

ON THE SHARPNESS OF THE FEWNOMIAL BOUND AND THE NUMBER OF COMPONENTS OF A FEWNOMIAL HYPERSURFACE

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ABSTRACT. We prove the existence of systems of n polynomial equations in n variables with a total of $n + k + 1$ distinct monomial terms possessing $\lfloor \frac{n+k}{k} \rfloor^k$ nondegenerate positive solutions. This shows that the recent upper bound of $\frac{e^2+3}{4} 2^{\binom{k}{2}} n^k$ for the number of nondegenerate positive solutions has the correct order for fixed k and large n . We also adapt a method of Perrucci to show that there are fewer than $\frac{e^2+3}{4} 2^{\binom{k+1}{2}} 2^n n^{k+1}$ connected components in a smooth hypersurface in the positive orthant of \mathbb{R}^N defined by a polynomial with $n+k+1$ monomials, where n is the dimension of the affine span of the exponent vectors. Our results hold for polynomials with real exponents.

1. INTRODUCTION

Khovanskii [5] showed there are at most $2^{\binom{n+k}{2}} (n+1)^{n+k}$ nondegenerate positive solutions to a system of n polynomial equations in n variables which are linear combinations of (the same) $n+k+1$ monomials. This fewnomial bound is also valid in the more general setting of linear combinations of monomials with real-number exponents. The underlying bounds are identical whether one uses integral or real exponents [8], and the arguments of [3] require that we allow real exponents.

While Khovanskii's fewnomial bound was not believed to be sharp, only recently have smaller bounds been found. The first breakthrough was due to Li, Rojas, and Wang [7] who showed that a system of two trinomials in two variables has at most 5 positive solutions — which is smaller than Khovanskii's bound of 5184. Bihan [2] showed that a system of n polynomials in n variables with $n+2$ monomials has at most $n+1$ nondegenerate positive solutions and proved the existence of such a system with $n+1$ positive solutions. Bihan and Sottile [3] generalized this to all k , giving the upper bound of $\frac{e^2+3}{4} 2^{\binom{k}{2}} n^k$ for the number of nondegenerate positive solutions, which is significantly smaller than Khovanskii's bound.

1.1. A Lower Bound for Fewnomial Systems. We show that the Bihan-Sottile upper bound [3] is near-optimal for fixed k and large n .

Theorem 1. *For any positive integers n, k with $n > k$, there exists a system of n polynomials in n variables involving $n+k+1$ distinct monomials and having $\lfloor \frac{n+k}{k} \rfloor^k$ nondegenerate positive solutions.*

We believe that there is room for improvement in the dependence on k , both in the upper bound of [3] and in the lower bound of Theorem 1.

2000 *Mathematics Subject Classification.* 14P99.

Key words and phrases. Fewnomials, connected component.

Rojas supported by NSF CAREER grant DMS-0349309.

Sottile supported by NSF CAREER grant DMS-0538734.

Proof. We will construct such a system when $n = km$, a multiple of k , from which we may deduce the general case as follows. Suppose that $n = km + j$ with $1 \leq j < k$ and we have a system of mk equations in mk variables involving $mk+k+1$ monomials with $(m+1)^k$ nondegenerate positive solutions. We add j new variables x_1, \dots, x_j and j new equations $x_1 = 1, \dots, x_j = 1$. Since the polynomials in the original system may be assumed to have constant terms, this gives a system with n polynomials in n variables having $n+k+1$ monomials and $(m+1)^k = \lfloor \frac{n+k}{k} \rfloor^k$ nondegenerate positive solutions. So let us fix positive integers k, m and set $n = km$.

Bihan [2] showed there exists a system of m polynomials in m variables

$$f_1(y_1, \dots, y_m) = \dots = f_m(y_1, \dots, y_m) = 0$$

having $m+1$ solutions, and where each polynomial has the same $m+2$ monomials, one of which we may take to be a constant.

For each $j = 1, \dots, k$, let $y_{j,1}, \dots, y_{j,m}$ be m new variables and consider the system

$$f_1(y_{j,1}, \dots, y_{j,m}) = \dots = f_m(y_{j,1}, \dots, y_{j,m}) = 0,$$

which has $m+1$ positive solutions in $(y_{j,1}, \dots, y_{j,m})$. As the sets of variables are disjoint, the combined system consisting of all km polynomials in all km variables has $(m+1)^k$ positive solutions. Each subsystem has $m+2$ monomials, one of which is a constant. Thus the combined system has $1 + k(m+1) = km+k+1 = n+k+1$ monomials. \square

Remark 2. Our proof of Theorem 1 is based on Bihan's nonconstructive proof of the existence of a system of n polynomials in n variables having $n+2$ monomials and $n+1$ nondegenerate positive solutions. While finding such systems explicitly is challenging in general, let us do so for $n \in \{2, 3\}$.

