Numerical Real Algebraic Geometry

Kinematics and Numerical Algebraic Geometry

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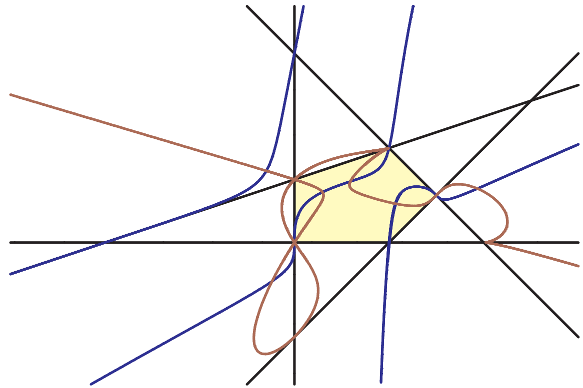


What is Numerical Algebraic Geometry?

It includes, at least, Numerical *Non*-Linear Algebra, and all this implies.

More specifically, numerical algebraic geometry is the numerical manipulation of polynomials and of sets/objects defined by polynomials.

Its importance is that it represents the future of computation in algebraic geometry, and is foundational for many current and future applications.

It represents the future, for computers are becoming wider, and not faster. The core numerical routine underlying numerical algebraic geometry, numerical path continuation, is trivially parallelizable, unlike most algorithms (e.g. Gröbner bases) that underlie the current dominant paradigm of symbolic computation.

Sources of Numerical Algebraic Geometry

Numerical Algebraic Geometry arose from Engineering Applications, particularly Kinematics, which we have already seen this morning in Charles's address, and will see more of tomorrow.

It has the potential to become the fundamental tool in applications that involve polynomials.

To acheive its potential, it needs a greater body of algorithms and continued software development, as well as successful applications to problems from other areas.

Real Challenges

Applications often demand real solutions, so it is natural to ask how do we compute the real solutions to a system of equations.

Dominant numerical algorithm for solving, homotopy continuation, necessarily computes all solutions, both real and complex.

Two classes of numerical algorithms for real solutions:

— Exclusion methods.
  Well-developed algorithms based on repeated subdivision.

— Semidefinite programming.
  Proposed by Lasserre, Laurent, and Rostalski.

A third method

Khovanskii-Rolle continuation is a third numerical method to compute real solutions.

— Based on proof of fewnomial bounds for real solutions.

— Uses 2 symbolic steps:

1) Gale duality reduces a (potentially high-degree) polynomial system to a system of rational functions on a different space.

2) Reducing this to solving some systems of low-degree polynomials & some path-continuation.

— Complexity is essentially the fewnomial bound.

Gale duality, via example

Suppose we have the system of polynomials,

$$\begin{aligned} v^2w^3 &= 1 - u^2v - uv^2w\,, \\ v^2w &= \tfrac{1}{2} - u^2v + uv^2w\,, \\ uvw^3 &= \tfrac{10}{11}(1 + u^2v - 3uv^2w)\,. \end{aligned} \qquad (1)$$

Observe that

$$(u^2v)^2 \cdot (v^2w^3)^3 \;=\; (uv^2w)^2 \cdot (v^2w) \cdot (uvw^3)^2 \quad \text{and}$$

$$(uv^2w)^3 \cdot (v^2w^3) \;=\; (u^2v) \cdot (v^2w)^3 \cdot (uvw^3)\,.$$

Substituting (1) into this, writing x for u^2v and y for uv^2w, and solving for 0, gives the Gale system of master functions

$$f := x^2(1-x-y)^3 - y^2(\tfrac{1}{2}-x+y)(\tfrac{10}{11}(1+x-3y))^2 = 0\,,$$

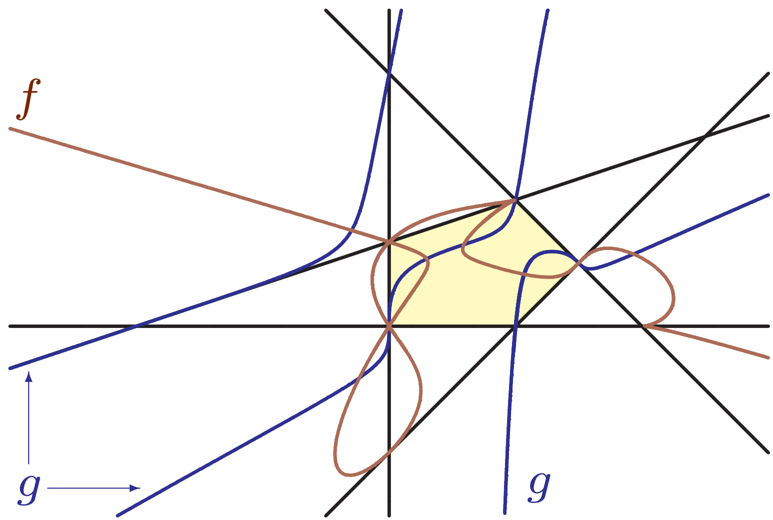
$$g := y^3(1-x-y) \;-\; x(\tfrac{1}{2}-x+y)^3\tfrac{10}{11}(1+x-3y) = 0\,.$$

