} \simeq \mathbb{RP}^n \,.$$

If $Y_{\mathcal{A}}$ and $\mathbb{RP}^n$ were orientible, $|\deg(f)|$ is a lower bound.

Orientability of real toric varieties

$Y_{\mathcal{A}}$ and $\mathbb{RP}^n$ are typically *not* orientible. This is improved by pulling back to the spheres $\mathbb{S}^{\mathcal{A}}$ and $\mathbb{S}^n$, which are oriented,

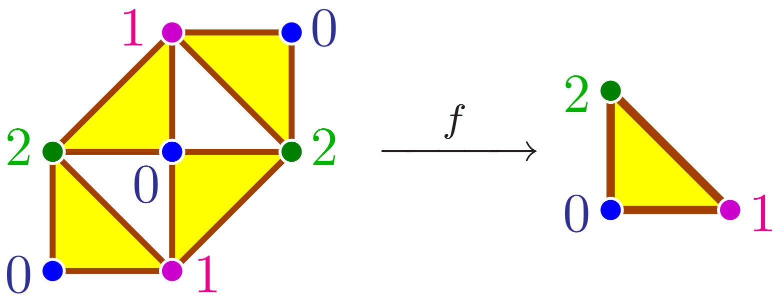
$$\begin{array}{ccccc} f^+ : Y^+_{\mathcal{A}} & \subset & \mathbb{S}^{\mathcal{A}} & \overset{f^+}{- - - \longrightarrow} & \mathbb{S}^n \\ \downarrow & & \downarrow & & \downarrow \\ f \quad : Y_{\mathcal{A}} & \subset & \mathbb{RP}^{\mathcal{A}} & \overset{f}{- - - \longrightarrow} & \mathbb{RP}^n \end{array}$$

The orientability of the spherical toric variety $Y^+_{\mathcal{A}}$ is characterized using the Newton polytope of $\mathcal{A}$. (Details omitted)

When $Y^+_{\mathcal{A}}$ is orientable, $|\deg(f^+)|$ is a lower bound on the number of real solutions. The challenge is to compute this degree.

Foldable triangulations

A triangulation of a polytope in $\mathbb{R}^n$ is *foldable* if it is 2-colorable, equivalently, if its vertices are $n{+}1$ colored.

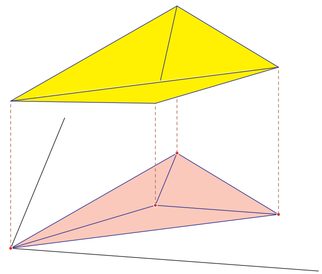


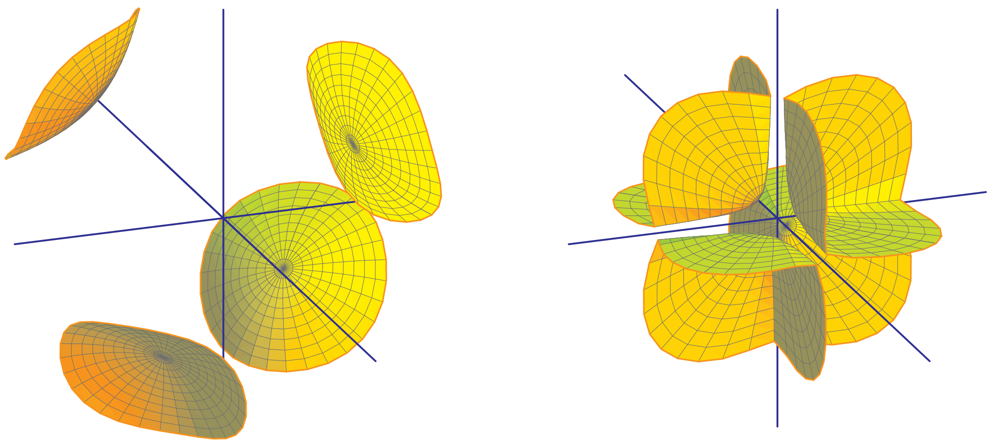
The vertex coloring defines a folding map to a simplex, whose degree is the *imbalance* of the folding. (Here $4-2=2$.)

A foldable triangulation of conv($\mathcal{A}$) gives a corresponding *Wronski projection* $f\colon \mathbb{S}^{\mathcal{A}} \to \mathbb{S}^n$. Restricting this to $Y_{\mathcal{A}}$, this leads to *Wronski polynomial systems* on $Y_{\mathcal{A}}$, which are the fibers of this map.

Toric degenerations

A regular unimodular triangulation induces a degeneration $t.Y_{\mathcal{A}}$ of the toric variety $Y_{\mathcal{A}}$ to a union of coordinate planes encoded in the triangulation.





