Jeux de tableaux

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Abstract. We study four operations defined on pairs of tableaux. Algorithms for the first three involve the familiar procedures of jeu de taquin, row insertion, and column insertion, respectively. The fourth operation of hopscotch is new, although specialised versions have appeared previously. Like the other three operations, hopscotch may be computed with a set of local rules in a growth diagram, and it preserves Knuth equivalence class. Each of these four operations gives rise to an *a priori* distinct theory of dual equivalence. We show that these four theories coincide. The four operations are linked via the involutive tableau operations of complementation and conjugation.

Résumé. Nous étudions quatre opérations définies sur les paires de tableaux de Young. Les algorithmes pour trois des opérations impliquent les procédures familières de jeu de taquin, l'insertion par rangs, et l'insertion par colonnes, respectivement. La quatrième opération de hopscotch est nouvelle, bien que les versions spécialisiés aient apparu précédemment. Comme les trois autres opérations, hopscotch peut être calculée avec une série de règles locales dans un diagramme de croissance, et elle conserve la classe d'équivalence de Knuth. Chacune de ces quatre opérations engendre un théorie *a priori* distincte d'équivalence duale. Nous montrons que ces quatre théories coïncident. Les quatre opérations sont reliées par les opérations involutives de complémentation et de conjugaison.

Introduction

Schützenberger's jeu de taquin [11] and Schensted insertion [10] have found their way into the standard toolkit of the combinatorial representation theorist. These algorithms were originally developed for tableaux of partition shape, but more recent work [9, 3] uses them to define operations on pairs of skew tableaux. A

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striking duality between Schensted insertion and Schützenberger's jeu de taquin was noted by Stembridge [14] in his theory of rational tableaux.

We introduce the involutive operation of complementation to formalise this duality, showing that internal row insertion and the jeu de taquin give complementary operations on pairs of tableaux. Our main tool is the growth diagram formulation for operations on tableaux introduced by Fomin [2] and rediscovered several times, most notably by Van Leeuwen [5]. We define a new operation hopscotch which is complementary to internal column insertion; it extends both Stroomer's column-sliding algorithm [15] and Tesler's [16] rightward- and leftward-shift games.

These four operations of jeu de taquin, internal row insertion, internal column insertion, and hopscotch all preserve Knuth equivalence. A consequence is that hopscotch gives rise to new algorithms to compute the Knuth class of a tableau. We also show that these four operations have the same dual equivalence theory.

This paper is organised as follows. Section 1 recalls our basic definitions and terminology and in Section 2 we introduce complementation and describe its behaviour with respect to Knuth equivalence and dual equivalence. Section 3 is devoted to operations on pairs of tableaux given by growth diagrams constructed from local rules, and introduces the notion of complementary operations on pairs of tableaux. In Section 4 we show that the two operations of jeu de taquin and internal row insertion are complementary. Finally, in Section 5 we define internal column insertion and its complementary operation hopscotch.

Complete proofs of the results reported here are found in the extended version "Complementary Tableaux Operations" [8].

1 Tableaux

If we have two comparable partitions $\alpha \subseteq \beta$ in Young's lattice, then we may form the *(skew) shape* β/α, which has *inner border* α and *outer border* β. We call β/α a *horizontal strip* if no column of β/α contains two or more boxes. That is, $\beta_1 \geq \alpha_1 \geq \beta_2 \geq \cdots \geq \beta_m \geq \alpha_m$. When $\alpha \subseteq \beta \subseteq \delta$, we say that the shape δ/β *extends* the shape β/α. If $\alpha \subset\!\!\!\cdot\; \beta$ is a cover, then β/α is a single box b, which we call an *inner corner* of δ/β. If instead $\beta \subset\!\!\!\cdot\; \delta$, then δ/β is a single box b, called an *outer corner* of β/α.

A *tableau* T with shape β/α is a chain $\alpha = \beta^0 \subseteq \beta^1 \subseteq \cdots \subseteq \beta^k = \beta$, where the successive skew shapes β^i/β^{i-1} are horizontal strips. The *content* of T is the sequence whose ith component is the number of boxes in β^i/β^{i-1}. A tableau is *standard* if each β^i/β^{i-1} consists of a single box. The *standard renumbering* of a tableau T is the refinement of the chain representing T where each horizontal strip is filled in 'left-to-right'. A tableau T *extends* another tableau S whenever the shape of T extends the shape of S.

