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NUMERICAL REAL ALGEBRAIC GEOMETRY

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ABSTRACT. Sottile's lectures from the Oberwolfach Seminar "New trends in algorithms for real algebraic geometry", November 23–28, 2009.

1. INTRODUCTION

We seek meaningful information about the non-degenerate solutions to a system

$$f_1(x_1, \ldots, x_n) \;=\; f_2(x_1, \ldots, x_n) \;=\; \cdots \;=\; f_N(x_1, \ldots, x_n) \;=\; 0 \tag{1.1}$$

of N polynomial equations in n variables. We are particularly interested in the following questions:

(a) What can be said about the number of solutions?
(b) How can we effecively compute them?

While systems with more equations that variables ($N > n$) arise naturally in algebraic geometry, we restrict ourselves to systems with the same number of variables as equatios, $N = n$, also called *square systems*.

Our ultimate goal will be to address these questions for the real solutions to a system of equations, but we will also address them for the complex solutions. It turns out that for both real and complex solutions, these two questions are linked. Table 1 gives the organization of this chapter in a 2×2 grid.

	Bounds	Numerical Algorithms
$\mathbb{C}$	I) Bernstein's Theorem and Mixed Volumes	II) Numerical Homotopy Continuation and Polyhedral Homotopies
$\mathbb{R}$	III) Fewnomial Bounds and Gale Duality	IV) Khovanskii-Rolle Continuation

TABLE 1. Organization

We will see that theoretical bounds lead to numerical algorithms with optimal complexity, in a suitible sense. We will also see that many important questions remain over $\mathbb{R}$.

Here, a solution to a system (1.1) is *non-degenerate* if and only if the differentials of the polynomials f_i are linearly independent at that solution. Non-degenerate solutions are

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particularly meaningful for numerical algorithms, as they are well-behaved under pertubations of the coefficients of the f_i. The polynomial system is *generic* if all solutions are non-degenerate.

2. Bernstein's Theorem and Mixed Volume

The most fundamental result about the number of solutions to a system of polynomial equations (1.1) is the classical theorem of Bézout [4].

Theorem 2.1 (Bézout, 1776)**.** *If, for each $i = 1, \dots, n$, the polynomial f_i in* (1.1) *has degree d_i, then there are at most $d_1 d_2 \cdots d_n$ non-degenerate complex solutions. If the polynomials are general given their degrees, then there are exactly $d_1 d_2 \cdots d_n$ solutions.*

We are interested in the case when the polynomials are *not* general given their degrees, which occurs quite often in nature. For example, the discriminant of a polynomial $f(x)$ is the polynomial in the coefficients of f that vanishes when $f(x)$ has a multiple root. The discriminant of $x^3 + ax^2 + bx + c$ is

$$4a^3c \;-\; a^2b^2 \;-\; 18abc \;+\; 4b^3 \;+\; 27c^2\,,$$

which is not a general polynomial of degree 4 in the variables a, b, c. In particular, it only has 5 monomials, whereas a general quartic in a, b, c has 35 monomials. Even worse, the exponents $(\alpha_a, \alpha_b, \alpha_c)$ of the monomials lie on the plane

$$\alpha_a + 2\alpha_b + 3\alpha_c \;=\; 6\,.$$

We consider instead polynomials with given combinatorial structure determined by their monomials. Vectors $\alpha \in \mathbb{Z}^n$ may be identified with monomials

$$\mathbb{Z}^n \ni \alpha \iff x^\alpha \;:=\; x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \;\in\; \mathbb{C}[x_1^{\pm}, x_2^{\pm}, \dots, x_n^{\pm}]\,,$$

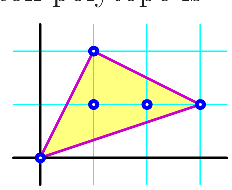
in the ring of Laurent polynomials. When $\alpha \in \mathbb{N}^n$ has non-negative components, this is a monomial in an ordinary polynomial ring. The *support* of a polynomial f is the set of exponent vectors of monomials in f. The *Newton polytope* $\mathrm{New}(f)$ of the polynomial f is the convex hull of its exponent vectors.

Example 2.2. The bivariate polynomial

$$1 \;+\; 2xy \;-\; 3xy^2 \;-\; 4x^2y \;+\; 5x^3y$$

has support $\{(0,0), (1,1), (1,2), (2,1), (3,1)\}$ and its Newton polytope is

$$\mathrm{conv}\{(0,0), (1,1), (1,2), (2,1), (3,1)\} \;=$$



Bernstein's Theorem concerns systems of polynomials (1.1) when the polynomials have fixed Newton polytopes. It is in terms of a quantity called mixed volume, which we now define.

The *Minkowski sum* of polytopes $P_1, \dots, P_m$,

$$P_1 + P_2 + \cdots + P_m \;:=\; \{x_1 + x_2 + \cdots + x_m \mid x_i \in P_i,\; i = 1, \cdots, m\}$$

is their pointwise sum. In Exercise 4 you are asked to prove the following easy fact. Given polynomials f and g,

$$\mathrm{New}(f \cdot g) \;=\; \mathrm{New}(f) + \mathrm{New}(g)\,.$$

Theorem-Definition 2.3. Suppose that $P_1, \ldots, P_n$ are polytopes in $\mathbb{R}^n$. Then

$$\mathrm{vol}(t_1P_1 + t_2P_2 + \cdots + t_nP_n)$$

is a homogeneous polynomial of degree n in $t_1, \ldots, t_n$. The coefficient of t_i^n in this polynomial is the volume of P_i, and the coefficient of $t_1t_2\cdots t_n$ in this polynomial is the *mixed volume*, $\mathrm{MV}(P_1, \ldots, P_n)$ of the polytopes $P_1, \ldots, P_n$.

We will give a proof of this result later in this lecture, and give an interpretation for all coefficients in this polynomial. We are now ready to state Bernstein's Theorem [2].

Theorem 2.4 (Bernstein [2])**.** *Let $f_1, \ldots, f_n \in \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$ be Laurent polynomials. Then the number of isolated solutions to the system*

$$f_1(x_1, \ldots, x_n) \;=\; f_2(x_1, \ldots, x_n) \;=\; \cdots \;=\; f_n(x_1, \ldots, x_n) \;=\; 0\,, \tag{2.5}$$

in the torus $(\mathbb{C}^\times)^n$ is at most the mixed volume of the Newton polytopes of the polynomials,

$$\mathrm{MV}(\mathrm{New}(f_1), \ldots, \mathrm{New}(f_n))\,,$$

and is equal to this mixed volume when the polynomials are general given their Newton polytopes.

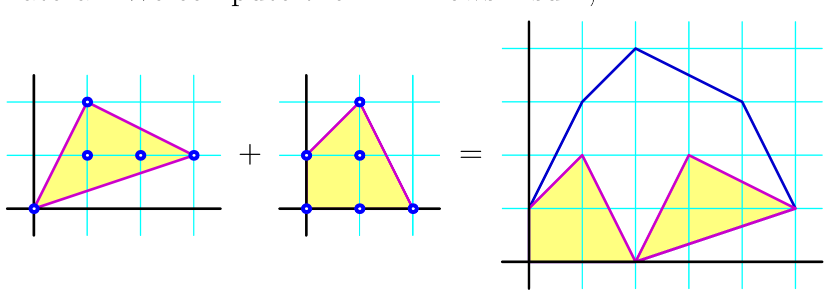
Example 2.6. For example, consider the system

$$\begin{aligned} f\;: &\qquad 1 + 2xy - 3xy^2 - 4x^2y + 5x^3y \;&=&\; 0 \\ g\;: &\quad 1 - 2x + 3x^2 - 5y + 7xy - 11xy^2 \;&=&\; 0\,. \end{aligned}$$

It has 8 solutions, all non-zero, but the product of its degrees is $4 \cdot 3 = 12$. We cpompute the corresponding mixed volume using Exercise 6, in which you are asked to show that if $P, Q \subset \mathbb{R}^2$ are polygons, then

$$\mathrm{MV}(P, Q) \;=\; \mathrm{area}(P + Q) \;-\; \mathrm{area}(P) \;-\; \mathrm{area}(Q)\,.$$

The Newton polygon P of f is the triangle of Example 2.2 and theeasy Newton polygon Q of g is a quadrilateral. We compute their Minkowski sum,



We see that P and Q both have area $5/2$ while $P + Q$ has area 13, so their mixed volume is $13 - 5/2 - 5/2 = 8$.

We follow an algorithmic proof of Bernstein's Theorem given in [11], which involves three steps.

(1) A local computation of the solutions to a system of binomials,
(2) a degeneration to the local computation,
(3) showing that this root count equals the mixed volume.

We treat these steps in the next three subsections.

2.1. **Binomial systems.** Let $\alpha^{(1)}, \beta^{(1)}, \ldots, \alpha^{(n)}, \beta^{(n)} \in \mathbb{Z}^n$ be exponent vectors of monomials and $a_1, b_1, \ldots, a_n, b_n \in \mathbb{C}^\times$ be non-zero complex numbers. We consider the system of binomial equations on $(\mathbb{C}^\times)^n$,

$$a_i x^{\alpha^{(i)}} + b_i x^{\beta^{(i)}} = 0 \quad \text{for } i = 1, \ldots, n\,. \tag{2.7}$$

Setting $c_i := -a_i/b_i$ and $\gamma^{(i)} = \beta^{(i)} - \alpha^{(i)}$ for each $i = 1, \ldots, n$, this is equaivalent to

$$x^{\gamma^{(i)}} = c_i \quad \text{for } i = 1, \ldots, n\,. \tag{2.8}$$

For example, the binomial system

$$7xy^2 - 5x^4y = 2x^2y^{-1} + 3x^{-1}y^2 = 0\,, \tag{2.9}$$

is equivalent to

$$x^3y^{-1} = 7/5 \qquad \text{and} \qquad x^{-3}y^3 = -2/3\,.$$

To solve (2.8), we introduce the homomorphism

$$\begin{array}{cccl} \varphi & : & (\mathbb{C}^\times)^n & \longrightarrow & (\mathbb{C}^\times)^n \\ & & x & \longmapsto & (x^{\gamma^{(1)}}, x^{\gamma^{(2)}}, \ldots, x^{\gamma^{(n)}})\,. \end{array}$$

Then our system (2.8) has the form $\varphi^{-1}(c)$, where $c = (c_1, \ldots, c_n) \in (\mathbb{C}^\times)^n$. The system has solutions when c lies in the image of φ and the solutions are in bijection with the kernel of φ.

The map φ of tori is obtained from the map of free abelian groups

$$\begin{array}{ccccl} \Gamma & : & \mathbb{Z}^n & \longrightarrow & \mathbb{Z}^n \\ & & \mathbf{e}_i & \longmapsto & \gamma^{(i)}\,, \quad \text{for } i = 1, \ldots, n\,, \end{array}$$

by applying the functor $\mathrm{Hom}(\bullet, \mathbb{C}^\times)$. (Here $\mathbf{e}_i$ is the ith standard basis vector.) It follows that the kernel of φ is

$$\mathrm{Hom}(\mathbb{Z}^n/\mathbb{Z}\{\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(n)}\}, \mathbb{C}^\times)\,.$$

We see that φ is surjective if and only if the exponents $\gamma^{(1)}, \ldots, \gamma^{(n)}$ span a full-rank sublattice of $\mathbb{Z}^n$, and when this occurs, the number of solutions is exactly the cardinality of the kernel. We summarize this computation.

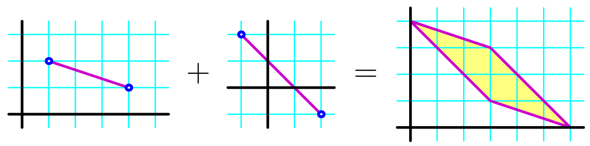
Lemma 2.10. *The number of solutions to the binomial system* (2.7) *is*

$$\left|\mathbb{Z}^n/\mathbb{Z}\{\beta^{(1)} - \alpha^{(1)}, \beta^{(2)} - \alpha^{(2)}, \ldots, \beta^{(n)} - \alpha^{(n)}\}\right|\,.$$

This number is exacty the volume of the parallelepiped spanned by the vectors $\gamma^{(1)}, \ldots, \gamma^{(n)}$, or more intrinsically, of the Minkowski sum of line segments

$$\overline{\alpha^{(1)}, \beta^{(1)}}\,,\ \overline{\alpha^{(2)}, \beta^{(2)}}\,,\ \ldots\,,\ \overline{\alpha^{(n)}, \beta^{(n)}}\,.$$

For example, the binomial system (2.9) has 6 solutions, which we may see from the parallelepiped obtained from the Minkowski sum of their exponent vectors,



2.2. **Toric degenerations.** We reduce counting the solutions to the system (2.5) in Bernstein's Theorem to the binomial systems of Subsection 2.1 using toric degenerations.

Suppose that $f_i(x)$ is one of the polynomials in (2.5),

$$f_i(x) \;=\; \sum_{\alpha\in \mathrm{New}(f_i)} c_{\alpha,i}x^\alpha\,.$$

For each monomial x^α in f_i, pick an integer $\nu_{\alpha,i}$ and multiply that monomial by $t^{\nu_{\alpha,i}}$, which gives the polynomial

$$f_i(x;t) \;=\; \sum_{\alpha\in \mathrm{New}(f_i)} c_{\alpha,i}t^{\nu_{\alpha,i}}x^\alpha\,.$$

We now consider the system of polynomial equations,

$$f_1(x;t) \;=\; f_2(x;t) \;=\; \cdots \;=\; f_n(x;t) \;=\; 0\,. \tag{2.11}$$

There are at least two ways to view this system.