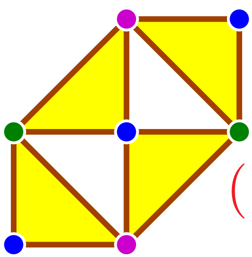
$t = 1$ $\qquad$ $t = 1/7$

Degree from toric degeneration

If the toric degeneration $Y_{\mathcal{A}}$ does not meet the center of the projection f, for $0 < t < 1$, then we deduce

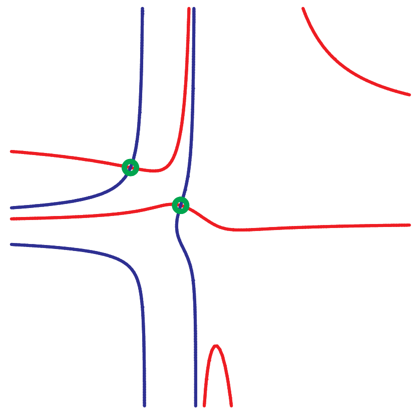
Theorem. *If $Y^+_{\mathcal{A}}$ is orientable, then the number of real solutions to a system of Wronski polynomials coming from a foldable triangulation on $\mathcal{A}$ is at least the imbalance of the folding.*

Wronski systems from the hexagon have at least 2 solutions:



$$3(1 + xy + x^2y^2) + 5(x + xy^2) \\ +(y + x^2y) = 0$$

$$(1 + xy + x^2y^2) - 2(x + xy^2) \\ -3(y + x^2y) = 0$$

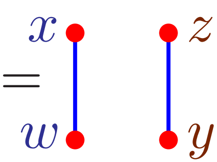


Polynomial systems from posets

What is any of this good for?

A partially ordered set P (poset) has natural Wronski polynomial systems with variables P. The toric varieties are oriented if and only if P is ranked modulo 2, and the lower bound is the *sign-imbalance* $\sigma(P)$ of P, studied by Stanley and Sjöstrand.

Example.

$P :=$ 

Wronski poly:

$$c_4\, wxyz + c_3(xyz + wxz) + c_2(wx + xz + yz) + c_1(x + z) + c_0$$

Monomials: order ideals

$\{\emptyset, x, z, wx, xz, yz, xyz, wxz, wxyz\}$

Linear extensions and sign imbalance

$wxyz$	$wyxz$	$ywxz$	$wyzx$	$ywzx$	$yzwx$	$\sigma(P)$
$+$	$-$	$+$	$+$	$-$	$+$	2

Inverse Wronski problem

The *Wronskian* of a (linear space of) univariate polynomials $f_1(t), \ldots, f_m(t)$ of degree $m+p-1$ is the determinant

$$Wr(f_1(t), \ldots, f_m(t)) \;:=\; \det\left(\left(\frac{d}{dt}\right)^i f_j(t)\right) ,$$

which has degree mp (and is considered up to a scalar).

Inverse Wronski problem: Given a (real) polynomial $F(t)$ of degree mp, which linear spaces have Wronskian $F(t)$?

Schubert (1884) computed the number of complex solutions. Mukhin, Tarasov, and Varchenko showed that if every root of $F(t)$ is real, then all spaces are real. (Shapiro Conjecture.)

Lower bounds in Schubert calculus

Assume $p \leq m$. If $m+p$ is odd, set $\sigma_{m,p}$ to be

$$\frac{1!2!\cdots(m-1)!(p-1)!(p-2)!\cdots(p-m+1)!(\frac{mp}{2})!}{(p-m+2)!(p-m+4)!\cdots(p+m-2)!\left(\frac{p-m+1}{2}\right)!\left(\frac{p-m+3}{2}\right)!\cdots\left(\frac{p+m-1}{2}\right)!}\,.$$

Set $\sigma_{m,p} = 0$ if $m+p$ is even. If $p > m$, then set $\sigma_{m,p} := \sigma_{p,m}$.

Eremenko-Gabrielov. *There are at least $\sigma_{m,p}$ real m-dimensional spaces of polynomials of degree $m+p-1$ with Wronskian a given general polynomial $F(t)$ of degree mp.*

They used Schubert induction.

This can be deduced from polynomial systems for the poset $C_m \times C_p$ (product of two chains) using SAGBI degenerations.

Data for $m = p = 3$

Observed numbers of real spaces versus $c :=$ number of complex conjugate pairs of roots of $F(t)$. Note that $\sigma_{3,3} = 0$.

c	0	2	4	6	8	10	12	14	16	18	20
1		1099		7975		42235		9081		6102	
2		24495		30089		25992		5054		3632	
3		39371		35022		15924		3150		1990	
4				76117		14481		3754		1375	

c	22	24	26	28	30	32	34	36	38	40	42
1	8827		1597		4207		1343		172		17362
2	4114		955		1586		832		63		3188
3	2183		494		622		367		35		842
4	2925		271		364		204		32		477

Tack!