If P is a tableau extended by Q, then the standard renumbering of P gives a set of instructions for applying jeu de taquin slides to Q [11, 12, 3]: The last cover of P is an inner corner of Q, and after applying the slide beginning with that corner, the next slide begins at the new inner corner given by the next to last cover of P, and so on. Write U for the resulting tableau. The sequence of outer corners obtained from this procedure form a tableau T. If we regard Q as a set of instructions for slides on P then we also obtain T after standard renumbering [3, 1]. If we set $\mathcal{J}(P,Q) = (T,U)$, then $\mathcal{J}$ defines an involution, which we call the *jeu de taquin*, on pairs of tableaux where one extends the other.

Two tableaux P and T are *Knuth equivalent* if one can be obtained from the other by a sequence of jeu de taquin slides. (This is equivalent to their reading words being Knuth equivalent [13].) Two tableaux Q and U with the same shape are *dual equivalent* if applying the same sequence of jeu de taquin slides to Q and to U always gives tableaux of the same shape. These are related by a result of Haiman [3]: The intersection of any Knuth equivalence class and any dual equivalence class is either empty or consists of a unique tableau.

2 Complementation

The classical version of the Robinson-Schensted-Knuth correspondence [6, 10, 4] gives information about the *polynomial* representations of GL_k. Stembridge introduced complementation to model the procedure of dividing by the determinantal representation of GL_k [14], thereby handling the rational representations.

Let T be a tableau with shape β/α and entries from the alphabet $[k]$, and fix $l \geq \beta_1$. Form the *complement* T^C of T as follows. For each $1 \leq j \leq l$, let A be the set-theoretic complement of the entries of column j of T in the set $[k]$. The set $\{k+1-a_i \mid a_i \in A\}$ written in increasing order beginning immediately below the last box in column j of T forms column j of T^C. For instance, if $k=4$ and $l=5$, then the two tableaux below are complements of each other.

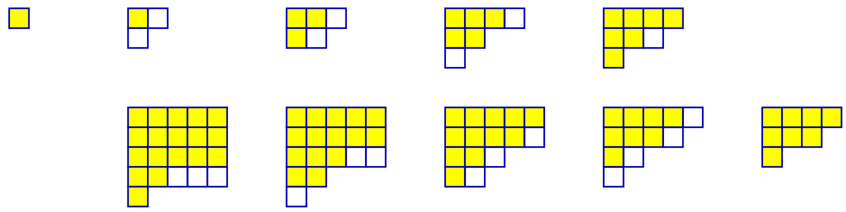
$$\begin{array}{|c|c|c|c|} \hline & 1 & 2 & 3 \\ \hline 1 & 2 & 4 & \\ \hline 3 & & & \\ \hline \end{array} \qquad\qquad \begin{array}{|c|c|c|c|c|} \hline & & & & 1 \\ \hline & & & 1 & 2 \\ \hline & 1 & 2 & 3 & 3 \\ \hline 1 & 2 & 4 & 4 & 4 \\ \hline 3 & & & & \\ \hline \end{array} \tag{1}$$

If we identify tableaux which differ by a vertical shift, we may also write T^C above T. This observation proves that complementation is involutive.

Theorem 1. *Let T be a tableau with shape β/α and entries from $[k]$ and suppose $l \geq \beta_1$. Then $T^{CC} = T$.*

Complementation depends upon both k and l. Our notation T^C intentionally disregards this dependence. We adopt the convention that we use the same integers k and l when two or more tableaux are to be complemented. Using an obvious extension of notation, we observe that the shape of T^C is $(l^k, \beta)/\alpha$.

We express complementation in terms of chains in Young's lattice. Let T be a tableau, written as a chain $\alpha = \beta^0 \subseteq \beta^1 \subseteq \cdots \subseteq \beta^k = \beta$, where β^i/β^{i-1} is the horizontal strip of i's in T. Then T^C is the chain $\beta \subseteq (l, \beta^{k-1}) \subseteq (l^2, \beta^{k-2}) \subseteq \cdots \subseteq (l^i, \beta^{k-i}) \subseteq \cdots \subseteq (l^k, \beta^0)$. It is an exercise that this agrees with the definition given above. In particular, it can immediately be seen that the diagram is as claimed. Each row of partitions below is one of the two complementary tableaux in (1). Here we write the second row in reverse order to illustrate the key feature of complementation—that the ith horizontal strip in the complement contains boxes in exactly the columns complementary to the corresponding $(k+1-i)$th horizontal strip in the original tableau.