(1) We may consider it as a system in $(\mathbb{C}^\times)^n\times\mathbb{C}$, where the parameter t lies in $\mathbb{C}$. Then it defines a curve in $(\mathbb{C}^\times)^n\times\mathbb{C}$, and the points with $t=1$ correspond to solutions of our original system (2.5). These are all fibres of the projection to the last coordinate. Their number is therefore the degree of the projection to the last coordinate, which we will compute by determining the number of branches of the curve for $|t|$ small. We may parametrize each branch by a Puiseaux series

$$X(t) \;=\; t^u.X_0 \;+\; \text{higher order terms in } t\,, \tag{2.12}$$

where $u\in\mathbb{Z}^n$, $t^u=(t_1^{u_1},\dots,t_n^{u_n})\in(C^\times)^n$, and $X_0\in(C^\times)^n$.

(2) Alternatively (in fact equivalently), we consider the polynomial $f(x;t)$ to lie in the Laurent ring $\mathbb{K}[x_1^\pm,\dots,x_n^\pm]$, where

$$\mathbb{K} \;=\; \bigcup_m \mathbb{C}((t^{\frac{1}{m}}))\,,$$

the field of Puiseaux series in t. This is an algebraically closed field of characteristic zero. Suppose that $X(t)\in(\mathbb{K}^\times)^n$ is a solution to the system of polynomials in the Puiseaux field. Then $X(t)$ is a (vector-valued) power series as in (2.12).

We first determine the exponents u of t. Let us expand the equation $0 = f_i(X(t);t)$ in powers of t,

$$\begin{aligned}0 \;=\; f_i(X(t);t) &= \sum_{\alpha\in\mathrm{New}(f_i)} c_{\alpha,i}t^{\nu_{\alpha,i}}X(t)^\alpha\\ &= \sum_{\alpha\in\mathrm{New}(f_i)} c_{\alpha,i}t^{\nu_{\alpha,i}}\bigl(t^u.X_0 \;+\; \text{higher order terms in } t\bigr)^\alpha\\ &= \sum_{\alpha\in\mathrm{New}(f_i)} c_{\alpha,i}t^{\nu_{\alpha,i}}t^{\alpha\cdot u}X_0^\alpha \qquad + \quad \text{higher order terms in } t.\end{aligned}$$

This implies that if we collect powers of t, then every coefficient vanishes, in particular, the coefficient of the minimal power of t vanishes. But that can only happen if there are two or more monomial terms of $f_i(x;t)$ for which this minimum is acheived.

Lemma 2.13. *The set u of leading exponents for solutions $X(t)$ (2.12) is a subset of the set*

$$\mathcal{T} \;:=\; \{u \mid \min\{\nu_{\alpha,i}+\alpha\cdot u \mid \alpha\in\mathrm{New}(f_i)\} \textit{ is acheived at least twice}\}\,.$$

For every weight vector $u\in\mathbb{Q}^n$ (or more generally $u\in\mathbb{R}^n$) let the initial form $\mathrm{ini}_u(f_i)$ be the sum of the terms $c_{\alpha,i}x^\alpha$ of f_i for which $\nu_{\alpha,i}+\alpha\cdot u$ is minimal among all terms of f. The *initial system* of (2.5) is the system

$$\mathrm{ini}_u(f_1) \;=\; \mathrm{ini}_u(f_2) \;=\; \cdots \;=\; \mathrm{ini}_u(f_n) \;=\; 0\,.$$

We have seen that if t^u is the lowest exponent in a solution $x(y)$ of the lifted system, then its coefficient $X_0\in(\mathbb{C}\times)^n$ is a solutio to the u-initial system, which implies that no initial form $\mathrm{ini}_u(f_i)$ is a monomial.

The set $\mathcal{T}$ is an example of a tropical prevariety cite. It, and in particular its cardinality, depends upon the choice of lifts $\nu_{\alpha,i}$. However, for generic lifts, this is a tropical complete intersection, which means that it consists of finitely many weights u, and for each weight and polynomial, the minimum is attained exactly twice. That is the initial system is a system of binomials.

Fix the lifts $\nu_{\alpha,i}$ so that the set $\mathcal{T}$ is a tropical complete intersection. Let u be a point of $\mathcal{T}$, and we now ask about the number of power series solutions (2.12) with initial exponents t^u. Write the initial forms as

$$\mathrm{ini}_u(f_i) \;:=\; c_{\alpha^{(i)},i}x^{\alpha^{(i)}} \;+\; c_{\beta^{(i)},i}x^{\beta^{(i)}}\,.$$

Then the initial coefficients X_0 of the power series (2.12) with initial exponent u satisfy the initial binomial system

$$0 \;=\; c_{\alpha^{(i)},i}X_0^{\alpha^{(i)}} \;+\; c_{\beta^{(i)},i}X_0^{\beta^{(i)}}\,, \qquad \text{for} \quad i=1,\ldots,n\,.$$

By Lemma 2.10 the solutions to this system are in bijection with

$$\mathbb{Z}^n/\mathbb{Z}\{\beta^{(i)}-\alpha^{(i)} \mid i=1,\ldots,n\}\,.$$

By the theory of Puiseaux series, each of these solutions X_0 to the initial binomial system may be developed into a unique Puiseaux series solution to the original (lifted) system (2.11).

Corollary 2.14. *The number of solutions to the original system* (2.5)*, which is equal to the number of solutions to the lifted system* (2.11)*, is equal to*

$$\sum_{u\in\mathcal{T}} \left|\mathbb{Z}^n/\mathbb{Z}\{\beta^{(i)}-\alpha^{(i)} \mid i=1,\ldots,n\}\right| .$$

2.3. **Mixed subdivisions.** We show that the root count in Corollary 2.14 may be interpreted as the mixed volume, and give a concrete realization of mixed volume and therefore a proof of Definition-Theorem 2.3.

For each $i=1,\ldots,n$ set $P_i=\mathrm{New}(f_i)$, the Newton polytope of the polynomial f_i, which lies in $\mathbb{R}^n$. We also define the lift of P_i,

$$P_i^+ \ := \ \mathrm{conv}\{(\alpha,\nu_{\alpha,i}) \mid \alpha\in P_i\}\,,$$

which is a polytope in $\mathbb{R}^{n+1}$. The *lower faces* of P_i^+ are the faces where a linear function of the form $(x,z)\mapsto x\cdot u+z$ is minimized on P_i^+. Write F_i^u for the face of P_i^+ on which this linear function is minimized. We may project these lower faces into $\mathbb{R}^n$. Their images forms a regular polyhedral subdivision of the Newton polytope P_i. Write Δ_i for this subdivision.

We may similarly consider the Minkowski sum of these lifts. Set $P^+ := P_1^+ + \cdots + P_n^+$. It also has lower faces, which also project to the Minkowski sum $P := P_1+\cdots+P_n$, forming a regular polyhedral subdivision of P. This subdivision is called a *mixed subdivision*, for its faces are Minkowski sums of faces of the subdivisions Δ_i. Indeed, projection commutes with Minkowski sum and a lower face F^u of P^+ on which a linear function $(u,1)$ acheives its minimum is necessarily the Minkowski sum of the faces F_i^u of P_i^+ on which $(u,1)$ acheives its minimum on P_i^+.

Complete this, following Huber-Sturmfels [11]. Include pictures and a nice running example. Maybe 2-3 pages?

Exercises on Bernstein's Theorem.

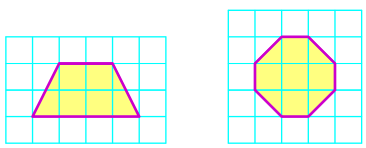
(1) What does Bernstein's Theorem say for univariate polynomials?
(2) Solve the following binomial systems
   (a) $x^5y^3-2xy^{-1} \ = \ 5xy-x^3y^4 \ = \ 0.$
   (b) $x^{11}y-3x^3y^{-5} \ = \ xy^{13}-7x^3y^4 \ = \ 0.$
   (c) $xyz^2-x^2y^{-1} \ = \ x^3y^3-2x^{-1}z^3 \ = \ y^3-xz^3 \ = \ 0.$
   (d) $x^3w-5y^2z^{-3} \ = \ y^3w^5-x^7z \ = \ xyz-7y^5w^9 \ = \ y^3-x^4yz^4w^2 \ = \ 0.$
(3) Determine the Newton polytope of each polynomial, and the mixed volume of each polynomial systyem. Check the conclusion of Bernstein's Theorem using a computer algebra system such as Singular, Macualay2, or CoCoA.
   (a) $1+2x+3y+4xy \ = \ 1-2xy+3x^2y-5xy^2 \ = \ 0.$
   (b) $$\begin{array}{rcl} 1+2x+3y-5xy+7x^2y^2 &=& 0\\ 1-2xy+4x^2y+8xy^2-16x^3y+32xy^3-64x^2y^2 &=& 0 \end{array}.$$
   (c) $$\begin{array}{rcl} 2+5xy-x^2y-6xy^2+4xy^3 &=& 0\\ 2x-y-2y^2-xy^2+2x^2y+x^2-5xy &=& 0 \end{array}.$$

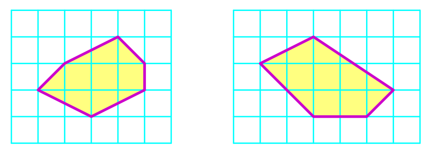
$$\text{(d)} \qquad \begin{array}{rcl} 1+x+y+z+xy+xz+yz+xyz &=& 0 \\ xy+2xyz+3xyz^2+5xz+7xy^2z+11yz+13x^2yz &=& 0 \\ 4-x^2y+2x^2z-xz^2+2yz^2-y^2z+2y^2x-8xyz &=& 0 \end{array} .$$

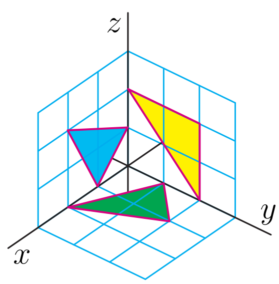
(4) Show that Newton polytopes are additive under polynomial multiplication,

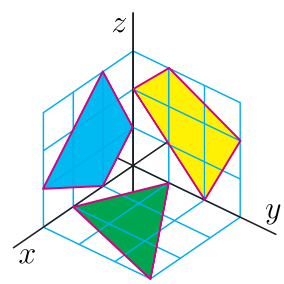
$$\mathrm{New}(f\cdot g) \;=\; \mathrm{New}(f)+\mathrm{New}(g)\,.$$

(5) Determine the mixed volumes of each of the lists of polytopes given below and verify Bernstein's Theorem by solving a randomly generated polynomial system with the given Newton polytopes.

(a) 

(b) 

(c) 

(d) 

(6) Prove the following formula for the mixed area of two polygons P and Q.

$$\mathrm{MV}(P,Q) \;=\; \mathrm{vol}(P+Q) \;-\; \mathrm{vol}(P) \;-\; \mathrm{vol}(Q)\,.$$

(7) Prove the following Lemma. If $F(t_1,\ldots,t_n)$ is a homogeneous polynomial of degree n, then the coefficient of $t_1t_2\cdots t_n$ is the alternating sum,

$$\sum_{x\in\{0,1\}^n} (-1)^{n-|x|}F(t)\,.$$

Use this to give an expression for the mixed volume $\mathrm{MV}(P_1,\ldots,P_n)$.

(8) Find integer lifting functions and compute the corresponding mixed subdivision of the lists of polytopes in Exercise 5.

(9) Deduce Bézout's Theorem from Bernstein's Theorem.

(10) Kushnirenko's Theorem is the special case of Bernstein's Theorem when all the Newton polytopes are equal. Formulate a statement of Kushnirenko's Theorem and deduce it from Bernstein's Theorem.

(11) For each $i=1,\ldots,k$ let $x^{(i)} := (x^{(i)}_1,\ldots,x^{(i)}_{n_i})$ be a list of n_i variables. A polynomial $f(x^{(1)},\ldots,x^{(k)})$ has *multidegree* $\mathbf{d}=(d_1,\ldots,d_k)$ if it has degree d_i in the variables $x^{(i)}$, for each $i=1,\ldots,k$. Use Kushnirenko's Theorem to deduce the bound for the number of common zeroes to a system of $n=n_1+\cdots+n_k$ equations that has multidegree $\mathbf{d}$.

3. Numerical Homotopy Continuation and Polyhedral Homotopies

Homotopy continuation is a basic numerical algorithm that is fundamental to the emerging field of numerical algebraic geometry. This aims to develop numerical algorithms to manipulate and study algebraic varieties. Besides algorithms to compute all solutions to a system of equations (the goal of this section), Numerical Algebraic Geometry has algorithms for the irreducible decomposition of a variety, for computing exceptional fibers of a map, for primary decomposition, and for finding the equations of an irreducible component of a variety. The standard references for numerical algebraic geometry are the book [19] and the survey [18].