Gale duality, continued

The original system is equivalent to the Gale system

$$f := x^2(1-x-y)^3 - y^2(\tfrac{1}{2}-x+y)(\tfrac{10}{11}(1+x-3y))^2 = 0\,,$$
$$g := y^3(1-x-y) \; - \; x(\tfrac{1}{2}-x+y)^3\tfrac{10}{11}(1+x-3y) = 0\,,$$

in the complement of the lines given by the linear factors.



Khovanskii-Rolle continuation

Given a system of master functions

$$\prod_{i=1}^{\ell+n} p_i(x)^{a_{i,j}} \;=\; 1 \qquad j = 1, \ldots, \ell\,, \qquad\qquad (*)$$

$(p_i(x)$ linear), we find solutions in the polyhedron

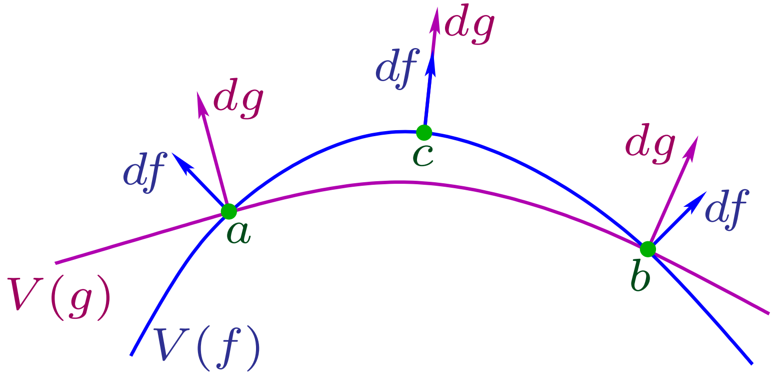
$$\Delta \;:=\; \{x \in \mathbb{R}^\ell \mid p_i(x) > 0\}\ .$$

The Khovanskii-Rolle Theorem (next slide) reduces solving $(*)$ to solving low degree polynomial systems, together with path continuation.

This is our new algorithm, which we now explain.

Khovanskii-Rolle Theorem

Theorem. *Between any two zeroes of* g *along the curve* $V(f)$: $f = 0$, *lies at least one zero of the Jacobian* $df \wedge dg$.



Starting where $V(f)$ meets the boundary of the polyhedron Δ and where the Jacobian vanishes on $V(f)$, tracing the curve $V(f)$ in both directions finds all solutions $f = g = 0$.

Degree reduction ($\ell = 2$)

A system of master functions

$$\prod_{i=1}^{2+n} p_i(x)^{a_{i,j}} \;=\; 1 \qquad j = 1, 2$$

in logarithmic form

$$\varphi_j \;:=\; \sum_{i=1}^{2+n} a_{i,j} \log p_i(x) \;=\; 0 \qquad j = 1, 2\,,$$

has Jacobians of low degree

$$J_2 \;:=\; \mathrm{Jac}(\varphi_1, \varphi_2) \qquad J_1 \;:=\; \mathrm{Jac}(\varphi_1, J_2)\;.$$

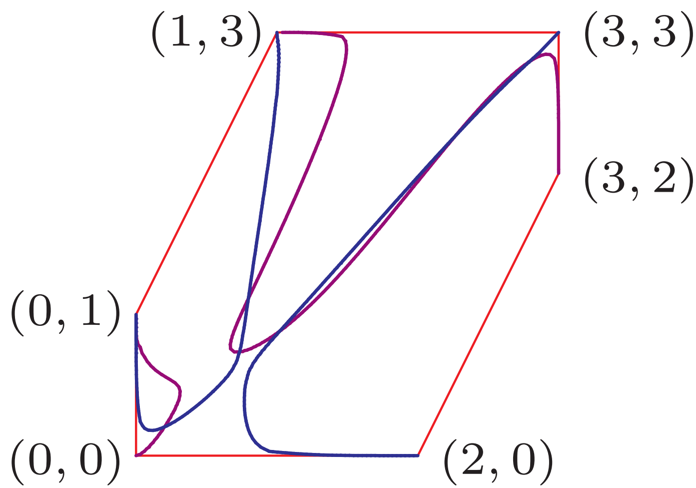
Here, $n = \deg(J_2)$ and $2n = \deg(J_1)$.