Complementation preserves both Knuth equivalence and dual equivalence.

Theorem 2. *Suppose T and U are tableaux with at most l columns and entries from the alphabet $[k]$. Then*

(1) T is Knuth equivalent to U if and only if T^C is Knuth equivalent to U^C.
(2) T is dual equivalent to U if and only if T^C is dual equivalent to U^C.

3 Growth diagrams and local rules

Many properties of tableau algorithms such as symmetry become clear when the algorithms are formulated in terms of growth diagrams governed by local rules [2, 7, 5]. We study algorithms related via complementation of their diagrams.

3.1 Growth diagrams

A *growth diagram* is a rectangular array of partitions where every row and every column is a tableau, with the additional restriction that all tableaux formed by the rows have the same content, as do all tableaux formed by the columns. The upper left quadrant of Figure 1 is a growth diagram where the horizontal tableaux have content $(1, 2, 2)$ and the vertical tableaux have content $(1, 1, 2)$.

Given an integer $l \geq \delta_1$, where δ is the lower right partition in the growth diagram, we complement the tableau represented by each row to obtain a new tableau, also written as a chain of shapes. These combine together to give a new growth diagram. We then complement the columns of these diagrams to obtain two more growth diagrams. Figure 1 shows the four growth diagrams we obtain from the upper left quadrant by this process. By Theorem 1, further complementation of the rows or columns yields no new growth diagrams, up to identifying tableaux with their vertical shifts in the plane.

1	2	31	33	33	431	442	4441
21	22	321	332	332	4321	4422	44421
211	221	3211	3321	3321	43211	44221	444211
221	222	3221	3322	3322	43221	44222	444221
221	222	3221	3322	3322	43221	44222	444221
4211	4221	43211	43321	43321	443211	444221	4444211
4421	4422	44321	44332	44332	444321	444422	4444421
4441	4442	44431	44433	44433	444431	444442	4444441

Fig. 1. Complementary growth diagrams with $k = 3$ and $l = 4$.

3.2 Local rules

A local rule is a rule for completing the missing corner of a 2×2 growth diagram given 3 partitions that are related by horizontal strips. Suppose we have three partitions α, β, and γ with γ/α and β/α horizontal strips, so that

$$\begin{matrix} \alpha & \beta \\ \gamma & \end{matrix}$$

is a (partial) growth diagram. An *insertion local rule* $\mathcal{L}$ is a rule that associates a fourth partition δ to such a partial growth diagram such that

$$\begin{matrix} \alpha & \beta \\ \gamma & \delta \end{matrix}$$

is a growth diagram. We write $\delta = \mathcal{L}(\alpha; \beta, \gamma)$. We further require that $\mathcal{L}$ is invariant under vertical shifts of the horizontal strips β/α and γ/α, that $\mathcal{L}$ is symmetric in β and γ, and that $\mathcal{L}$ is reversible in that α can be reconstructed from β, γ, and δ. In order for the reversal map to be defined for all β, γ, and δ, we must allow tableaux to have rows indexed by nonpositive integers. In practice, we prepend one part of size l to each shape in the growth diagram before applying the reversal map $\mathcal{L}$. This amounts to a vertical shift of the tableaux in the plane. We choose to denote this reversal map also by $\mathcal{L}$, and write $\alpha = \mathcal{L}(\beta, \gamma; \delta)$.

A *switching local rule* $\mathcal{L}$ is a rule for completing the missing lower left corner γ of a partial growth diagram

$$\begin{matrix} \alpha & \beta \\ & \delta \end{matrix} .$$

We write $\gamma = \mathcal{L}(\alpha, \beta, \delta)$. By symmetry, a switching local rule also gives a rule for completing a missing upper right corner of a partial growth diagram. In order to make $\mathcal{L}$ reversible, we insist that the rule be symmetrical: $\gamma = \mathcal{L}(\alpha, \beta, \delta) \iff \beta = \mathcal{L}(\alpha, \gamma, \delta)$. The context in which we apply a local rule will always make clear which kind of rule, insertion or switching, we are using.