The importance of numerical algebraic geometry is that numerical computation is the future of computation in algebraic geometry. The reason for this is simple. Current and future increases in computer power will come from the greater use of parallelism, instead of increased clock speed, which was the case until in the early 21st century. Symbolic algorithms based upon Gröbner bases, which is the dominant paradigm for computation in algebraic geometry, do not appear to be parallelizable, and so they will not be able to take advantage of increases in computer power due to parallelism. In contrast, algorithms based on numerical homotopy are trivially parallelizable, and are therefore poised to reap the benefits of our parallel future. There is also an intellectual appeal to numerical computation. In many cases, we are able to compute precise information (number of real solutions, degree of embedded component, monodromy group, genus of curve, or Chern numbers) using the inherently imprecise methods of numerical analysis and computation with floating point numbers.

The basic idea of numerical homotopy continuation is simple: When faced with the task of solving a system of polynomial equations,

$$f_1(x_1, x_2, \ldots, x_n) \ = \ f_2(x_1, x_2, \ldots, x_n) \ = \ \cdots \ = \ f_n(x_1, x_2, \ldots, x_n) \ = \ 0\,, \tag{3.1}$$

DON'T, at least not directly. Instead, solve a different system, perhaps one whose solutions are easy to find (or whose solutions you already now), and then use these solutions to find solutions to your original system (3.1).

There are two steps towards making this idea work.

(1) Design. Embedd your system (3.1) into a family of systemsparametrized by $t \in \mathbb{C}$ called a *homotopy*,

$$H(x; t) \ = \ 0\,,$$

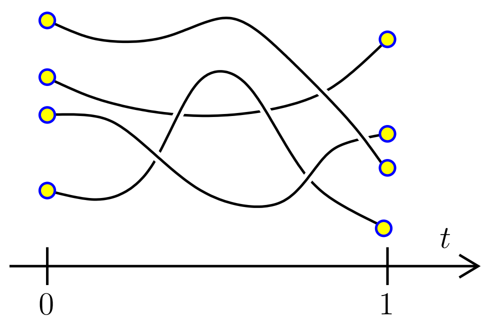
which satisfies,

- $H(x; 1)$ is the original system, called the *target system*, and
- $H(x, 0)$ is the system you know how to solve, called the *start system*.

(2) Tracking. Use numerical continuation to follow solutions from $t = 0$ to $t = 1$.

We discuss these steps separately in the following two subsections.

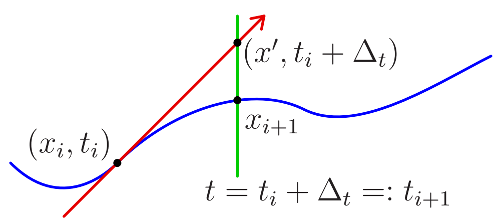
3.1. **Tracking.** Treat the complex numbers $\mathbb{C}$ as a two-dimensional real vector space $\mathbb{R}^2$ via $(a, b) \mapsto a + b\sqrt{-1}$. Restricting to $t \in \mathbb{R}$, we may regard $H(x, t) = 0$ as a system of $2n$ real equations on $\mathbb{R}^{2n} \times \mathbb{R}$. Assuming that the polynomials of the homotopy are sufficiently general, this system defines a collection of smooth directed paths in $\mathbb{R}^{2n} \times [0,1]$, which we display schemetically.



Then, starting with the solutions to the start system at $t = 0$, we track each path from $t = 0$ to $t = 1$ to find solutions to the target system.

Path-tracking consists of iteratively constructing a sequence of points (x_0, t_0), (x_1, t_1), $\ldots$, (x_s, t_s) that lie on or near the desired path, and where $0 = t_0 < t_1 < \cdots < t_s = 1$. The transition from point (x_i, t_i) to (x_{i+1}, t_{i+1}) uses a predictor-corrector algorithm, which proceeds in two steps: Given the initial point (x_i, t_i) and some *steplength* Δ_t, the algorithm first *predicts* a point $(x', t_i + \Delta_t)$ near the path, and then it freezes the value of t at $t_{i+1} = t_i + \Delta_t$, and employs Newton's method to *correct* this to a point (x_{i+1}, t_{i+1}) sufficiently close to the path. Such predictor-corrector algorithms are standard in numerical analysis.

There are several schemes for the predictor step; we describe a first-order predictor in which $(x', t_i + \Delta_t)$ lies on the tangent line to the arc through (x_i, t_i). For the correction step, we freeze $t = t_{i+1} = t_i + \Delta_t$ for the Newton iterations.



More specifically, let H_x be the $2n \times 2n$ Jacobian matrix of partial derivatives of the functions H with respect to the variables x and H_t be the vector of partial derivatives of H with respect to t. Then the tangent line to the arc $H(x; t) = 0$ at a point (x_i, t_i) has direction the kernel of the matrix $[H_x(x_i, t_i) \,:\, H_t(x_i, t_i)]$. The point $(x_i + \Delta_x, t_i + \Delta_t)$ along the tangent line is found by solving the equation

$$[H_x(x_i, t_i) \,:\, H_t(x_i, t_i)] \cdot \begin{bmatrix} \Delta_x \\ \Delta_t \end{bmatrix} \;=\; 0\,.$$

Multiplying on the left by $H_x(x_i, t_i)^{-1}$ and solving for Δ_x, we get

$$\Delta_x \;=\; -\Delta_t H_x(x_i, t_i)^{-1} H_t(x_i, t_i)\,. \tag{3.2}$$

For the corrector step, we add the equation $t = t_{i+1}$ to the other equations $H(x, t) = 0$ to get a square system and then use Newton's method to refine the approximate solution

(x', t_{i+1}). Newton's method uses the linear approximation

$$\Lambda(x' + \Delta_x) \;=\; H(x', t_{i+1}) + H_x(x', t_{i+1}) \cdot \Delta_x$$

to find Δ_x so that $\Lambda(x' + \Delta_x) = 0$, which is

$$\Delta_x \;=\; -H_x(x', t_{i+1})^{-1} \cdot H(x', t_{i+1})\,.$$

When the point (x', t_{i+1}) is sufficiently close to a solution of $H(x, t_{i+1})$, Newton's method converges quite rapidly, doubling the precision at each step.

If the correction fails to converge after a fixed number (e.g.two) iterations of Newton's method, the stepsize is halved and the algorithm returns to the prediction, recomputing Δ_x (3.2) and then applying Newton iterations. Conversely, if correction succeeds, Δ_t is doubled. This *adaptive steplength* heuristic provides a balance between speed and security in actual path tracking.

Remark 3.3. Since (x_i, t_i) does not necessarily lie on the path but instead is close to it, we are not actually computing points on a tangent line to the curve $H(x, t) = 0$, but rather on the tangent line to the nearby curve $H(x, t) = H(x_i, t_i)$.

Observe that both the predictor and the corrector steps require that we compute the inverse of the Jacobian matrix H_x at the points generated in the algorithm. Thus the numerical efficiency and accuracy of these computations depends strongly on how close this Jacobian is to being singular. This is typically measured by the *condition number* of the Jacobian matrix H_x, which is

$$\|H_x\| \cdot \|H_x\|^{-1}\,,$$

where $\|\boldsymbol{.}\|$ is any (fixed) matrix norm. In numerical analysis, $\|\boldsymbol{.}\|$ is typically the matrix norm induced by the usual Euclidean norm on $\mathbb{R}^{2n}$ so that $\|A\|$ is the least singular value of A, and the condition number of A is then the ratio of its largest to its smallest singular values.

3.2. **Homotopies.** While path-tracking is a more-or-less standard procedure in numerical analysis, finding a homotopy to solve a given set of equations is more of an art form. Fortunately, in algebraic geometry we often study varieties that naturally live in families, and this extends to natural families of systems of equations. We also often study varieties by degenerating them into simpler varieties—such degenerations were at the heart of the proof of Bernstein's Theorem that we gave in Section 2. Such families and degenerations of systems lead naturally to homotopies that are optimal for solving the equations from a given family.

The general challenge when designing a homotopy to solve a system of polynomials is to find a homotopy that connects the target system to a start system that we can solve, while ensuring that each solution in target system is connected to a solution in the start system. This guarantees that all solutions to the target system will be found by tracing paths from the solutions to the start system. Geometrically, $H(X, t) = 0$ should define a curve in $\mathbb{C}^n \times \mathbb{C}$ whose components project dominantly onto the last factor and which is smooth above the point $0 \in \mathbb{C}$. Restricting this last coordinate to a real curve γ in $\mathbb{C}$ connecting 0 to 1 gives paths that may be tracked.

The homotopy is *optimal* if each solution to the start system is connected to a unique solution to the target system along a smooth arc. This is illustrated below in Figure 1. Optimality ensures that there are no extraneous paths to be followed when finding all

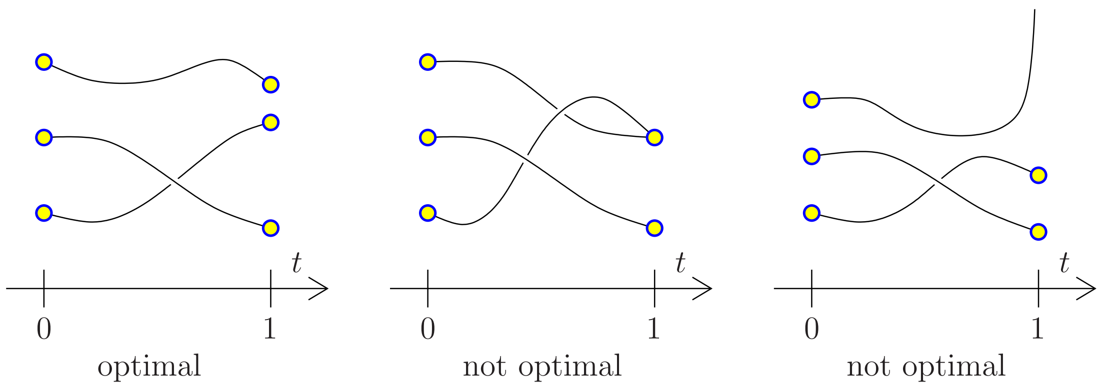


Figure 1. Optimal and non-optimal homotopies

solutions to the target system. We describe three optimal homotopies.

3.2.1. *Cheater Homotopy.* As we noted, many systems of equations occur in well-defined families in which each system has the same number of solutions. This situation is common in enumerative geometry which also typcally gives rise to non-square ($N > n$ in (1.1)) systems. Given such a family, if you have one solved instance, then this may be bootstrapped to solve other related systems.

For example, suppose that $F(x, s) = 0$ is a family of systems depending on a parameter s, which we will take to lie in some variety S. If we have the solutions to $F(x, s_0) = 0$ for some fixed $s_0 \in S$, but want solutions for a given parameter s_1, we may find a real path $\gamma\colon [0, 1] \to S$ with $\gamma(0) = s_0$ and $\gamma(1) = s_1$, and then construct the homotopy $H(x, t) = F(x, \gamma(t))$. If the two points s_0 and s_1 correspond to general members of the family and if the path γ is chosen sufficiently general, then this will define an optimal homotopy.

This selection is particularly straightforward when the space of parameters is $\mathbb{C}^n$, for then, we may simply let γ be the straight line connecting s_0 to s_1, $\gamma(t) = (1-t)s_0 + ts_1$. This gives the *straight-line homotopy*

$$H(x, t) \;=\; (1-t)F(x, s_0) \;+\; tF(x, s_1)\,.$$

In the exercises, you are invited to explore a well-known shortcoming in the straight-line homotopy, together with a remedy which is often used in practice.

Cheater homotopies got their name because they do not involve clever design; one simply exploits the properties of a given family of systems. The name is however misleading, for cheater homotopies are an integral part of any numerical homotopy software. One reason is that the software first solves a random instance of the system of equations, which almost surely has no singularities and has the expected number of solutions. These solutions are then used as the input for a cheater homotopy that connects this general system to the particular one whose solutions are desired. In this way special (and computationally expensive) routines to handle possible non-optimality of the paths are only employed in the second homotopy. Another reason is that numerical homotopy continuation is used for far more than merely solving systems of equations, and these other uses often rely upon cheater homotopies. (Their ubitquity is one reason they are also called parameter homotopies.)

3.2.2. *Bézout Homotopy.* The Bézout homotopy is conceptually the simplest homotopy and is a special case of a cheater homotopy. To solve a system

$$f_1(x_1,\ldots,x_n) \;=\; f_2(x_1,\ldots,x_n) \;=\; \cdots \;=\; f_n(x_1,\ldots,x_n) \;=\; 0\,, \tag{3.4}$$

where the polynomial f_i has degree d_i for $i=1,\ldots,n$, we instead solve the system

$$x_1^{d_1} \;-\; 1 \;=\; x_2^{d_2} \;-\; 1 \;=\; \cdots \;=\; x_n^{d_n} \;-\; 1 \;=\; 0\,. \tag{3.5}$$

This has the $d_1d_2\cdots d_n$ solutions

$$(\zeta_1^{k_1},\zeta_2^{k_2},\ldots,\zeta_n^{k_n}) \qquad \text{for all } 0\leq k_i\leq d_i-1\,,\ i=1,\ldots,n\,,$$

where $\zeta_i = e^{2\pi/d_i}$ is a primitive d_ith root of unity. The Bézout homotopy is then the straight-line homotopy between the two systems (3.4) and (3.5). That is,

$$h_i(x,t) \;:=\; (1-t)f_i(x)+t(x^{d_i}-1)\,, \qquad \text{for} \quad i=1,\ldots,n\,.$$

3.2.3. *Polyhedral Homotopy.* This homotopy is substantially more refined than the previous two. It is based on the proof we saw of Bernstein's Theorem, and it is optimal for solving sparse systems of equations. Both it and the algorithmic proof of Bernstein's Theorem were introduced in the seminal article [11]. The current fastest continuation software for solving systems of equations as of 2009 relies on polyhedral homotopy

Follow the Huber-Sturmfels paper

Make sure to state a Theorem about the optimality of polyhedral homotopy. Mention the combinatorial complexity of computing mixed decompositions. ?Include Schubert example to illustrate the non-univerality of polyhedral homotopies?