An example

Consider the system with $\ell = 2$ and $n = 4$:

$$f_1 := \frac{(3500)^{12} x^{27} (3-x)^8 (3-y)^4}{y^{15}(4-2x+y)^{60}(2x-y+1)^{60}} = 1\,,$$

$$f_2 := \frac{(3500)^{12} x^{8} y^{4} (3-y)^{45}}{(3-x)^{33}(4-2x+y)^{60}(2x-y+1)^{60}} = 1\,.$$



Low-Degree Jacobians

If $\varphi_i := \log(f_i)$, then $J_2 := \mathrm{Jac}(\varphi_1, \varphi_2) \cdot \prod p_i(x,y) \ =$

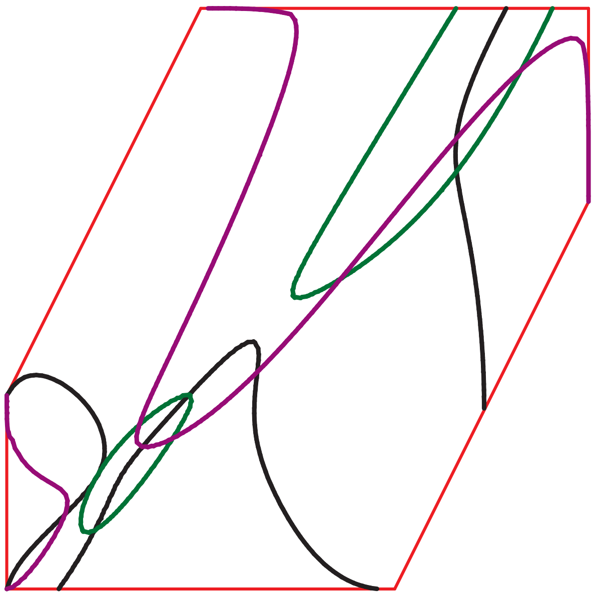
$$2736 - 15476x + 2564y + 32874x^2 - 21075xy + 6969y^2 - 10060x^3$$
$$-7576x^2y + 8041xy^2 - 869y^3 \ + \ 7680x^3y - 7680x^2y^2 + 1920xy^3 \,.$$

(polynomial of degree $n = 4$.) $J_1 := \mathrm{Jac}(\varphi_1, \Gamma_2) \cdot \prod p_i(x,y)^2 \ =$

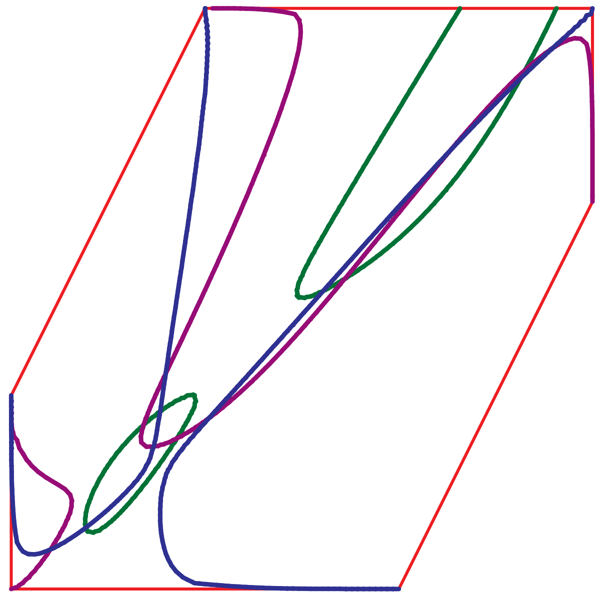
$$8357040x - 2492208y - 25754040x^2 + 4129596xy - 10847844y^2$$
$$-37659600x^3 + 164344612x^2y - 65490898xy^2 + 17210718y^3 \ + 75054960x^4$$
$$-249192492x^3y + 55060800x^2y^2 + 16767555xy^3 - 2952855y^4 \ - 36280440x^5$$
$$+143877620x^4y + 35420786x^3y^2 - 80032121x^2y^3 + 19035805xy^4 - 1128978y^5$$
$$+5432400x^6 - 33799848x^5y - 62600532x^4y^2 + 71422518x^3y^3 - 13347072x^2y^4$$
$$-1836633xy^5 + 211167y^6 + 2358480x^6y + 21170832x^5y^2 - 13447848x^4y^3$$
$$-8858976x^3y^4 + 7622421x^2y^5 - 1312365xy^6 - 1597440x^6y^2 - 1228800x^5y^3$$
$$+4239360x^4y^4 - 2519040x^3y^5 + 453120x^2y^6 \,.$$

(A polynomial of degree $8 = 2n$.)

Completing the example



Follow $V(J_2) \cap \partial\Delta$ and $J_1 = J_2 = 0$ along $V(J_2)$ to find $J_2 = \varphi_1 = 0$.



Follow $V(\varphi_1) \cap \partial\Delta$ and $\varphi_1 = J_2 = 0$ along $V(\varphi_1)$ to find $\varphi_1 = \varphi_2 = 0$.