3.3 Tableaux algorithms

An insertion local rule $\mathcal{L}$ determines a bijection on pairs of tableaux which share a common border as follows. Given a pair (P, Q) of tableaux sharing an inner border, write P across the first row of an array and Q down the first column. Use $\mathcal{L}$ to fill in the array and obtain a growth diagram. Define $\mathcal{L}(P,Q) := (T,U)$ where T is the tableau of the last row and U the tableau of the last column in this diagram. A switching local rule $\mathcal{L}$ determines a similar bijection, where initially we write P across the first row and Q down the last column.

Theorem 3. *An insertion local rule $\mathcal{L}$ determines a bijection*

$$\left\{\begin{array}{l}\textit{Tableaux } P \textit{ and } Q \textit{ which}\\ \textit{share an inner border and}\\ \textit{occupy columns } 1,\ldots,l.\end{array}\right\} \stackrel{\mathcal{L}}{\longleftrightarrow} \left\{\begin{array}{l}\textit{Tableaux } T \textit{ and } U \textit{ which}\\ \textit{share an outer border and}\\ \textit{occupy columns } 1,\ldots,l.\end{array}\right\}$$

where P and T have the same content, as do Q and U.

A switching local rule $\mathcal{L}$ determines a bijection

$$\left\{\begin{array}{l}\textit{Tableaux } P \textit{ and } Q\\ \textit{where } Q \textit{ extends } P.\end{array}\right\} \stackrel{\mathcal{L}}{\longleftrightarrow} \left\{\begin{array}{l}\textit{Tableaux } T \textit{ and } U\\ \textit{where } T \textit{ extends } U.\end{array}\right\}$$

where P and T have the same content, as do Q and U. Also, P and U have the same inner border and Q and T have the same outer border.

We combine complementation of growth diagrams with this construction of tableau algorithms. Given a local rule $\mathcal{L}$, define its *complement* $\mathcal{L}^C$ by

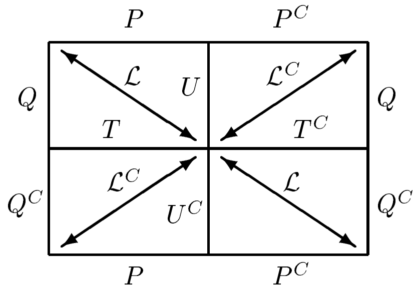
$$\mathcal{L}^C(P,Q) = (T,U^C), \quad \text{where} \quad \mathcal{L}(P,Q^C) = (T,U)\,, \tag{2}$$

when this is well-defined.

One set of conditions on $\mathcal{L}$ which ensure this is well-defined is the following.

I. $\mathcal{L}$ does not increase the number of columns in tableaux.
II. We have $\mathcal{L}(P,Q) = (T,U)$ if and only if $\mathcal{L}(P^C,Q^C) = (T^C,U^C)$.

The growth diagrams for $\mathcal{L}$ and $\mathcal{L}^C$ fit into a diagram of four growth diagrams as in Section 3.1, displayed schematically below.



3.4 Dual equivalence

Let $\mathcal{L}$ be an insertion local rule. Given tableaux P and Q sharing a border, set $(T,U) = \mathcal{L}(P,Q)$. This passage from P to T (which is determined by Q) is an $\mathcal{L}$-*move* applied to P. Two tableaux P and P' with the same shape are

$\mathcal{L}$*-dual equivalent* if applying the same sequence of $\mathcal{L}$-moves to P and to P' gives tableaux of the same shape. We similarly define $\mathcal{L}$-moves and $\mathcal{L}$-dual equivalence for a switching local rule $\mathcal{L}$. It follows that if Q and Q' are $\mathcal{L}$-dual equivalent, then the $\mathcal{L}$-moves determined by Q and Q' are identical.