Exercises on Homotopy Continuation.

(1) Suppose that we use the straight-line homotopy

$$H(x;t) \;=\; t(x^2-x-1) \,+\, (1-t)(x^2-1) \qquad \text{for} \quad t\in[0,1]$$

to compute the golden ratio numerically. Sketch the path taken by the roots of $H(x;t)$ for $t\in[0,1]$. Discuss the suitability of the predictor-corrector path tracking algorithm for this homotopy.

(2) For an arbitrary complex number $\theta\in\mathbb{C}$, consider the homotopy

$$H(x;t) \;=\; t(x^2-x-1) \,+\, \theta(1-t)(x^2-1) \qquad \text{for} \quad t\in[0,1]\,.$$

Sketch the paths taken by the roots of $H(x;t)$ for $t\in[0,1]$ for different values of θ. (For example, $\theta\in\{e^{i\frac{3\pi}{4}}, e^{i\frac{\pi}{2}}, e^{i\frac{\pi}{4}}, e^{i\frac{\pi}{6}}\}$.)

(3) Interpreting the coefficents of vectors $\mathbf{v}$ and $\mathbf{w}$ in a convex combination $\tau\mathbf{v}+(1-\tau)\mathbf{w}$ as points in $\mathbb{CP}^1$, plot curves $t\mathbf{v}+\theta(1-t)\mathbf{w}$ for $t\in[0,1]$ and different values of θ, say those in the previous exercise together with $\theta=1$, in $\mathbb{CP}^1$. That is, plot $t/(t(1-\theta)+\theta)$ for $t\in[0,1]$.

(4) Download and install PHCPack, Hom4PS2, and Bertini onto a computer, and test them on a suite of polynomial systems in three variables of your choosing. Compare their performance. Push your computer, and look for the limits of computability.

Bonus: Install Singular or Macaulay2 or CoCoA, and try to compute the dimension and degrees of the varieties defined by the equations from the first part of this exercise.

4. Fewnomial Bounds and Gale Duality

The first two sections dealt with bounds for the number of complex solutions to systems of equations and optimal numerical algorithms based on those bounds. In this section and the next, we turn to the real numbers, beginning with bounds on the number of real solutions to a system of equations. The gold standard in this topic is Descartes' bound which is derived from his rule of signs. Both appeared in his 1637 treatise *La Géométrie* [10].

Theorem 4.1 (Descartes)**.** *Suppose that $a_0 < a_1 < \cdots < a_l$ are integers. Then the number of positive solutions ($x \in \mathbb{R}_{>0}$) to*

$$c_0x^{a_0} \ + \ c_1x^{a_1} \ + \ \cdots \ + \ c_lx^{a_l} \ = \ 0\,,$$

where the coefficients c_j are nonzero, is at most the number of sign changes in the coefficients,

$$\#\{i \mid c_{i-1}\cdot c_j < 0\}\,.$$

We deduce Descartes' bound.

Corollary 4.2. *The number of positive roots of a univariate polynomial with $k+1$ terms is at most k.*

Descartes' bound is sharp, as may be seen from the polynomial

$$\prod_{i=1}^{k}(x-i)\,,$$

which has degree k, and hence $k{+}1$ terms, and k positive roots.

In 1980, Khovanskii gave a multivariate generalization of Descartes' bound [12, 13].

Theorem 4.3. *The number of non-degenerate positive solutions to a system*

$$f_1(x_1,x_2,\ldots,x_n) \ = \ f_2(x_1,x_2,\ldots,x_n) \ = \ \cdots \ = \ f_n(x_1,x_2,\ldots,x_n) \ = \ 0\,, \tag{4.4}$$

of n polynomials in n variables with a total of $n+l+1$ monomial terms is at most

$$2^{\binom{n+l}{2}}(n+1)^{n+l}\,.$$

This was a remarkable theorem, as it showed that the number of real solutions has a completely different character than the number of complex solutions. It is also a special case of a much more general bound he established involving quasi-polynomials and Pfaff functions. Consequently, Khovanskii's bound is enormous. For example, when $n = l = 2$, the bound is $2^{\binom{4}{2}}3^4 = 5184$ for the number of positive solutions to a system of two equations in two unknowns, having a total of five monomial terms.

Let $\chi(n,l)$ be the maximum number of non-degenerate positive solutions to a system of n polynomials in n variables having a total of $l+n+1$ monomials. Khovanskii showed that $\chi(n,l) \leq 2^{\binom{n+l}{2}}(n+1)^{n+l}$. While it was widely believed there was considerable slack in this bound, no improvements were found until quite recently. Not only has a smaller bound been found, but also a lower bound in the form of a construction which shows that this new bound is asymptotically sharp, in a certain sense.

Theorem 4.5 ([3, 5, 6, 7])**.** $$\frac{e^2+3}{4}2^{\binom{l}{2}}n^l \;>\; \chi(n,l) \;\geq\; l^{-l}n^l\,.$$

This upper bound is an improvement over the bound of Khovanskii; when $(n,l)=(2,2)$, this bound becomes

$$\frac{e^2+3}{4}2^{\binom{2}{2}}\cdot 2^2 \;\geq\; 2.597\cdot 2^1\cdot 2^2 \;\geq 20.77\,.$$

This can be further improved to 15 by simplifying a complicated expression from which the bound of Theorem 4.5 was obtained.

A reason for this parametrization (n variables, $n{+}l{+}1$ monomials) is that when $l=0$, there is a change of coordinates so that the system (4.4) becomes a system of linear equations, which has either 1 or 0 positive solutions. For the case case $l=1$, we have a bound $n{+}1\geq\chi(n,l)$ [3], which is sharp by Bihan's construction of systems for any n having $n{+}1$ positive solutions [5]. The general case was treated in [7], giving the upper bound,

$$\frac{e^2+3}{4}2^{\binom{l}{2}}n^l \;>\; \chi(n,l)\,.$$

The method employed to derive these new bounds is to first transform the system into an equavalent Gale dual system and then use Khovanskii's generalization of Rolle's Theorem to estimate the number of solutions to this new system. We will derive these new bounds in the next two subsections. A construction of systems in km variables with $km{+}k{+}1$ distinct monomials having $(m{+}1)^k$ positive solutions was given in [6] using the construction in [5]. This gives the lower bound $\chi(n,l)\geq l^{-1}n^l$. We present this construction in the last subsection in this section.

These bounds show that, for l fixed and n large, the number $\chi(n,l)$ has order n^l. It remains an open problem to tighten the inequality of Theorem 4.5 and also to give any asymptotic bound when n is fixed and l is large. The first place to look for an improvement is when $(n,l)=(2,2)$ for which there is a gap $15\geq\chi(2,2)\geq 5$ between known bounds and constructions. The exercises explore these bounds.

4.1. **Gale duality for polynomial systems.** The upper bound for $\chi(n,l)$ of Theorem 4.5 is actually an upper bound for the number of positive solutions to a system of rational functions, called a Gale system. We establish that upper bound in the next subsection. Here, we show how to transform a system of polynomials into an equivalent Gale system.

Let $\mathcal{A}=\{0,a_1,\ldots,a_{n+l}\}\subset\mathbb{Z}^n$ be a collection of exponent vectors for Laurent polynomials. Suppose that we have a system of n polynomial equations (4.4) in which every polynomial f_i has support $\mathcal{A}$, and therefore the system has $n{+}l{+}1$ monomials in all. We want to estimate the non-degenerate positive solutions to this system. This number will not decrease if we perturb the coefficients of the polynomials, so we may assume that all solutions are simple, and the coefficients are general.

We can solve these equations for the monomials $x^{a_1},\ldots,x^{a_n}$ to get an equivalent system

$$\begin{aligned} x^{a_i} \;&=\; c_{i,0}+c_{i,1}x^{a_{n+1}}+\cdots+c_{i,l}x^{a_{n+l}} \\ &=\; p_i(x^{a_{n+1}},x^{a_{n+2}},\ldots,x^{a_{n+l}})\,,\qquad \text{for } i=1,\ldots,n\,. \end{aligned} \tag{4.6}$$

where $p_i(y_1,\ldots,y_l)=c_{i,0}+c_{i,1}y_1+\cdots+c_{i,l}y_l$ is an affine linear polynomial for $i=1,\ldots,n$. For $i=1,\ldots,l$, define the affine polynomial

$$p_{n+i}(y_1,y_2,\ldots,y_l) \ := \ y_i\,.$$

An integer linear relation on $\mathcal{A}$,

$$b_ia_1+b_2a_2+\cdots+b_{n+l}a_{n+l} \ = \ 0\,, \tag{4.7}$$

where $b=(b_1,\ldots,b_{n+l})\in\mathbb{Z}^{n+l}$, gives the consequence of (4.6),

$$\begin{aligned} 1 \quad = \quad \prod_{i=1}^{n+l}(x^{a_i})^{b_i} \quad &= \quad \prod_{i=1}^{n+l}\big(p_i(x^{a_{n+1}},\ldots,x^{a_{n+l}})\big)^{b_i} \\ &=: \quad \big(p(x^{a_{n+1}},\ldots,x^{a_{n+l}})\big)^{b}\,. \end{aligned}$$

Thus if $b^{(1)},\ldots,b^{(l)}\in\mathbb{Z}^{n+l}$ is a basis for the free abelian group of all linear relations (4.7), we have the consequence of (4.6),

$$1 \ = \ \big(p(x^{a_{n+1}},\ldots,x^{a_{n+l}})\big)^{b^{(j)}} \qquad j=1,\ldots,l\,.$$

Let us introduce new variables $y=(y_1,\ldots,y_l)\in\mathbb{C}^l$ and consider the system

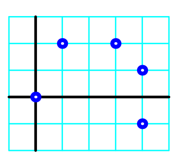
$$1 \ = \ \big(p(y_1,y_2,\ldots,y_l)\big)^{b^{(j)}} \qquad j=1,\ldots,l\,, \tag{4.8}$$

in the set where $\prod_i p_i(y)\neq 0$. We call this the *Gale (dual) system*, as its exponents come from the vector configuration Gale dual to the exponents $\mathcal{A}$. The reason for introducing this new Gale system (4.8) is that it is equivalent to the original system (4.4).

Theorem 4.9 ([8]). *Suppose that $\mathcal{A}$ spans the integer lattice of all exponents, that is, $\mathbb{Z}\mathcal{A}=\mathbb{Z}^n$. Then the association $y_i=x^{a_{n+i}}$ for $i=1,\ldots,l$ gives a scheme-theoretic isomorphism between solutions to the polynomial systems* (4.4) *or* (4.6) *in $(\mathbb{C}^\times)^n$ and solutions to the Gale system* (4.8) *in the region $\{y\in\mathbb{C}^l\mid p_i(y)\neq 0,\ i=1,\ldots,n+l\}$.*

When $\mathbb{Z}\mathcal{A}$ has odd index in $\mathbb{Z}^n$, this association $y_i=x^{a_{n+i}}$ is a scheme-theoretic isomorphism between real solutions to the two systems, and in general the association is an isomorphism between positive solutions to the polynomial systems (4.4) *and solutions to the Gale system in $\Delta:=\{y\mid p_i(y)>0,\ i=1,\ldots,n+l\}$.*

Example 4.10. Let $\mathcal{A}=\{(4,-1),(3,2),(4,1),(1,2),(0,0)\}\subset\mathbb{Z}^2$, which we display below.



Suppose that we have the system of polynomial equations

$$\begin{aligned} 2x^4y^{-1}-3x^3y^2-4x^4y+xy^2-\tfrac{1}{2} \quad &= \quad 0 \\ x^3y^2+2x^4y-xy^2-\tfrac{1}{2} \quad &= \quad 0 \end{aligned} \tag{4.11}$$

in the torus $(\mathbb{C}^\times)^2$, which has support $\mathcal{A}$. Here $n = l = 2$ and $m = 0$. We may diagonalize this to obtain

$$\begin{array}{rclcl} x^3y^2 &=& x^4y^{-1} - x^4y - \frac{1}{2} &=:& p_1(x^4y^{-1}, x^4y)\,, \\ xy^2 &=& x^4y^{-1} + x^4y - 1 &=:& p_2(x^4y^{-1}, x^4y)\,. \end{array}$$

Let s, t be new variables and set $p_3(s,t) := s$ and $p_4(s,t) := t$.