The system of $n = 2$ equations with 2 variables

$$x^2y - (1 + 4x^2) = xy - (4 + x^2) = 0,$$

has $4 = 2 + 1 + 1$ monomials and exactly 3 complex solutions, each of which is nondegenerate and lies in the positive quadrant. We give numerical approximations, computed with the computer algebra system SINGULAR [4, 10],

$$(2.618034, 4.1459), \quad (1, 5), \quad (0.381966, 10.854102).$$

The system of $n = 3$ equations with 3 variables

$$yz - (1/8 + 2x^2) = xy - (1/220 + x^2) = z - (1 + x^2) = 0,$$

has $5 = 3 + 1 + 1$ monomials and exactly 4 complex solutions, each of which is nondegenerate and lies in the positive octant,

$$(0.076645, 0.1359, 1.00587), \quad (0.084513, 0.13829, 1.00714), \\ (0.54046, 0.54887, 1.2921), \quad (1.29838, 1.30188, 2.6858).$$

1.2. An Upper Bound for Fewnomial Hypersurfaces. Khovanskii also considered smooth hypersurfaces in the positive orthant $\mathbb{R}_{>}^n$ defined by polynomials with $n+k+1$ monomials. He showed [6, Sec. 3.14, Cor. 4] that the total Betti number of such a fewnomial hypersurface is at most

$$(2n^2 - n + 1)^{n+k} (2n)^{n-1} 2^{\binom{n+k}{2}}.$$

Li, Rojas, and Wang [7] bounded the number of connected components of such a hypersurface by $n(n+1)^{n+k+1} 2^{n-1} 2^{\binom{n+k+1}{2}}$. Perrucci [9] lowered this bound to $(n+1)^{n+k} 2^{1+\binom{n+k}{2}}$.

His method was to bound the number of compact components and then use an argument based on the faces of the n -dimensional cube to bound the number of all components. We improve Perrucci's method, using the n -simplex and the bounds of Bihan and Sottile [3] to obtain a new, lower bound.

Theorem 3. *A smooth hypersurface in $\mathbb{R}_{>}^N$ defined by a polynomial with $n+k+1$ monomials whose exponent vectors have n -dimensional affine span has fewer than*

$$\frac{e^2 + 3}{4} \cdot 2^{\binom{k+1}{2}} 2^n n^{k+1}$$

connected components.

Both our method and that of Perucci estimate the number of compact components in the intersection of the hypersurface with linear spaces which are chosen to preserve the sparse structure so that the fewnomial bound still applies. For this same reason, this same method was used by Benedetti, Loeser, and Risler [1] to bound the number of components of a real algebraic set in terms of the degrees of its defining polynomials.

Observe that if the exponent vectors of the monomials in a polynomial f in N variables have n -dimensional affine span, then there is a monomial change of variables on $\mathbb{R}_{>}^N$ so that f becomes a polynomial in the first n variables, and thus our hypersurface becomes a cylinder

$$\mathbb{R}_{<}^{N-n} \times \{x \in \mathbb{R}_{>}^n \mid f(x) = 0\},$$

which shows that it suffices to prove Theorem 3 when $n = N$. That is, for smooth hypersurfaces in \mathbb{R}^n defined by polynomials with $n+k+1$ monomial terms whose exponent vectors affinely span \mathbb{R}^n .

Let $\kappa(n, k)$ be the maximum number of compact connected components of such a smooth hypersurface and let $\tau(n, k)$ be the maximal number of connected components of such a hypersurface. We deduce Theorem 3 from the following combinatorial lemma.