Theorem 4. *Let $\mathcal{L}$ be a local rule. If (2) defines a complementary mapping $\mathcal{L}^C$, then $\mathcal{L}$-dual equivalence coincides with $\mathcal{L}^C$-dual equivalence. If $\mathcal{L}$ satisfies condition II, then a tableau P is $\mathcal{L}$-dual equivalent to Q if and only if P^C is $\mathcal{L}$-dual equivalent to Q^C.*

Proof. Complementing the second coordinates of a sequence of $\mathcal{L}$-moves applied to a tableau P gives a sequence of $\mathcal{L}^C$-moves applied to P. Thus $\mathcal{L}$-dual equivalence coincides with $\mathcal{L}^C$-dual equivalence. When $\mathcal{L}$ satisfies condition II, complementing both coordinates of a sequence of $\mathcal{L}$-moves applied to a tableau P gives a sequence of $\mathcal{L}$-moves applied to P^C. Thus P is $\mathcal{L}$-dual equivalent to Q if and only if P^C is $\mathcal{L}$-dual equivalent to Q^C. $\blacksquare$

4 Internal row insertion

We now apply the formalism of Section 3 to show Schützenberger's jeu de taquin and internal row insertion (a modification of the procedure introduced in [9]) are complementary operations on pairs of tableaux. We first define the insertion local rule $\mathcal{R}$ for internal row insertion. Suppose $\alpha \subset \beta$ and $\alpha \subset \gamma$ with β/α and γ/α horizontal strips. Choose $l > \beta_1, \gamma_1$, set $\alpha_0 = \beta_0 = \gamma_0 = \delta_0 = l$, and for $i > 0$,

$$\delta_i \;:=\; \max\{\beta_i, \gamma_i\} \;+\; \min\{\beta_{i-1}, \gamma_{i-1}\} \;-\; \alpha_{i-1}\,. \tag{3}$$

Define $\mathcal{R}(\alpha; \beta, \gamma) := \delta$. Observe that α is determined by β, γ, and δ.

Lemma 1. *The rule (3) defines an insertion local rule $\mathcal{R}$.*

Let *internal row insertion* be the tableau operation determined by $\mathcal{R}$. Observe that $\mathcal{R}$ satisfies the conditions I and II of Section 3.3, and so it has a complement. We formalise the duality between jeu de taquin slides and row insertion discovered by Stembridge [14].

Theorem 5. *The tableau operation $\mathcal{R}^C$ coincides with the jeu de taquin $\mathcal{J}$.*

The local rule $\mathcal{R}$ is described in terms of horizontal strips filled with entries a and b as follows. Given horizontal strips β/α filled with a's and γ/α filled with b's, we transfer any a's and b's which occupy the same boxes in row i into the next row, beginning with $\max\{\beta_i, \gamma_i\}$. The reverse operation is similarly described.

Example 1. In Figure 1 the upper-left and lower-right growth diagrams are examples of $\mathcal{R}$ while the remaining two growth diagrams are examples of the complementary action $\mathcal{J}$. For instance the lower left growth diagram shows that

$$\mathcal{J}\left(\begin{array}{cccc} & & \mathbf{a} & \mathbf{a} \\ & & \mathbf{b} & \mathbf{b} \\ & \mathbf{b} & \mathbf{c} & \mathbf{c} \\ \mathbf{a} & \mathbf{1} & \mathbf{2} & \\ \mathbf{2} & \mathbf{3} & \mathbf{3} & \end{array}\right) \quad = \quad \begin{array}{cccc} & & \mathbf{2} & \mathbf{a} \\ & & \mathbf{3} & \mathbf{b} \\ & \mathbf{1} & \mathbf{a} & \mathbf{c} \\ \mathbf{2} & \mathbf{3} & \mathbf{b} & \\ \mathbf{a} & \mathbf{b} & \mathbf{c} & \end{array}\,.$$

In Appendix 1 to Chapter 7 in [13], Fomin gives the following local rule for the jeu de taquin: If $\alpha \lessdot \beta \lessdot \delta$, then either the interval $[\alpha, \delta]$ in Young's lattice contains a fourth partition $\gamma \neq \beta$, or else it is a chain, in which case we set $\gamma := \beta$. One may verify from the definition of $\mathcal{J} = \mathcal{R}^C$ that $\gamma = \mathcal{J}(\alpha, \beta, \delta)$.

Similarly, suppose $\alpha \lessdot \beta$ and $\alpha \lessdot \gamma$. In Section 7.13 of [13], Stanley gives Fomin's local rules for Robinson-Schensted insertion.