Note that

$$(x^3y^2)^{-1}(xy^2)^3(x^4y^{-1})^2(x^4y)^{-2} \;=\; (x^3y^2)^3(xy^2)^{-1}(x^4y^{-1})(x^4y)^{-3} \;=\; 1\,,$$

and so the vectors $b^{(1)} := (-1, 3, 2, -2)$ and $b^{(2)} := (3, -1, 1, -3)$ are integer linear relations on $\mathcal{A}$. Since $\mathbb{Z}\mathcal{A} = \mathbb{Z}^2$ and $b^{(1)}, b^{(2)}$ are a basis for the free abelian group of integer linear relations on $\mathcal{A}$, by Theorem 4.9 the polynomial system (4.11) in $(\mathbb{C}^\times)^2$ is equivalent to the Gale system

$$\begin{array}{rcl} s^2t^{-2}(s-t-\frac{1}{2})^{-1}(s+t-1)^3 &=& 1 \\ st^{-3}(s-t-\frac{1}{2})^3(s+t-1)^{-1} &=& 1\,, \end{array} \tag{4.12}$$

where $st(s+t-1)(s-t-\frac{1}{2}) \neq 0$. We display these two systems in Figure 2, drawing also the excluded lines. Each system has three solutions with the positive solution to the polynomial equation coresponding to the solutions of the Gale system lying in the quadrilateral Δ.

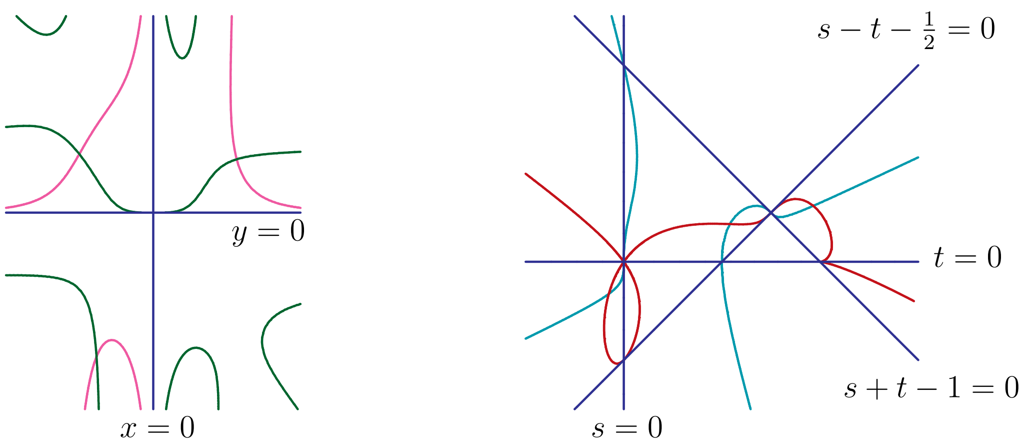


FIGURE 2. The polynomial system (4.11) and the Gale system (4.12).

Remark 4.13. There is no relation between the two pairs of curves in Figure 2. Only the three common points of intersection are related by the Gale duality of Theorem 4.9.

4.2. **Proof of fewnomial upper bound.** By Gale duality (Theorem 4.9), to establish the upper bound in Theorem 4.5, it suffices to prove the following result.

Theorem 4.14. *Let $p_1(y), \ldots, p_{n+l}(y)$ be affine linear polynomials that, together with the constant polynomial 1, span the space of affine linear polynomials on $\mathbb{R}^l$. For any linearly independent vectors $\mathcal{B} = \{b^{(1)}, \ldots .b^{(l)}\} \subset \mathbb{Z}^{n+l}$, the number of solutions to*

$$1 \;=\; \big(p(y)\big)^{b^{(j)}} \qquad \textit{for } j = 1, \ldots, l\,, \tag{4.15}$$

in the region $\Delta := \{y \in \mathbb{R}^l \mid p_i(y) > 0$ *for* $i = 1, \ldots, n+l\}$ *is at most*

$$\frac{e^2+3}{4} 2^{\binom{l}{2}} n^l \,.$$

Our proof of this Theorem is as easy as 1-2-3.

4.2.1. *One idea.* The primary tool in establishing these bounds is Khovanskii's generalization of Rolle's classical theorem from elementary calculus. We begin with some definitions. Let Δ be an open subset of $\mathbb{R}^l$ whose boundary is piecewise algebraic. For functions $f_1, \ldots, f_s$ on Δ, let

$$\mathcal{V}_\Delta(f_1, \ldots, f_s) \ := \ \{y \in \Delta \mid f_i(y) = 0 \ \text{ for } i = 1, \ldots, s\}\,,$$

be their common zeroes. If C is a curve in Δ, let $\mathrm{ubc}_\Delta(C)$ be the number of unbounded components of C in Δ.

Theorem 4.16 (Khovanskii-Rolle)**.** *Let* $f_1, \ldots, f_l$ *be smooth functions defined on a domain* $\Delta \subset \mathbb{R}^l$ *where the first* $l-1$ *functions,* $f_1, \ldots, f_{l-1}$ *define a smooth curve* $C = \mathcal{V}_\Delta(f_1, \ldots, f_{l-1})$ *with finitely many components which meets the zero set* $\mathcal{V}_\Delta(J)$ *of the Jacobian determinant* J *of* $f_1, \ldots, f_l$ *in finitely many points. Then* $\mathcal{V}_\Delta(f_1, \ldots, f_l)$ *is finite with*

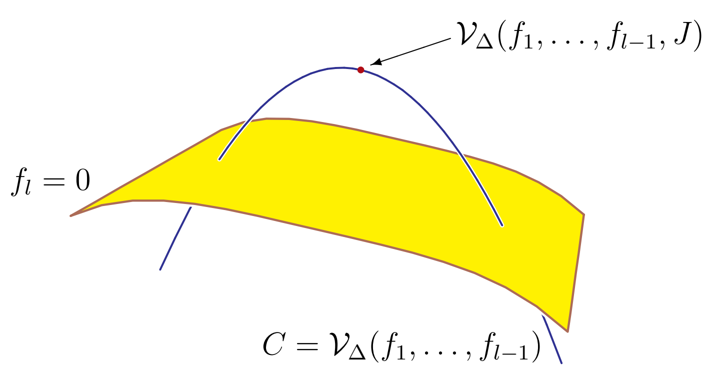
$$\#\mathcal{V}_\Delta(f_1, \ldots, f_l) \ \le \ \mathrm{ubc}_\Delta(C) + \#\mathcal{V}_\Delta(f_1, \ldots, f_{l-1}, J)\,. \tag{4.17}$$

This is not the Khovanskii-Rolle Theorem in its complete generality, but rather in the form which we use. This form follows from the usual Rolle Theorem. Suppose that $f_l(a) = f_l(b) = 0$, for points a, b on the same component of C. Let $s(t)$ be the arclength along this component of C, measured from a point $t_0 \in C$, and consider the map,

$$\begin{array}{rcl} C & \longrightarrow & \mathbb{R}^2 \\ t & \longmapsto & (s(t), f_l(t))\,. \end{array}$$

This is the graph of a differentiable function $g(s)$ which vanishes when $s = s(a)$ and $s = s(b)$, so there is a point $s(b)$ between $s(a)$ and $s(b)$ where its derivative aso vanishes, by the usual Rolle Theorem. But then c lies between a and b on that component of C, and the vanishing of $g'(s(c))$ is equivalent to the Jacobian J vanishing at c.

Thus along any arc of C connecting two zeroes of f_l, the Jacobian vanishes at least once.



4.2.2. *Two Reductions.* We make two assumptions about the degree 1 polynomials $p_i(y)$ and the exponents $b^{(j)}$ that appear in (4.15).

(1) Without any loss of generality, we may assume that the hyperplanes defined by $p_i(y) = 0$ for $i = 1, \ldots, n{+}l$ meet with nornal crossings, and that this is also true at infinity—they meet the hyperplane at infinity with normal crossings. This means that any j of the hyperplanes (including the hyperplane at infinity) either have empty intersection or they meet in an affine space of codimension j.

   This may be arranged by simply perturbing the affine linear polynomials $p_i(y)$, which will not affect the number of nondegenerate real solutions to (4.15).

(2) Without any loss of generality, we may assume that the matrix $\mathcal{B}$ whose rows are $b^{(1)}, \ldots, b^{(l)}$ is general in that each of its minors is non-singular.

   This may also be arranged by simply perturbing the exponents $b^{(j)}$. However, the perturbation requires some justification. First observe that we may replace the affine linear polynomials $p_i(y)$ by their absolute values, $|p_i(y)|$, and get the system

$$1 \;=\; \bigl|p(y)\bigr|^{b^{(j)}} \qquad \text{for } j = 1, \ldots, l\,,$$

   which defines the same set in Δ, but may have more zeroes outside of Δ.

   In these new equations, it does not matter that the exponents are integral, and they may be perturbed to obtain real number exponents whose matrix $\mathcal{B}$ has all minors non-singular. We may even assume that these new exponents $b^{(j)}$ are rational vectors, as any real number is approximated by rationals. Lastly, we may clear denominators in these rational exponents to obtain integer exponents, and then safely drop the absolute values and assume that we have our original system (4.15), but whose exponents are sufficiently general.

4.2.3. *Three Lemmata.* We will find it useful to replace the system (4.15) of rational functions by equivalent systems of polynomials and of logarithms. For each $j = 1, \ldots, l$, let $b^{(j)}_{\pm}$ be the coordinatewise maximum of $\pm b^{(j)}$ and the 0 vector, and define the polynomial

$$f_j(y) \;:=\; \bigl(p(y)\bigr)^{b^{(j)}_+} \;-\; \bigl(p(y)\bigr)^{b^{(j)}_-} \;=\; \bigl(p^{b^{(j)}} - 1\bigr)\bigl(p(y)\bigr)^{b^{(j)}_-}\,, \tag{4.18}$$

as well as the transcendental function

$$\psi_j(y) \;:=\; \log\bigl|p(y)\bigr|^{b^{(j)}} \;=\; \sum_{i=1}^{n+l} b^{(j)}_i \log|p_i(y)|\,.$$

We use these functions to define varieties μ_j for each $j = 1, \ldots, l$. Set

$$\mu_j \;:=\; \mathcal{V}(f_1, \ldots, f_j)\,.$$

This is an algebraic variety, which, outside of the hyperplanes $p_i(y) = 0$, is also defined as $\mathcal{V}(\psi_1, \ldots, \psi_l)$. We will also regard the polyhedron Δ as possibly having one of its facets be the hyperplane at infinity, so that it has $n{+}l{+}1$ facets.

Lemma 4.19. *The variety μ_j has codimension j and is smooth outside the hyperplanes $p_i(y) = 0$. It meets the boundary $\partial\Delta$ of Δ in a union of codimension j faces of Δ and in the neighborhood of any point of $\mu_j \cap \partial\Delta$, μ_j has exactly one branch in Δ.*

The smoothness is guaranteed by the genericity of the degree 1 polynomials, while the other properties make use of the assumptions on the exponents. This is deduced in [7].

Define Jacobians $J_l, J_{l-1}, \ldots, J_1$ by downward induction on j by setting J_j to be the Jacobian determinant of the functions $\psi_1, \ldots, \psi_j,\ J_{j+1}, \ldots, J_l$.

Lemma 4.20. *The Jacobian J_j is a rational function with denominators the degree 1 polynomials $p_i(y)$. If we clear denominators,*

$$\widetilde{J}_j \ = \ J_j \cdot \Bigl(\prod_{i=1}^{n+l} p_i(y)\Bigr)^{2^{l-j}} \ ,$$

we obtain a polynomial of degree $2^{l-j} \cdot n$.

This follows from expanding the Jacobian determinant using the Binet-Cauchy Theorem, and a proof is given in [7].

For the third lemma, set $C_j := \mu_{j-1} \cap \mathcal{V}_(\widetilde{J}_{j+1}, \ldots, \widetilde{J}_l)$, which is defined in Δ by μ_{j-1} and the Jacobians $J_{j+1}, \ldots, J_l$. We may ensure this is a smooth curve by perturbing the polynomials $\widetilde{J}_i$ without invalidating the conclusions of the Khovanskii-Rolle Theorem. (Details for this are similar to the arguments given in [9].) Since $\mathcal{V}_\Delta(f_j) = \mathcal{V}_\Delta(\psi_j)$, the Khovanskii-Rolle Theorem implies that

$$\begin{aligned} \#\mathcal{V}_\Delta(f_1, \ldots, f_l) \ &\leq \ \mathrm{ubc}(C_l) \ + \ \#\mathcal{V}_\Delta(f_1, \ldots, f_{l-1},\ J_l) \\ &\leq \ \mathrm{ubc}(C_l) \ + \ \cdots \ + \ \mathrm{ubc}(C_1) \ + \ \#\mathcal{V}_\Delta(J_1, \ldots, J_l)\,. \end{aligned} \tag{4.21}$$

Observe that by Lemma 4.19, the points where C_j meets the boundary of Δ are a subset of the points on the codimension j faces F of Δ where the polynomials $\widetilde{J}_{j+1}, \ldots, \widetilde{J}_l$ vanish. By our assumption on the degree 1 polynomials, these faces are where j of the $p_i(y)$ vanish.

The last lemma provides three estimates for these numbers. The first two use Bézout's Theorem, replacing the Jacobian J_i by the polynomial $\widetilde{J}_j$ (see the discussion following Proposition 5.9), and the last is a simple combinatorial argument.