Lemma 4. $\tau(n, k) \leq \sum_{i=0}^{n-1} \binom{n+1}{i} \kappa(n-i, k+1).$

Proof of Theorem 3. Bihan and Sottile [3] proved that

$$\kappa(n, k) \leq \frac{e^2 + 3}{8} 2^{\binom{k}{2}} n^k.$$

Substituting this into Lemma 4 bounds $\tau(n, k)$ by

$$\begin{aligned} & \frac{e^2 + 3}{8} 2^{\binom{k+1}{2}} \sum_{i=0}^{n-1} \binom{n+1}{i} (n-i)^{k+1} \\ & < \frac{e^2 + 3}{8} 2^{\binom{k+1}{2}} n^{k+1} \sum_{i=0}^{n+1} \binom{n+1}{i} = \frac{e^2 + 3}{8} 2^{\binom{k+1}{2}} n^{k+1} 2^{n+1}. \quad \square \end{aligned}$$

Proof of Lemma 4. Let f be a polynomial in the variables x_1, \dots, x_n which has $n+k+1$ distinct monomials whose exponent vectors affinely span \mathbb{R}^n and suppose that $f(x) = 0$ defines a smooth hypersurface X in $\mathbb{R}_{>}^n$. We may apply a monomial change of coordinates to $\mathbb{R}_{>}^n$ and assume that $1, x_1, x_2, \dots, x_n$ are among the monomials appearing in f .

Suppose that $\varepsilon := (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n) \in \mathbb{R}_{>}^{1+n}$ with $\varepsilon_0 \varepsilon_1 \cdots \varepsilon_n \neq 1$. Define hypersurfaces H_0, H_1, \dots, H_n of $\mathbb{R}_{>}^n$ by

$$\begin{aligned} H_0 &:= \{x \in \mathbb{R}_{>}^n \mid x_1 \cdots x_n \varepsilon_0 = 1\} \quad \text{and} \\ H_i &:= \{x \mid x_i = \varepsilon_i\} \quad \text{for } i = 1, \dots, n. \end{aligned}$$

The transformation $\text{Log}: \mathbb{R}_{>}^n \rightarrow \mathbb{R}^n$ defined on each coordinate by $x_i \mapsto \log(x_i)$ sends the hypersurfaces H_0, H_1, \dots, H_n to hyperplanes in general position. That is, if $S \subset \{0, 1, \dots, n\}$ and we define $H_S := \bigcap_{i \in S} H_i$, then $\text{Log}(H_S)$ is an affine linear space of dimension $n - |S|$. Moreover, the complement of the union of hypersurfaces H_i has $2^{n+1} - 1$ connected components, exactly one of which is bounded.

If we restrict f to some H_S , we will obtain a new polynomial f_S in $n - |S|$ variables with at most $1 + (n - |S|) + (k + 1)$ monomials. Indeed, if $i \in S$ with $i \neq 0$, then the equation $x_i = \varepsilon_i$ allows us to eliminate both the variable and the monomial x_i from f . If however $0 \in S$, then we pick an index $j \notin S$ and use $x_1 \cdots x_n \varepsilon_0 = 1$ to eliminate the variable x_j , which will not necessarily eliminate a monomial from f . For almost all ε , the polynomial f_S defines a smooth hypersurface X_S of H_S .

We may choose ε small enough so that every compact connected component of X lies in the bounded region of the complement of the hypersurfaces H_i , and every noncompact connected component of X meets some hypersurface H_i . Shrinking ε if necessary, we can ensure that every bounded component of X_S lies in the bounded region of the complement of $H_j \cap H_S$ for $j \notin S$, and every unbounded component meets some $H_j \cap H_S$ for $j \notin S$.

Given a connected component C of X , the subsets $S \subset \{0, 1, \dots, n\}$ such that C meets H_S form a simplicial complex. If S represents a maximal simplex in this complex, then $C \cap H_S$ is a union of compact components of X_S , and $|S| < n$ as H_S is not a point. Thus the number of connected components of X is bounded by the sum of the numbers of compact components of X_S for all $S \subset \{0, 1, \dots, n\}$ with $n > |S|$. Since each f_S has at most $1 + (n - |S|) + (k + 1)$ monomials, this sum is bounded by the sum in the statement of the lemma.

Remark 5. If f contains a monomial $x^a := x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ with no $a_i = 0$, then we can alter the proof of Lemma 4 to obtain a bound of

$$\frac{e^2 + 3}{4} \cdot 2^{\binom{k}{2}} 2^n n^k$$

connected components for the hypersurface defined by f .

The basic idea is that if we redefine H_0 to be

$$H_0 := \{x \in \mathbb{R}_{>}^n \mid x^a \varepsilon_0 = 1\}.$$

then the polynomials f_S on H_S have only $1 + (n - |S|) + k$ monomials, and so we estimate the number of compact components of X_S by $\kappa(n - i, k)$ instead of $\kappa(n - i, k + 1)$.

ACKNOWLEDGMENT

We thank Alicia Dickenstein and Daniel Perucci whose comments inspired us to find the correct statement and proof of Theorem 3.

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