(1) If $\beta \neq \gamma$, then let $\delta := \beta \vee \gamma$, the unique partition covering both β and γ.
(2) If $\beta = \gamma$, then β/α is a single box in the ith row, and we define δ so that δ/β is a single box in the $(i+1)$st row.

One may verify from the definition (3) of $\mathcal{R}$ that $\delta = \mathcal{R}(\alpha; \beta, \gamma)$.

The skew insertion procedure of Sagan and Stanley [9] mixes Schensted insertion with internal insertion $\mathcal{R}$. If tableaux T and U share a common inner border, then $\mathcal{R}$ acts on the pair (T, U) in the same way as the forward direction of the procedure of Theorem 6.11 of [9] when the matrix word π is empty.

5 Internal Column Insertion and Hopscotch

For standard tableaux (and hence for all tableaux via standard renumbering), internal column insertion is equivalent to internal row insertion. When combined with complementation we get a new operation, which we call hopscotch.

5.1 Local rules for column insertion

The local-rules formulation of internal column insertion for standard tableaux is the matrix transpose of that for $\mathcal{R}$: Simply replace 'row' by 'column' in the Fomin rules at the end of Section 4.

Consider now the reverse procedure. Given $\beta \lessdot \delta$ and $\gamma \lessdot \delta$, we define α by

1. If $\gamma \neq \beta$, then α is their greatest lower bound, $\beta \wedge \gamma$.
2. If $\gamma = \beta$, and the box δ/β is in column j, then β/α is a box in column $j-1$.

This fails when δ/β is a box in the first column. This forces us to generalise our notions of shape and tableaux to allow creation of new columns as necessary. Henceforth we define a *shape* to be a finite weakly decreasing sequence of integers (positive or negative), called *parts*. This set of shapes with a fixed length forms a poset under componentwise comparison, extending Young's lattice of partitions. Using $\subset$ for this partial order, we define skew shapes β/α for $\alpha \subset \beta$ as before.

A horizontal strip is (as before), a skew shape β/α with at most one box in each column. We define a *tableau* of shape β/α with entries from $[k]$ to be a chain $\alpha = \beta^0 \subset \beta^1 \subset \cdots \subset \beta^k = \beta$ where each β^i/β^{i-1} is a horizontal strip. We can convert a tableau T (as defined here) into a tableau as defined in Section 1 merely by adding a large enough number to each part of the shapes that define T, in effect shifting T horizontally.

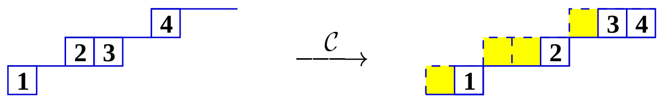
5.2 Internal column insertion for column-strict tableaux

For shapes α, β, and γ with β/α and γ/α horizontal strips, we define the shape δ recursively as follows. Suppose that the shapes α, β, and γ each have m parts. Set $d_m = 0$, and for $i = m, m-1, \ldots, 2$ set

$$\begin{aligned} \delta_i &:= \min(\beta_i + \gamma_i + d_i - \alpha_i,\ \alpha_{i-1}), \\ d_{i-1} &:= \max(\beta_i + \gamma_i + d_i - \alpha_i - \alpha_{i-1},\ 0). \end{aligned} \tag{4}$$

and set $\delta_1 := \beta_1 + \gamma_1 + d_1 - \alpha_1$. Define $\mathcal{C}(\alpha; \beta, \gamma) := \delta$.

For example, suppose $\alpha = (5, 2, 0)$ and $\beta = \gamma = (6, 4, 1)$. Then we have



and so $\delta = (8, 5, 2)$. Here $d_3 = d_2 = 0$, but $d_1 = 1$.

Example 2. We apply $\mathcal{C}$ to the tableaux P and Q of the upper left growth diagram in Figure 1 to obtain the following growth diagram. Here each $d_i = 0$.

$$\begin{array}{llll} 1 & 2 & 31 & 33 \\ 11 & 21 & 32 & 43 \\ 21 & 31 & 42 & 53 \\ 221 & 321 & 431 & 541 \end{array} \tag{5}$$

Lemma 2. *The rule (4) defines a local rule $\mathcal{C}$.*

Internal column insertion $\mathcal{C}$ is the tableau operation determined by $\mathcal{C}$.