Lemma 4.22. *With the above definitions we have,*

(1) $\#\mathcal{V}_\Delta(J_1, \ldots, J_l) \ \leq \ 2^{\binom{l}{2}} n^l\,.$

(2) $\mathrm{ubc}(C_j) \ \leq \ \dfrac{1}{2}\dbinom{n+l+1}{j} n^{l-j} 2^{\binom{l-j}{2}}\,.$

(3) $\dbinom{n+l+1}{j} n^{l-j} 2^{\binom{l-j}{2}} \ \leq \ 2^{\binom{l}{2}} n^l \cdot \dfrac{2^{j-1}}{j!}\,.$

We complete the proof of Theorem 4.14. By (4.21) and Lemma 4.22,

$$\begin{aligned} \#\mathcal{V}_\Delta(f_1, \ldots, f_l) \ &\leq \ \tfrac{1}{2}\Bigl(\textstyle\sum_{j=1}^{l} \binom{n+l+1}{j} n^{l-j} 2^{\binom{l-j}{2}}\Bigr) \ + \ 2^{\binom{l}{2}} n^l \\ &\leq \ 2^{\binom{l}{2}} n^l \cdot \Bigl(1 + \tfrac{1}{2}\textstyle\sum_{j=1}^{l} \frac{2^{j-1}}{j!}\Bigr) \\ &< \ 2^{\binom{l}{2}} n^l \cdot \Bigl(1 + \tfrac{1}{2}\textstyle\sum_{j=1}^{\infty} \frac{2^{j-1}}{j!}\Bigr) \\ &= \ 2^{\binom{l}{2}} n^l \cdot \Bigl(1 + \tfrac{e^2-1}{4}\Bigr) \ = \ \tfrac{e^2+3}{4} 2^{\binom{l}{2}} n^l\ . \end{aligned} \tag{4.23}$$

4.3. **Lower bound for $\chi(n,l)$.** We establish the lower bound for $\chi(n,l)$ of Theorem 4.5, which is based on Bihan's construction of a system of n polynomials in n variables with $n+2$ distinct monomials that has $n{+}1$ positive solutions.

Theorem 4.24 ([6]). *For any positive integers n, l with $n > l$, there exists a system of n polynomials in n variables involving $n{+}l{+}1$ distinct monomials and having $\lfloor \frac{n+l}{l} \rfloor^l$ nondegenerate positive solutions.*

Proof. We will construct such a system when $n = lm$, a multiple of l, from which we may deduce the general case as follows. Suppose that $n = lm + k$ with $1 \leq k < l$ and we have a system of ml equations in ml variables involving $ml{+}l{+}1$ monomials with $(m{+}1)^l$ nondegenerate positive solutions. We add k new variables $x_1, \ldots, x_k$ and k new equations $x_1 = 1, \ldots, x_k = 1$. Since the polynomials in the original system may be assumed to have constant terms, this gives a system with n polynomials in n variables having $n{+}l{+}1$ monomials and $(m{+}1)^l = \lfloor \frac{n+l}{l} \rfloor^l$ nondegenerate positive solutions. So let us fix positive integers l, m and set $n := lm$.

Bihan [5] showed there exists a system of m polynomials in m variables

$$f_1(y_1, \ldots, y_m) \;=\; \cdots \;=\; f_m(y_1, \ldots, y_m) \;=\; 0$$

having $m{+}1$ positive solutions, and where each polynomial has the same $m{+}2$ monomials, one of which we may take to be a constant.

For each $k = 1, \ldots, l$, let $y_{k,1}, \ldots, y_{k,m}$ be m new variables and consider the system

$$f_1(y_{k,1}, \ldots, y_{k,m}) \;=\; \cdots \;=\; f_m(y_{k,1}, \ldots, y_{k,m}) \;=\; 0\,,$$

which has $m{+}1$ positive solutions in $(y_{k,1}, \ldots, y_{k,m})$. As the sets of variables are disjoint, the combined system consisting of all lm polynomials in all lm variables has $(m{+}1)^l$ positive solutions. Each subsystem has $m{+}2$ monomials, one of which is a constant. Thus the combined system has $1 + l(m{+}1) = lm{+}l{+}1 = n{+}l{+}1$ monomials. ∎

4.4. **Open Questions.** We close this section with three questions.

(1) In the bound $\chi(n,l) \leq \frac{e^2+3}{4} 2^{\binom{l}{2}} n^l$, the term $2^{\binom{l}{2}}$ is perhaps unnecessarily large. Can this be lowered, or perhaps replaced by l^n?
(2) We suggested the term l^n, as there are constructions with $O(l^n)$ solutions. Is it possible to construct a system of fewnomials with more than l^n solutions?
(3) Can these new fewnomial methods be used to get better bounds on the topology of fewnomial systems, perhaps along the lines of Basu's chapter?

Exercises on Fewnomials.

(1) Compute the number of non-zero complex solutions to the system

$$10x^{106} \;+\; 11y^{53} \;-\; 11y \;=\; 10y^{106} \;+\; 11x^{53} \;-\; 11x \;=\; 0\,.$$

How many of them are real?
(2) Give a system involving the monomials 1, x, y, xyz^8, z, z^2, z^3 that has 9 or more real solutions. Can you find a system with more real solutions?

(3) Exhibit a system of two polynomials in the variables x, y involving a total of 4 different monomials that has three positive real solutions.

(4) Exhibit a system of three polynomials in the variables x, y, z involving a total of 5 different monomials that has four positive real solutions.

(5) Find a system Gale dual to each system of functions below. For each, also compute the number of complex solutions (non-zero, and off the hyperplanes), the number of real solutions, and the number of positive solutions.

$$(a)\qquad \left\{\begin{array}{rcl} 1+3x^3y-6x^2y^2+5y^3 &=& 0\\ 4+5x^3y-yx^2y^2-9y^3 &=& 0 \end{array}\right.$$

$$(b)\qquad \left\{\begin{array}{rcl} 2-3xy^3+4x^2z+x^2y^2-6y^3+11xy^2z^5 &=& 0\\ 6+2xy^3-x^2z+2x^2y^2+7y^3-6xy^2z^5 &=& 0\\ -1-xy^3+3x^2z-3x^2y^2+2y^3+3xy^2z^5 &=& 0 \end{array}\right.$$

$$(c)\qquad \left\{\begin{array}{rcl} x^{-1}(x+y-1)y^{-1}(x+2y-6)^2(y+2z-6) &=& 1\\ x(x+y-1)^{-3}y^2(x+2y-6)(y+2z-6)^{-2} &=& 1 \end{array}\right.$$

$$(d)\qquad \left\{\begin{array}{rcl} x^2(x+y-1)^3y(x+2y-6)^2(y+2z-6) &=& 1\\ x^{-4}(x+y-1)^3y(x+2y-6)^{-2}(y+2z-6)^{-3} &=& 1 \end{array}\right.$$

$$(e)\qquad \left\{\begin{array}{rcl} x(x+y-1)^7y(x+2y-6)^5(y+2z-6)^3 &=& 1\\ x^{-6}(x+y-1)^{-4}y^2(x+2y-6)^{-3}(y+2z-6)^{-6} &=& 1 \end{array}\right.$$

(6) Challenge project: Construct a system of two polynomials in two variables, having a total of 5 monomials that has six or more real positive solutions. That is, show that $\chi(2,2) \geq 5$.

5. Khovanskii-Rolle Continuation

The method of numerical homotopy continuation was presented in Section 3. This finds all complex solutions to a system of polynomial equations. The need to avoid singularities forces homotopy to be an intrincically complex-number algorithm. If only real solutions are desired, as is often the case in applications, this may be rather inefficient, especially for fewnomial solutions which may have only a handful of real solutions among their many complex solutions. The many-body problem in celestial mechanics gives problems of this nature. In [16], the polyhedral homotopy method as implemented in HOM4PS-2.0 [15] was used to classify isolated stable central configurations of five equal masses. The Bernstein number for this problem is $439,690,761$, which was the number of paths followed. However, only 258 led to positive real (hence physically meaningful) solutions.

There are, however, numerical algorithms which only find real solutions to systems of equations and for which the computation is largely to entirely real. One such class are subdivision algorithms, which recursively subdivide a domain searching for real solutions. We discuss these briefly below. A completely different type of algorithm was developed by Lasserre, Laurent, and Rostalski [14] to compute the real radical of an ideal. This is described in Laurent's chapter. A third algorithm is Khovanskii-Rolle continuation [1] to which we devote this section.

A subdivision algorithm is based on an exclusion test, which is a certificate that a system of polynomials has no common solutions in a given region. For example, the positivestellensatz can give a certificate that a list of polynomials has no solutions in a given semialgebraic set. The elementary step in an exclusion algorithm involves trying to generate a certificate for a system of polynomials to have no solutions on a domain Δ. If the algorithm succeeds in generating a certificate, it stops; otherwise, it subdivides Δ into smaller domains and calls itself on each subdomain.

In this way, a tree is generated whose leaves represent the smallest subdomains yet computed which may contain common zeroes. When a threshold in their size is reached, the algorithm terminates and outputs these smallest domains as approximate solutions, perhaps calling another algorithm to certify that these domains contain solutions.

Subdivision algorithms are only as good as their exclusion tests. One very attractive class of tests are based on the Bernstein basis from geometric modeling as explained in Roy's chapter on certificates of positivity. Such a subdivision algorithm was proposed and implemented by Mourrain and Pavone [17].

In all three of these algorithms, as well as in the polyhedral homotopy algorithm, we see how theoretical advances lead to good numerical algorithms. For Khovanskii-Rolle continuation, the theoretical advance is the Khovanskii-Rolle Theorem 4.16, which gives an inductive step to find solutions to a system $f_1 = \cdots = f_l = 0$ using curve-tracing and solutions to the system $f_1 = \cdots = f_l = J = 0$, where J is the Jacobian of the first system. The Khovanskii-Rolle algorithm solves a fewnomial system by first converting it into a Gale dual system, which is solved using this inductive step organized globally as in the proof of the fewnomial bound in Subsection 4.2. Here, we first describe curve-tracing, then give the inductive step based on the Khovanskii-Rolle Theorem before presenting the full algorithm for solving Gale systems, and we close with a carefully worked example.

5.1. **Curve-tracing.** The fundamental computation in numerical homotopy continuation is tracking a *directed*, implicitly-defined path in $\mathbb{R}^{2n}\times[0,1]$ from $t=0$ to $t=1$ using perdictor-corrector methods. For Khovanskii-Rolle continuation, the fundamental computation is tracing an *undirected* curve using predictor-corrector methods. While similar, a difference is that in path-tracking, as employed in homotopy methods, there are several safeguards to help ensure that the algorithm correctly tracks the paths.

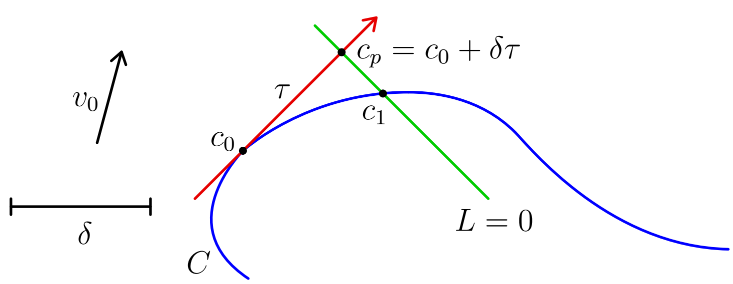
We may trace a curve C that is implicitly defined in a domain $\Delta\subset\mathbb{R}^l$ by $l-1$ functions,

$$C \;=\; \mathcal{V}_\Delta(f_1,\dots,f_{l-1})\,,$$

starting from a point $c_0\in C$. For this, we also need a *steplength* δ and a vector v_0 encoding the approximate direction of motion along C. Using a tangent predictor, we first compute a unit length tangent vector τ to the curve C at c_0—this generates the kernel of the Jacobian at c_0,

$$\frac{\partial(f_1,\dots,f_{l-1})}{\partial(x_1,\dots,x_l)}(c_0)\ .$$

There are two such unit tangents and we choose the one pointing in the direction of v_0, in that $v_0\cdot\tau>0$. Then we set $c_p:=c_0+\delta\tau$.



The point c_p is near the curve C. To improve this approximation, choose an affine equation $L(x)=0$ that vanishes at c_p and is transverse to the tangent direction τ, for example

$$L(x) \;:=\; \tau\cdot(x-c_p)\,.$$

Adding L to the functions $f_1,\dots,f_{l-1}$ gives a square system F with c_p close to a solution of $F(x)=0$. Newton iterations beginning with c_p yield a nearby point c_1 on C. Then the point c_1 and the direction τ may be used as input for a subsequent predictor step to continue tracing C in the same direction. As with path-tracking, adaptive steplength may be employed to improve the efficiency of curve-tracing.

In this way, given a point c on a curve C and tangent direction at c, we may trace the component of C containing c in the given direction until some stopping criterion is attained, such as leaving the domain Δ or returning to c, or finding a solution on C to another equation.

5.2. **Continuation step.** Recall the inequality in the Khovanskii-Rolle Theorem,

$$\#\mathcal{V}_\Delta(f_1,\dots,f_l)\;\le\;\mathrm{ubc}_\Delta(C)+\#\mathcal{V}_\Delta(f_1,\dots,f_{l-1},\,J)\,, \tag{4.17}$$

where $f_1,\dots,f_l$ are functions defined in a domain $\Delta\subset\mathbb{R}^n$ with Jacobian J and where C is the curve cut out by $f_1,\dots,f_{l-1}$. We used this inequality to derive the fewnomial bound. The main idea in Khovanskii-Rolle continuation comes from the proof of the inequality (4.17), which implies that, given the points $\mathcal{V}_\Delta(f_1,\ldots,f_{l-1},\,J)$ and the points where C meets the boundary of Δ, we may use curve-tracing along C to find all points in $\mathcal{V}_\Delta(f_1,\ldots,f_l)$.