Theorem 6. *Internal column insertion is a bijection*

$$\left\{ \begin{array}{l} \textit{Tableaux } P \textit{ and } Q \textit{ which} \\ \textit{share an inner border.} \end{array} \right\} \xleftrightarrow{\ \mathcal{C}\ } \left\{ \begin{array}{l} \textit{Tableaux } T \textit{ and } U \textit{ which} \\ \textit{share an outer border.} \end{array} \right\}$$

with $\mathcal{C} \circ \mathcal{C}$ the identity such that if $\mathcal{C}(P, Q) = (T, U)$, then P is Knuth equivalent to T and Q is Knuth equivalent to U. Also, the outer border of P equals the inner border of U, and outer border of Q equals the inner border of T.

Furthermore, $\mathcal{C}$-dual equivalence coincides with $\mathcal{J}$-dual equivalence.

5.3 Hopscotch and stable tableaux

We would like to define the new tableau operation $\mathcal{H}$ of hopscotch to be the operation complementary to internal column insertion $\mathcal{C}$. That is, if Q extends P, then we would set

$$\mathcal{H}(P, Q) \ := (T, U^C) \qquad \text{where} \qquad \mathcal{C}(P, Q^C) \ = \ (T, U)\,. \tag{6}$$

This does not work as condition I of Section 3.3 fails to hold for $\mathcal{C}$.

We solve this problem by building an asymmetry into $\mathcal{H}$ so that one of P or Q is a new object called a stable tableau, which is an ordinary tableau complemented with respect to infinitely many columns.

A *stable shape* α is a finite weakly decreasing sequence of integers and the symbols ∞ and $\overline{\infty}$, where $\overline{\infty} < n < \infty$ for all integers n. If we regard a shape to have arbitrarily many trailing $\overline{\infty}$s, then stable shapes form a lattice under componentwise comparison, which contains our earlier poset of shapes. We form the stable skew shape β/α as before. If the finite parts of α and β occur in the same rows, then we can ignore initial ∞'s and trailing $\overline{\infty}$'s and regard β/α as an ordinary skew shape. In this way, stable skew shapes and ordinary skew shapes may extend each other.

A *stable horizontal strip* β/α is a pair of stable shapes $\alpha \subseteq \beta$ where, if α_i is the first finite part of α and β_m the last finite part of β, then

$$\alpha_{i-1} = \infty = \beta_i > \alpha_i \geq \beta_{i+1} \geq \cdots \geq \alpha_{m-1} \geq \beta_m > \alpha_m = \overline{\infty} = \beta_{m+1}\,.$$

If we consider β/α as a collection of boxes in the plane, then β/α has at most one box in each column, and it has 2 half-infinite rows of boxes extending from either end. Equivalently, if we let $\alpha^S := (\infty, \alpha)$, then β/α is a stable horizontal strip if and only if $\beta \subseteq \alpha^S$, and removing the ∞'s and $\overline{\infty}$'s from β and α^S, α^S/β is an ordinary horizontal strip with boxes in the empty columns of β/α.

A *stable tableau* T with stable shape β/α and entries from $[k]$ is a chain $\alpha = \beta^0 \subseteq \beta^1 \subseteq \cdots \subseteq \beta^k = \beta$ of stable shapes, where the successive stable skew shapes β^i/β^{i-1} are stable horizontal strips. The appropriate notion of complementation for stable tableaux involves complementing all columns. Thus if $T : \beta^0 \subseteq \beta^1 \subseteq \cdots \subseteq \beta^k$ is a stable tableau, its *stable complement* T^S is the chain $\beta^k \subset (\infty, \beta^{k-1}) \subset \cdots \subset (\infty^k, \beta^0)$. Since T was a stable tableau, each shape β^i has one more part of ∞ than its predecessor β^{i-1} and the same number of finite parts, and so we may remove the same number of parts of ∞ from each shape (∞^{k-i}, β^i) of T^S and obtain an ordinary tableau. In Figure 2, we show a tableau T and its stable complement T^S.

Theorem 7. *Let T be a tableau with diagram β/α and entries from $[k]$. Then T^S is a stable tableau with stable shape $(\infty^k, \alpha)/\beta$ and entries from $[k]$. Similarly, if T is a stable tableau with entries from $[k]$, then T^S is an ordinary tableau with entries from $[k]$. In either case, we have $T^{SS} = T$.*

(For the last assertion, we identify tableaux that differ by a vertical shift.)

Definition 1. *We define hopscotch, $\mathcal{H}$, to be the stable complement of internal column insertion. That is, given P and Q with P a tableau and Q a stable tableau with either P extending Q or Q extending P, then we set*

$$\mathcal{H}(P;Q) \ := \ (T;U^S) \qquad \textit{where} \qquad \mathcal{C}(P,Q^S) \ = \ (T,U)\,.$$

Suppose P and Q are as in Definition 1, and we have $\mathcal{H}(P;Q) = (T;U)$. We call the passage from the tableau P to the tableau T a *$\mathcal{H}$-move applied to P*

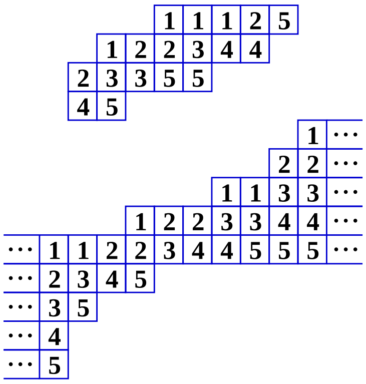


Fig. 2. (Stable) complementary tableaux.

(which depends upon the stable tableau Q). Likewise, the passage from Q to the stable tableau U is also called a $\mathcal{H}$-move. As in Section 3.4, this gives rise to the notion of $\mathcal{H}$-dual equivalence for tableaux and also for stable tableaux.

Theorem 8. *The tableau operation hopscotch gives a bijection*

$$\left\{\begin{array}{l}\text{Pairs } (P;Q) \text{ with } P \text{ a}\\ \text{tableau, } Q \text{ a stable tableau,}\\ \text{and } Q \text{ extending } P.\end{array}\right\} \xleftrightarrow{\mathcal{H}} \left\{\begin{array}{l}\text{Pairs } (T;U) \text{ with } T \text{ a}\\ \text{tableau, } U \text{ a stable tableau,}\\ \text{and } T \text{ extending } U.\end{array}\right\}$$

with $\mathcal{H}\circ\mathcal{H}$ the identity. If $\mathcal{H}(P;Q) = (T;U)$, then P is Knuth equivalent to T and Q^S is Knuth equivalent to U^S.

Furthermore, two tableaux P and T are $\mathcal{H}$-dual equivalent if and only if P and T are $\mathcal{J}$-dual equivalent, and two stable tableaux Q and U are $\mathcal{H}$-dual equivalent if and only if the tableaux Q^S and U^S are $\mathcal{J}$-dual equivalent.

5.4 Hopscotch and Tesler's Shift Games

We give local rules for $\mathcal{H}$ when the ordinary tableau is standard, that is when the horizontal strip is a single box. As hopscotch commutes with standard renumbering (because internal column insertion does), this enables the computation of $\mathcal{H}(P;Q)$ when P is a column-strict tableau.

Definition 2. *Let $\alpha \subset \beta \Subset \delta$ be shapes with β/α a stable horizontal strip. Let c denote the column of δ/β, and H denote the set of columns of β/α. Define the stable shape γ with $\alpha \Subset \gamma \subset \delta$ as follows.*

1. *If $c \in H$, then define γ so that α and γ also differ in column c. (So δ/γ will be the same set of columns H.)*
2. *If $c \notin H$, then let c' denote the smallest element of H greater than c, and define γ so that γ and α differ in column c'. (So δ/γ will be the set of columns $H' := H \cup \{c'\}\backslash\{c\}$.)*

Theorem 9. *Given stable shapes $\alpha \subset \beta \subset \delta$ with β/α a stable horizontal strip, the stable shape γ defined in Definition 2 satisfies $\gamma = \mathcal{H}(\alpha, \beta, \delta)$.*

In his study of semi-primary lattices, Tesler [16] defines certain leftward- and rightward-shift games applied to standard skew tableaux. These give rise to an algorithm to construct a tableau T of partition shape from a standard skew tableau P. He then uses a geometric argument to show that P and T are Knuth equivalent. In fact, Tesler's algorithm may be regarded as a special case of hopscotch and this gives a combinatorial proof that his algorithm computes the unique tableau of partition shape Knuth equivalent to a skew tableau P.

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