Algorithm 5.1 (Continuation Algorithm)**.**

Let $\Delta \subset \mathbb{R}^l$ be a bounded domain with piecewise smooth boundary. Suppose that $f_1,\ldots,f_{l-1}$ are differentiable functions defined on Δ that define a smooth curve C which meets the boundary of Δ in finitely many points, and that g is another differentiable function defined on Δ with finitely many zeroes on C. Let J be the Jacobian determinant of the functions $f_1,\ldots,f_{l-1},g$.

Input: Solutions S of $f_1=\cdots=f_{l-1}=J=0$ in the interior of Δ and the set T of points where C meets the boundary of Δ.

Output: All solutions U to $f_1=\cdots=f_{l-1}=g=0$ in the interior of Δ.

— For each s in S, follow C in both directions from s until one of the following occurs:

(5.2)
(1) A solution $u\in U$ to $g=0$ is found,
(2) A solution $s'\in S$ to $J=0$ is found, or
(3) A point $t\in T$ where C meets the boundary of Δ is found.

— For each t in T, follow C in the direction of the interior of Δ until one of the stopping criteria (5.2) occurs.

Each point $u\in U$ is found twice.

Remark 5.3. The stopping criteria (5.2) may be dected by monitoring the signs of g, J, and the functions defining the boundary of Δ.

Proof of correctness. Singular solutions to $g(s)=0$ on C are also solutions to $J=0$, and these may be detected by monitoring the signs of g and J—one of which will change sign at a zero of either. By the Khovanskii-Rolle Theorem, if we remove all points from C where $J=0$ to obtain a curve C°, then no two solutions of $g=0$ will lie in the same connected component of C°. No solution can lie in a compact component (oval) of C°, as this is also an oval of C, and the number of solutions $g=0$ on an oval is necessarily even, so the Khovanskii-Rolle Theorem would imply that $J=0$ has a solution on this component.

Thus the solutions of $g=0$ lie in different connected components of C° and the boundary of each such component consists of solutions to $J=0$ on C and/or points where C meets the boundary of Δ. This shows that the algorithm finds all solutions $g=0$ on C, and that it will find each twice, once for each end of the component of C° on which it lies. □

Each unbounded component of C has two ends that meet the boundary of Δ. Writing $\mathcal{V}(f_1,\ldots,f_{l-1},g)$ for the common zeroes in Δ to $f_1,\ldots,f_{l-1},g$, and $\mathrm{ubc}(C)$ for the unbounded components of C, we obtain the inequality

$$2\#\mathcal{V}(f_1,\ldots,f_{l-1},g)\ \leq\ \#\text{paths followed}\ =\ 2\#\mathrm{ubc}(C)\ +\ 2\#\mathcal{V}(f_1,\ldots,f_{l-1},J)\,, \tag{5.4}$$

as each solution is obtained twice. This gave the bound in the Khovanskii-Rolle Theorem on the number of common zeroes of $f_1,\ldots,f_{l-1},g$ in Δ and it will be used to estimate the number of paths followed in the Khovanskii-Rolle algorithm.

5.3. **Khovanskii-Rolle algorithm.** The Khovanskii-Rolle continuation algorithm finds all the non-degenerate positive solutions to a system

$$f_1(x_1,\dots,x_n) \;=\; f_2(x_1,\dots,x_n) \;=\; \cdots \;=\; f_n(x_1,\dots,x_n) \;=\; 0$$

of polynomial equations, without also computing any complex solutions, or any non-positive real solutions. It does this by first converting this polynomial system into a Gale dual system. In fact, Khovanskii-Rolle continuation is essentially an algorithm to compute all solutions to a system of equations of the form

$$1 \;=\; \big(p(y_1,y_2,\dots,y_l)\big)^{b^{(j)}} \qquad j=1,\dots,l\,, \tag{5.5}$$

in the polyhedron

$$\Delta \;:=\; \{y\in\mathbb{R}^l \mid p_i(y)>0,\ i=0,\dots,n{+}l\}\,, \tag{5.6}$$

which we assume is bounded. Here, the affine polynomials $p_0(y),\dots,p_{n+l}(y)$ and vectors $b^{(1)},\dots,b^{(l)}\in\mathbb{R}^{n+l}$ are general in the sense of Subsection 4.2.2. (The affine polynomial p_0 comes from the hyperplane at infinity, as we may have applied a projective transformation to ensure the polyhedron Δ is bounded.)

We first make some observations and definitions. In Δ, we may take logarithms of the functions in (5.5) to obtain the equivalent system

$$\psi_j(y) \;:=\; \log\big(p(y_1,y_2,\dots,y_l)\big)^{b^{(j)}} \;=\; \sum_{i=0}^{l+n} b_i^{(j)}\log p_i(y) \;\;=\;0 \qquad \text{for } j=1,\dots,l\,.$$

The Khovanskii-Rolle algorithm will find solutions to these equations by iteratively applying Algorithm 5.1. For $j=l,l{-}1,\dots,2,1$ define the function $\widetilde{J}_j$ to be the Jacobian determinant of $(\psi_1,\dots,\psi_j;\ \widetilde{J}_{j+1},\dots,\widetilde{J}_l)$. Also, let $f_1(y),\dots,f_j(y)$ be the polynomials defined in (4.18). In the domain Δ, $\psi_i(y)=1$ is equivalent to $f_j(y)=0$.

Algorithm 5.7 (Khovanskii-Rolle Continuation)**.**

Suppose that ψ_j and $\widetilde{J}_j$ for $j=1,\dots,l$ are as above. For each $j=0,\dots,l$, let S_j be the solutions in Δ to

$$\psi_1 \;=\; \cdots \;=\; \psi_j \;=\; \widetilde{J}_{j+1} \;=\; \cdots \;=\; \widetilde{J}_l \;=\; 0\,,$$

and for each $j=1,\dots,l$, let T_j be the set of solutions to

$$f_1 \;=\; \cdots \;=\; f_{j-1} \;=\; 0 \qquad \text{and} \qquad \widetilde{J}_{j+1} \;=\; \cdots \;=\; \widetilde{J}_l \;=\; 0 \tag{5.8}$$

in the boundary of Δ.

Input: Solutions S_0 of $\widetilde{J}_1=\cdots=\widetilde{J}_l=0$ in Δ, and sets $T_1,\dots,T_l$.
Output: Solution sets $S_1,\dots,S_l$.

For each $j=1,\dots,l$, apply the Continuation Algorithm 5.1 with

$$(f_1,\dots,f_{l-1}) \;=\; (\psi_1,\dots,\psi_{j-1}\,,\ \widetilde{J}_{j+1},\dots,\widetilde{J}_l)$$

and $g=\psi_j$. The inputs for this are the sets S_{j-1} and T_j, and the output is the set S_j.

The last set computed, S_l, is the set of solutions to the system of master functions (5.5) in Δ. We illustrate the algorithm in Subsection 5.4.

Proof of correctness. Note that for each $j = 1, \ldots, l$, the jth step finds the solution set S_j, and therefore preforms as claimed by the correctness of the Continuation Algorithm 5.1. The correctness for the Khovanskii-Rolle algorithm follows by induction on j. $\square$

The Khovanskii-Rolle algorithm requires the precomputation of the sets S_0 and $T_1, \ldots, T_l$. The feasibility of this task follows from the proof of the fewnomial bound in Subsection 4.2. By our assumptions on the generality of the polynomials p_i, a face of Δ of codimension k is the intersection of Δ with k of the hyperplanes $p_i(y) = 0$. We collect some consequences of the lemmas of Subsetion 4.2.3.

Proposition 5.9. *Let $p_i, i = 0, \ldots, l{+}n$, $\psi_j, \widetilde{J}_j, j = 1, \ldots, l$, and Δ be as above.*

(1) *The solutions T_j to* (5.8) *in the boundary of Δ are among the solutions to*
$$\widetilde{J}_{j+1} = \cdots = \widetilde{J}_l = 0$$
in the codimension-j faces of Δ.

(2) $\widetilde{J}_j \cdot \prod_{i=0}^{l+n} p_i(y)^{2^{l-j}}$ *is a polynomial of degree $2^{l-j}n$.*

Thus solving the system (5.8) in the boundary of Δ is equivalent to solving (at most) $\binom{l+n+1}{j}$ systems of the form

$$p_{i_1} \;=\; \cdots \;=\; p_{i_j} \;=\; \widetilde{J}_{j+1} \;=\; \cdots \;=\; \widetilde{J}_l \;=\; 0 \tag{5.10}$$

and then discarding the solutions y for which $p_i(y) < 0$ for some i. Replacing each Jacobian $\widetilde{J}_j$ by the polynomial $J_j := \widetilde{J}_j \cdot \prod_{i=0}^{l+n} p_i(y)^{2^{l-j}}$, and using the affine equations $p_{i_1}(y) = \cdots = p_{i_j} = 0$ to eliminate j variables, we see that (5.10) is a polynomial system with Bézout number
$$n \cdot 2n \cdot 4n \cdots\; 2^{l-j-1}n \;=\; 2^{\binom{l-j}{2}}n^{l-j}$$
in $l - j$ variables. Thus the number of solutions T_j to the system (5.8) in the boundary of Δ is at most $2^{\binom{l-j}{2}}n^{l-j}\binom{l+n+1}{j}$.

Likewise S_0 consists of solutions in Δ to the system (5.10) when $j = 0$ and so has at most the Bézout number $2^{\binom{l}{2}}n^l$ solutions. Thus the inputs to the Khovanskii-Rolle Algorithm are solutions to polynomial systems in l or fewer variables.

Theorem 5.11. *The Khovanskii-Rolle Continuation Algorithm finds all solutions to the system* (5.5) *in the bounded polyhedron Δ* (5.6)*, when the affine polynomials $p_i(y)$ and exponents $b_i^{(j)}$ are general as in Subsection* 4.2.2*. It accomplishes this by solving auxiliary polynomial systems* (5.10) *and following implicit curves. For each $j = 0, \ldots, l{-}1$, it will solve at most $\binom{l+n+1}{j}$ polynomial systems in $l{-}j$ variables, each having Bézout number $2^{\binom{l-j}{2}}n^{l-j}$. The sum of the Bézout numbers of these systems is at most $\frac{e^2+1}{2}2^{\binom{l}{2}}n^l$. In all, it will trace at most*

$$l\,2^{\binom{l}{2}+1}n^l \;+\; \sum_{j=1}^{l}(l{+}1{-}j)2^{l-j}n^{l-j}\binom{l+n+1}{j} \quad<\quad l\,\frac{e^2+3}{2}2^{\binom{l}{2}}n^l \tag{5.12}$$

implicit curves in Δ.

Proof. The first statement is a restatement of the correctness of the Khovanskii-Rolle Continuation Algorithm. For the second statement, we enumerate the number of paths, following the discussion after Proposition 5.9. Let s_j, t_j be the number of points in S_j and T_j, respectively, and let r_j be the number of paths followed in the jth step of the Khovanskii-Rolle Continuation Algorithm.

By (5.4), $r_j = t_j + 2s_{j-1}$ and $s_j \leq \frac{1}{2}t_j + s_{j-1}$. So $r_j \leq t_j + \cdots + t_1 + 2s_0$, and

$$r_1 + \cdots + r_l \;\leq\; 2s_0 + \sum_{j=1}^{l}(l{+}1{-}j)t_j\,.$$

Using the estimates

$$s_0 \;\leq\; 2^{\binom{l}{2}}n^l \qquad \text{and} \qquad t_j \;\leq\; 2^{\binom{l-j}{2}}n^{l-j}\tbinom{l+n+1}{j}\,, \tag{5.13}$$

we obtain the estimate on the left of (5.12). Since $l{+}1{-}j \leq l$, we bound it by

$$2l\Big(2^{\binom{l}{2}}n^l \;+\; \frac{1}{2}\sum_{j=1}^{l}2^{l-j}n^{l-j}\tbinom{l+n+1}{j}\Big)\,,$$

which is bounded by $l\,\frac{e^2+3}{2}2^{\binom{l}{2}}n^l$, by the computation (4.23). ☐

Remark 5.14. The bound (5.12) on the number of paths to be followed is not sharp. First, not every system of the affine polynomials

$$p_{i_1}(y) \;=\; p_{i_2}(y) \;=\; \cdots \;=\; p_{i_j}(y) \;=\; 0\,,$$

defines a face of Δ. Even when this defines a face F of Δ, only the solutions to (5.10) that lie in F contribute to T_j, and hence to the number of paths followed. Thus any slack in the estimates (5.13) reduces the number of paths to be followed. Since these estimates lead to the fewnomial bound for s_l, we see that the Khovanskii-Rolle Continuation Algorithm naturally takes advantage of any lack of sharpness in the fewnomial bound.

This may further be improved if Δ has $m < l + n + 1$ facets for then the binomial coefficients in the exponents of 2 in (5.12) become $\binom{m}{2}$.

5.4. **An Example.** We explain how to use the Khovanskii-Rolle continuation algorithm to find all positive solutions to the following system of Laurent polynomials (written in diagonal form)

$$\begin{aligned}
cd &= \tfrac{1}{2}be^2 + 2a^{-1}b^{-1}e - 1 & cd^{-1}e^{-1} &= \tfrac{1}{2}(1 + \tfrac{1}{4}be^2 - a^{-1}b^{-1}e)\\
bc^{-1}e^{-2} &= \tfrac{1}{4}(6 - \tfrac{1}{4}be^2 - 3a^{-1}b^{-1}e) & bc^{-2}e &= \tfrac{1}{2}(8 - \tfrac{3}{4}be^2 - 2a^{-1}b^{-1}e)\\
ab^{-1} &= 3 - \tfrac{1}{2}be^2 + a^{-1}b^{-1}e\;.
\end{aligned} \tag{5.15}$$

We first convert it into a dual Gale system as in Section 4.1. Its set of exponent vectors are the columns of the matrix

$$\mathcal{A} := \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 2 & -1 & -2 & 1 & 0 \end{pmatrix}.$$

Since

$$\mathcal{A}\begin{pmatrix} -1 & 1 & -2 & 1 & -2 & 2 & -1 \\ 1 & 6 & -3 & 6 & -2 & 7 & 1 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

we have the following identity among the monomials

$$\begin{aligned} (a^{-1}b^{-1}e)^{-1}(cd)^1(be^2)^{-2}(cd^{-1}e^{-1})^1(bc^{-1}e^{-2})^{-2}(bc^{-2}e)^2(ab^{-1})^{-1} &= 1, \\ (a^{-1}b^{-1}e)^1\ (cd)^6(be^2)^{-3}(cd^{-1}e^{-1})^6(bc^{-1}e^{-2})^{-2}(bc^{-2}e)^7(ab^{-1})^1 &= 1. \end{aligned} \tag{5.16}$$

A Gale system dual to (5.15) is obtained from the identity (5.16) by first substituting x for the term $\frac{1}{4}be^2$ and y for $a^{-1}b^{-1}e$ in (5.15) to obtain

$$\begin{aligned} cd &= 2x + 2y - 1 & cd^{-1}e^{-1} &= \tfrac{1}{2}(1 + x - y) \\ bc^{-1}e^{-2} &= \tfrac{1}{4}(6 - x - 3y) & bc^{-2}e &= \tfrac{1}{2}(8 - 3x - 2y) \\ ab^{-1} &= 3 - 2x + y\ , \end{aligned}$$

Then, we substitute these affine linear polynomials for the monomials in (5.16) to obtain the system of rational functions

$$\begin{aligned} y^{-1}(2x{+}2y{-}1)\ (4x)^{-2}\left(\tfrac{1+x-y}{2}\right)\left(\tfrac{6-x-3y}{4}\right)^{-2}\left(\tfrac{8-3x-2y}{2}\right)^{2}(3{-}2x{+}y)^{-1} &= 1, \\ y(2x{+}2y{-}1)^6(4x)^{-3}\left(\tfrac{1+x-y}{2}\right)^{6}\left(\tfrac{6-x-3y}{4}\right)^{-2}\left(\tfrac{8-3x-2y}{2}\right)^{7}(3{-}2x{+}y) &= 1. \end{aligned} \tag{5.17}$$

If we solve these for 0, they become $f = g = 0$, where

$$\begin{aligned} f &:= (2x{+}2y{-}1)(1{+}x{-}y)(8{-}3x{-}2y)^2 - 8yx^2(6{-}x{-}3y)^2(3{-}2x{+}y), \\ g &:= y(2x{+}2y{-}1)^6(1{+}x{-}y)^6(8{-}3x{-}2y)^7(3{-}2x{+}y) - 32768\,x^3(6{-}x{-}3y)^2. \end{aligned} \tag{5.18}$$

Figure 3 shows the curves defined by f and g and the lines given by the affine linear factors in f and g. The curve $f = 0$ has the three branches indicated and the other arcs belong to $g = 0$. The central heptagon is the domain Δ in which the affine polynomial factors in (5.17) are all positive. The six solutions in Δ to $f = g = 0$ correspond under the Gale duality of Theorem 4.9 to the positive solutions of the original system of Laurent polynomials (5.15).

We solve the system of Laurent monomials by solving the system (5.17) using the Khovanskii-Rolle algorithm. For $y \in \Delta$, we may take logarithms of the system (5.17) to obtain the

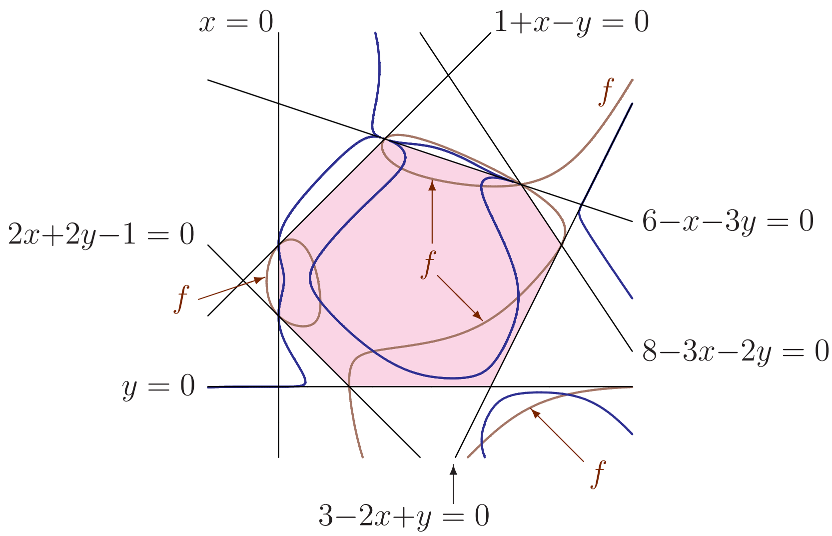


FIGURE 3. Curves and lines

system $\psi_1(y)=\psi_2(y)=0$, where

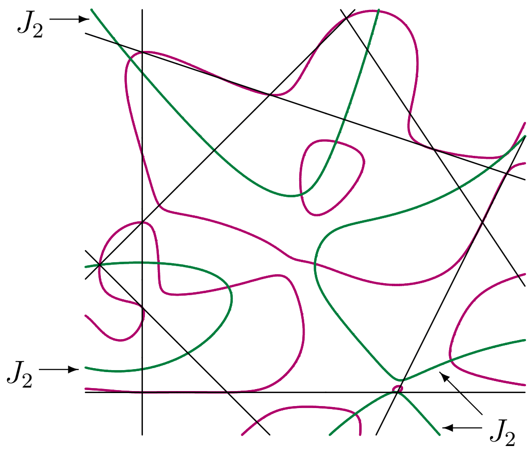
$$\begin{aligned}
\psi_1(y) &= -\log(y)+\log(2x+2y-1)-2\log(4x)+\log\left(\tfrac{1+x-y}{2}\right)\\
&\qquad -2\log\left(\tfrac{6-x-3y}{4}\right)+2\log\left(\tfrac{8-3x-2y}{2}\right)-\log(3-2x+y)\,.\\
\psi_2(y) &= \log(y)+6\log(2x+2y-1)-3\log(4x)+6\log\left(\tfrac{1+x-y}{2}\right)\\
&\qquad -2\log\left(\tfrac{6-x-3y}{4}\right)+7\log\left(\tfrac{8-3x-2y}{2}\right)+\log(3-2x+y)\,.
\end{aligned}$$

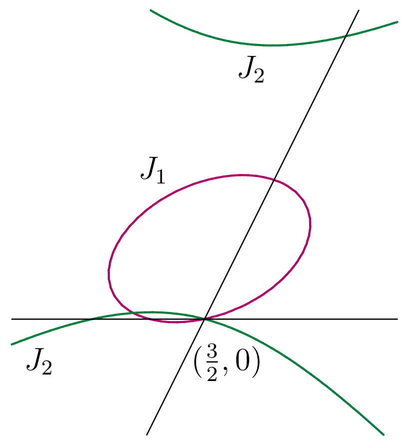
The polynomial forms J_2, J_1 of the Jacobians are (omitting the middle 60 terms from J_1),

$$\begin{aligned}
J_2 &= -168x^5-1376x^4y+480x^3y^2-536x^2y^3-1096xy^4+456y^5+1666x^4+2826x^3y\\
&\quad +3098x^2y^2+6904xy^3-1638y^4-3485x^3-3721x^2y-15318xy^2-1836y^3\\
&\quad +1854x^2+8442xy+9486y^2-192x-6540y+720\,.\\
J_1 &= 10080x^{10}-168192x^9y-611328x^8y^2-\quad\cdots\quad+27648x+2825280y\,.
\end{aligned}$$

We now describe the Khovanskii-Rolle algorithm on this example.

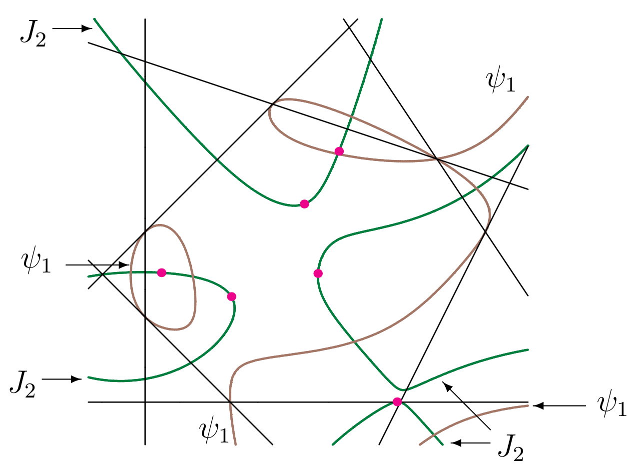
Precomputation. We first find all solutions S_0 to $J_1=J_2=0$ in the heptagon Δ, and all solutions $J_2=0$ in its boundary. Below are the curves $J_2=0$ and $J_1=0$ and the heptagon. The curve $J_2=0$ consists of the four indicated arcs. The remaining arcs in this picture belong to $J_1=0$. On the right is an expanded view in a neighborhood of the lower right vertex, $(\frac{3}{2}, 0)$.





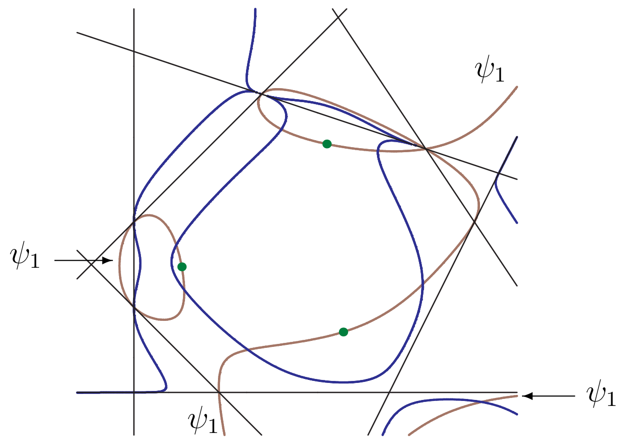
A numerical computation finds 50 common solutions to $J_1 = J_2 = 0$ with 26 real. Only six solutions lie in the interior of the heptagon with one on the boundary at the vertex $(3/2, 0)$. There are 31 points where the curve $J_2 = 0$ meets the lines supporting the boundary of the heptagon, but only eight lie in the boundary of the hexagon. This may be seen in the pictures above.

First continuation step. Beginning at each of the six points in Δ where $J_1 = J_2 = 0$, the algorithm traces the curve in both directions, looking for a solution to $\psi_1 = J_2 = 0$. Beginning at each of the eight points where the curve $J_2 = 0$ meets the boundary, it follows the curve into the interior of the heptagon, looking for a solution to $\psi_1 = J_2 = 0$. In tracing each arc, it either finds a solution, a boundary point, or another point where $J_1 = J_2 = 0$. We may see that in the picture below, which shows the curves $\psi_1 = 0$ and $J_2 = 0$, as well as the points on the curve $J_2 = 0$ where J_1 also vanishes.



In this step, $2 \cdot 6 + 8 = 20$ arcs are traced. The three solutions of $J_2 = \psi_1 = 0$ will each be found twice, and 14 of the tracings will terminate with a boundary point or a point where $J_1 = 0$.

Second Continuation step. This step begins at each of the three points where $\psi_1 = J_2 = 0$ that were found in the last step, as well as at each of the six points where $\psi_1 = 0$ meets the boundary of the heptagon (necessarily in some vertices).



Curve tracing can be carried out even in the presence of singularities [1, §4.3], as in the case of the curves initiating at vertices. In this case, this final round of curve-tracing revealed all six solutions within the heptagon, as anticipated. Furthermore, each solution was discovered twice, again, as anticipated.

By Theorem 5.11, the bound on the number of paths followed (using $7 = l + n$ in place of $l + n + 1$ in the binomials as in Remark 5.14) is

$$2 \cdot 2 \cdot 2^{\binom{2}{2}} \cdot 5^2 \ + \ 2 \cdot 2^{\binom{1}{2}} \cdot 5^1 \cdot \binom{7}{1} \ + \ 1 \cdot 2^{\binom{0}{2}} \cdot 5^0 \cdot \binom{8}{2} \ = \ 298\,.$$

By Theorem 3.10 in [7], the fewnomial bound in this case is

$$2 \cdot 5^2 + \lfloor \tfrac{(5+1)(5+3)}{2} \rfloor \ = \ 74\,.$$

In contrast, we only traced $20+12 = 32$ curves to find the six solutions in the heptagon. The reason for this discrepancy is that this fewnomial bound is pessimistic and the Khovanskii-Rolle Continuation Algorithm exploits any slack in it.

Exercises on Khovanskii-Rolle continuation.

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